[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 171–176 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Definition 7.]

Definition 7.

Ἐπίπεδος ἐπιφάνειά ἐστιν, ἥτις ἐξ ἴσου ταῖς ἐφ᾽ ἑαυτῆς εὐθείαις κεῖται.

A plane surface *is a surface which lies evenly with the straight lines on itself.*

The Greek follows exactly the definition of a straight line *mutatis mutandis*, i.e., with ταῖς…εὐθείαις for τοῖς…σημείοις. Proclus remarks that, in general, all the definitions of a straight line can be adapted to the plane surface by merely changing the *genus*. Thus, for instance, a plane surface is "a *surface* the middle of which covers the ends" (this being the adaptation of Plato's definition of a straight line). Whether Plato actually gave this as the definition of a surface or not, I believe that Euclid's definition of a plane surface as *lying evenly with the straight lines on itself* was intended simply to express the same idea without any implied appeal to vision (just as in the corresponding case of the definition of a straight line).

As already noted under Def. 4, Proclus tries to read into Euclid's definition the Archimedean assumption that "of surfaces which have the same extremities, if those extremities are in a plane, the plane is the least." But, as I have stated, his interpretation of the words seems impossible, although it is adopted by Simplicius also (see an-Nairīzī).

Ancient alternatives.

The other ancient definitions recorded are as follows.

1. *The surface which is stretched to the utmost* (ἐπ᾽ ἄκρον τεταμένη): a definition which Proclus describes as equivalent to Euclid's definition (on Proclus' own view of that definition). Cf. Heron, Def. 9, "(a surface) which is right (and) stretched out" (ὀρθὴ οὖσα ἀποτεταμένη), words which he adds to Euclid's definition.

2. *The least surface among all those which have the same extremities.* Proclus is here (p. 117, 9) obviously quoting the Archimedean *assumption*.

3. *A surface all the parts of which have the property of fitting on* (*each other*) (Heron, Def. 9).

4. *A surface such that a straight line fits on all parts of it* (Proclus, p. 117, 8), or *such that the straight line fits on it all ways*, i.e. however placed (Proclus, p. 117, 20).

With this should be compared:

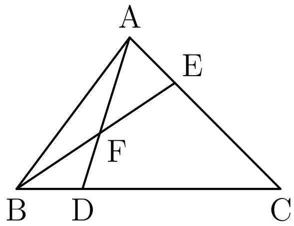
5. "(*A plane surface is*) *such that, if a straight line pass through two points on it, the line coincides wholly with it at every spot, all ways,*" i.e. however placed (one way or the reverse, no matter how), ἧς ἐπειδὰν δύο σημείων ἅψηται εὐθεῖα, καὶ ὅλη αὐτῇ κατὰ πάντα τόπον παντοίως ἐφαρμόζεται (Heron, Def. 9). This appears, with the words κατὰ πάντα τόπον παντοίως omitted, in Theon of Smyrna (p. 112, 5, ed. Hiller), so that it goes back at least as far as the 1st c. A.D. It is of course the same as the definition commonly attributed to Robert Simson, and very widely adopted as a substitute for Euclid's.

This same definition appears also in an-Nairīzī (ed. Curtze, p. 10) who, after quoting Simplicius' explanation (on the same lines as Proclus') of the meaning of Euclid's definition, goes on to say that "others defined the plane surface as that in which it is possible to draw a straight line from any point to any other."

Difficulties in ordinary definitions.

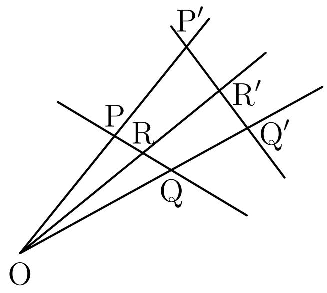
Gauss observed in a letter to Bessel that the definition of a plane surface as *a surface such that, if any two points in it be taken, the straight line joining them lies wholly in the surface* (which, for short, we will call "Simson's" definition) contains more than is necessary, in that a plane can be obtained by simply projecting a straight line lying in it from a point outside the line but also also lying on the plane; in fact the definition includes a theorem, or postulate, as well. The same is true of Euclid's definition of a plane as the surface which "lies evenly with (*all*) the straight lines on itself," because it is sufficient for the definition of a plane if the surface "lies evenly" with those lines which pass through a fixed point on it and each of the several points of a straight line also lying in it but not passing through the point. But from Euclid's point of view it is immaterial whether a definition contains more than the necessary minimum *provided* that the *existence* of a thing possessing all the attributes contained in the definition is afterwards proved. This however is not done in regard to the plane. No proposition about the nature of a plane as such appears before Book XI., although its existence is presupposed in all the geometrical Books I.–IV. and VI.; nor in Book XI. is there any attempt to prove, e.g., by construction, the existence of a surface conforming to the definition. The explanation may be that the existence of the plane as defined was deliberately assumed from the beginning like that of points and lines, the existence of which, according to Aristotle, must be assumed as principles unproved, while the existence of everything else must be proved; and it may well be that Aristotle would have included plane surfaces with points and lines in this statement had it not been that he generally took his illustrations from *plane* geometry (excluding solid).

But, whatever definition of a plane is taken, the evolution of its essential properties is extraordinarily difficult. Crelle, who wrote an elaborate article *Zur Theorie der Ebene* (read in the Academie der Wissenschaften in 1834) of which account must be taken in any full history of the subject, observes that, since the plane is the field, as it were, of almost all the rest of geometry, while a proper conception of it is necessary to enable Eucl. I. 1 to be understood, it might have been expected that the theory of the plane would have been the subject of at least the same amount of attention as, say, that of parallels. This however was far from being the case, perhaps because the subject of parallels (which, for the rest, presuppose the notion of a plane) is *much easier* than that of the plane. The nature of the difficulties as regards the plane have also been pointed out recently by Mr Frankland (*The First Book of Euclid's Elements*, Cambridge, 1905): it would appear that, whatever definition is taken, whether the simplest (as containing the minimum necessary to determine a plane) or the more complex, e.g. Simson's, some postulate has to be assumed in addition before the fundamental properties, or the truth of the other definitions, can be established. Crelle notes the same thing as regards Simson's definition, containing *more* than is necessary. Suppose a plane which lies the triangle ABC. Let AD join the vertex A to any point D on BC, and BE the vertex B to any point E on CA.



Then, according to the definition, AD lies wholly in the plane of the triangle, so does BE. But if both AD and BE are to lie wholly in the one plane AD, BE must intersect, say at F: if they did not, there would be two planes in question, not one. But the fact that the lines intersect and that, say, AD does not pass above or below BE, is by no means self-evident.

Mr Frankland points out the similar difficulty as regards the simpler definition of a plane as the surface generated by a straight line passing always through a fixed point and always intersecting a fixed straight line. Let OPP', OQQ' drawn from O intersect the straight line X at P, Q respectively. Let R be any third point on X: then it needs to be proved that OR intersects $P'Q'$ in some point, say R'.



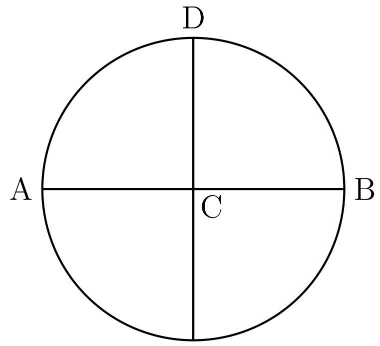
Without some postulate, however, it is not easy to see how to prove this, or even to prove that $P'Q'$ intersects X.

Crelle's essay. Definitions by Fourier, Deahna, Becker.

Crelle takes as the standard of a good definition that it shall be, not only as simple as possible, but also the best adapted for deducing, with the aid of the simplest possible principles, further properties belonging to the thing defined. He was much attracted by a very lucid definition, due, he says, to Fourier, according to which *a plane is formed by the aggregate of all the straight lines which, passing through one point on a straight line in space, are perpendicular to that straight line.* (This is really no more than an adaptation from Euclid's proposition XI. 5, to the effect that, if one of four concurrent straight lines be at right angles to each of the other three, those three are in one plane, which proposition is also used in Aristotle, *Meteorologica* III. 3, 373 a 13.) But Crelle confesses that he had not been able to deduce the necessary properties from this and had had to substitute the definition, already mentioned, of a plane as *the surface containing, throughout their whole length, all the straight lines passing through a fixed point and also intersecting a straight line in space*; and he only claims to have proved, after a long series of propositions, that the "Fourier"- or "perpendicular"-surface and the *plane* of the other definition just given are identical, after which the properties of the "Fourier"-surface can be used along with those of the plane. The advantage of the Fourier definition is that it leads easily, by means of the two propositions that triangles are equal in all respects (1) when two sides and the included angle are respectively equal and (2) when all three sides are respectively equal, to the property expressed in Simson's definition. But Crelle uses to establish these two congruence-theorems a number of propositions about *equal angles*, *supplementary* angles, *right* angles, *greater* and *less* angles; and it is difficult to question the soundness of Schotten's criticism that these notions in themselves really presuppose that of the plane. The difficulty due to Fourier's use of the word "perpendicular," if that were all, could no doubt be got over. Thus Deahna in a dissertation (Marburg, 1837) constructed a plane as follows. Presupposing the notions of a straight line and a sphere, he observes that, if a sphere revolve about a diameter, all the points of its surface which move describe closed curves (circles). Each of these circles, during the revolution, moves along itself, and one of them divides the surface of the sphere into two congruent parts. The aggregate then of the lines joining the centre to the points of this circle form the *plane.* Again J. K. Becker (*Die Elemente der Geometrie*, 1877) pointed out that the revolution of a right angle about one side of it produces a conical surface which differs from all other conical surfaces generated by the revolution of other angles in the fact that *the particular cone coincides with the cone vertically opposite to it*: this characteristic might therefore be taken in order to get rid of the use of the *right angle.*

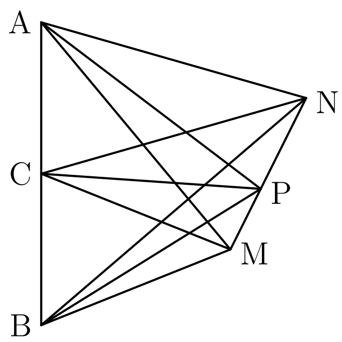
W. Bolyai and Lobachewsky.

Very similar to Deahna's equivalent for Fourier's definition is the device of W. Bolyai and Lobachewsky (described by Frischauf, *Elemente der absoluten Geometrie*, 1876). They worked upon a fundamental idea first suggested, apparently, by Leibniz. Briefly stated, their way of evolving a *plane* and a *straight line* was as follows. Conceive an infinite number of pairs of concentric spheres described about two fixed points in space, O, O', as centres, and with equal radii, gradually increasing: these pairs of equal spherical surfaces intersect respectively in homogeneous curves (circles), and the "Inbegriff" or aggregate of these curves of intersection forms a *plane.* If A be a point on one of these circles (k say), suppose points M, M' to start simultaneously from A and to move in opposite directions *at the same speed* till they meet at B, say: B then is "opposite" to A, and A, B divide the circumference into two equal halves. If the points A, B be held fast and the whole system be turned about them until O takes the place of O', and O' of O, the circle k will occupy the same position as before (though turned a different way). Two opposite points P, Q say, of each of the other circles will remain stationary during the motion as well as A, B: the "Inbegriff" or aggregate of all such points which remain stationary forms a *straight line.* It is next observed that the *plane* as defined can be generated by the revolution of the straight line about OO', and this suggests the following construction for a plane. Let a circle as one of the curves of intersection of the pairs of spherical surfaces be divided as before into two equal halves at A, B. Let the arc ADB be similarly bisected at D, and let C be the middle point of AB. This determines a straight line CD which is then *defined* as "perpendicular" to AB. The revolution of CD about AB generates a *plane.* The property stated in Simson's definition is then proved by means of the congruence-theorems proved in Eucl. I. 8 and I. 4. The first is taken as proved, practically by



considerations of symmetry and homogeneity. If two spherical surfaces, not necessarily equal, with centres O, O' intersect, A and its "opposite" point B are taken as before on the curve of intersection (a circle) and, relatively to OO', the point A is taken to be convertible with B or any other point on the homogeneous curve. The second (that of Eucl. I. 4) is established by simple application. Rausenberger objects to these proofs on the grounds that the first *assumes* that the two spherical surfaces intersect in one single curve, not in several, and that the second compares *angles*: a comparison which, he says, is possible only in a *plane*, so that a plane is really presupposed. Perhaps as regards the particular comparison of angles Rausenberger is hypercritical; but it is difficult to regard the supposed proof of the theorem of Eucl. I. 8 as sufficiently rigorous (quite apart from the use of the uniform *motion* of points for the purpose of bisecting lines).

Simson's property is proved from the two congruence-theorems thus. Suppose that AB is "perpendicular" (as defined by Bolyai) to two generators CM, CN of a plane, or suppose CM, CN respectively to make with AB two angles congruent with one another. It is enough to prove that, if P be any point on the straight line MN, then CP, just as much as CM, CN respectively, makes with AB two angles congruent with one another and is therefore a generator. We prove successively the congruence of the following



pairs of triangles:

$$\begin{array}{ll} ACM, & BCM \\ ACN, & BCN \\ AMN, & BMN \\ AMP, & BMP \\ ACP, & BCP, \end{array}$$

whence the angles ACP, BCP are congruent.

Other views.

Enriques and Amaldi (*Elementi di geometria*, Bologna, 1905), Veronese (in his *Elementi*) and Hilbert all assume as a *postulate* the property stated in Simson's definition. But G. Ingrami (*Elementi di geometria*, Bologna, 1904) proves it in the course of a remarkable series of closely argued propositions based on a much less comprehensive postulate. He evolves the theory of the plane from that of the triangle, beginning with a triangle as a mere *three-side* (trilatero), i.e. a *frame*, as it were. His postulate relates to the *three-side* and is to the effect that each "(rectilineal) segment" joining a vertex to a point of the opposite side meets every segment similarly joining each of the other two vertices to the points of the sides opposite to them respectively, and, conversely, if a point be taken on a segment joining a vertex to a point of the opposite side, and if a straight line be drawn from another vertex to the point on the segment so taken, it will if produced meet the opposite side. A *triangle* is then defined as the figure formed by the aggregate of all the segments joining the respective vertices of a *three-side* to points on the opposite sides. After a series of propositions, Ingrami evolves a plane as *the figure formed by the "half straight-lines" which project from an internal point of the triangle the points of the perimeter*, and then, after two more theorems, proves that a plane is determined by any three of its points which are not in a straight line, and that *a straight line which has two points in a plane has all its point in it.*

The argument by which Bolyai and Lobachewsky evolved the plane is of course equivalent to the definition of the plane as *the locus of all points equidistant from two fixed points in space.*

Leibniz in a letter to Giordano defined a plane as *that surface which divides space into two congruent parts.* Adverting to Giordano's criticism that you could conceive of surfaces and lines which divided space or a plane into two congruent parts without being *plane* or *straight* respectively, Beez (*Über Euklidische und Nicht-Euklidische Geometrie*, 1888) pointed out that what was wanted to complete the definition was the further condition that the two congruent spaces could be *slid along each other* without the surfaces ceasing to coincide, and claimed priority for his completion of the definition in this way. But the idea of *all the parts* of a plane fitting exactly on *all other points* is ancient, appearing, as we have seen, in Heron, Def. 9.