[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 165–169 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Definition 4.]

Definition 4.

Ἐὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἲσου τοῖς ἐαυτῆς σημείοις κεῖται.

A straight line *is a line which lies evenly with the points on itself.*

The only definition of a straight line authenticated as pre-Euclidean is that of Plato, who defined it as "*that of which the middle covers the ends*" (relatively, that is, to an eye placed at either end and looking along the straight line). It appears in the *Parmenides* 137 E: "straight is whatever has its middle in front of (i.e. so placed as to obstruct the view of) both its ends" (εὐθύ γε οὗ ἂν τὸ μέσον ἀμφοῖν τοῖν ἐσχάτοιν ἐπίπροσθεν ᾖ). Aristotle quotes it in equivalent terms (*Topics* VI. 11, 148 b 27), οὗ τὸ μέσον ἐπιπροσθεῖ τοῖς πέρασιν; and, as he does not mention the name of its author, but states it in combination with the definition of a line as the extremity of a surface, we may assume that he used it as being well known. Proclus also quotes the definition as Plato's in almost identical terms, ἧς τὰ μέσα τοῖς ἄκροις ἐπιπροσθεῖ (p. 109, 21). This definition is ingenious, but implicitly appeals to the sense of sight and involves the postulate that the line of sight is straight. (Cf. the Aristotelian *Problems* 31, 20, 959 a 39, where the question is why we can better observe straightness in a row, say, of letters with one eye than with two.) As regards the straightness of "visual rays," ὄψεις, cf. Euclid's own *Optics*, Deff. 1, 2, assumed as *hypotheses*, in which he first speaks of the "straight lines" drawn from the eye, avoiding the word ὄψεις), and then says that the figure contained by the *visual rays* (ὄψεις) is a cone with its vertex in the eye.

As Aristotle mentions no definition of a straight line resembling Euclid's, but gives only Plato's definition and the other explaining it as the "extremity of a surface," the latter being evidently the current definition in contemporary textbooks, we may safely infer that Euclid's definition was a new departure of his own.

Proclus on Euclid's definition.

Coming now to the interpretation of Euclid's definition, εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ᾽ ἑαυτῆς σημείοις κεῖται, we find any number of slightly different versions, but none that can be described as quite satisfactory; some authorities, e.g. Savile, have confessed that they could make nothing of it. It is natural to appeal to Proclus first; and we find that he does in fact give an interpretation which at first sight seems plausible. He says (p. 109, 8 sq.) that Euclid "shows by means of this that the straight line alone [of all lines] occupies a distance (κατέχειν διάστημα) equal to that between the points on it. For, as far as one of the points is distant from another, so great is the length (μέγεθος) of the straight line of which they are the extremities; and this is the meaning of lying ἐξ ἴσου to (or with) the points on it" [ἐξ ἴσου being thus, apparently, interpreted as "at" (or "over") "an equal distance"]. "But if you take two points on the circumference (of a circle) or any other line, the distance cut off between them along the line is greater than the interval separating them. And this is the case with every line except the straight line. Hence the ordinary remark, based on a common notion, that those who journey in a straight line only travel the necessary distance, while those who do not go straight travel more than the necessary distance." (Cf. Aristotle, *De caelo*, I. 4, 271 a 13, "we always call the distance of anything the straight line" drawn to it.) Thus Proclus would interpret somewhat in this way: "a straight line is that which represents extension equal with (the distances separating) the points on it." This explanation seems to be an attempt to graft on to Euclid's definition the *assumption* (it is a λαμβανόμενον, not a definition) of Archimedes (*On the sphere and cylinder* 1. *ad init.*) that "of all the lines which have the same extremities the straight line is the least." For this purpose ἐξ ἴσου has apparently to be taken as meaning "at an equal distance," and again "lying at an equal distance" as equivalent to "extending over (or representing) an equal distance." This is difficult enough in itself, but is seen to be an impossible interpretation when applied to the similar definition of a plane by Euclid (Def. 7) as a surface "which lies evenly with the straight lines on itself." In that connexion Proclus tries to make the same words ἐξ ἴσου κεῖται mean "extends over an equal *area* with." He says namely (p. 117, 2) that, "if two straight lines are set out" on the plane, the plane surface "occupies a space equal to that between the straight lines." But two straight lines do not determine by themselves any space at all; it would be necessary to have a *closed* figure with its boundaries in the plane before we could arrive at the equivalent of the other assumption of Archimedes that "of surfaces which have the same extremities, if those extremities are in a plane, the plane is the least [in area]." This seems to be an impossible sense for ἐξ ἴσου even on the assumption that it means "at an equal distance" in the present definition. The necessity therefore of interpreting ἐξ ἴσου similarly in both definitions makes it impossible to regard it as referring to *distance* or *length* at all. It should be added that Simplicius gave the same explanations as Proclus (an-Nairīzī, p. 5).

The language and construction of the definition.

Now let us consider the actual wording and grammar of the phrase ἥτις ἐξ ἴσου τοῖς ἐφ᾿ ἑαυτῆς σημείοις κεῖται. As regards the expression ἐξ ἴσου we note that Plato and Aristotle (whose use of it seems typical) commonly have it in the sense of "on a footing of equality": cf. οἱ ἐξ ἴσου in Plato's *Laws* 777 D, 919 D; Aristotle, *Politics* 1259 b 5 ἐξ ἴσου εἶναι βούλεται τὴν φύσιν, "tend to be on an equality in nature," *Eth. Nic.* VIII. 12, 1161 a 8 ἐνταῦθα πάντες ἐξ ἴσου, "there all are on a footing of equality." Slightly different are the uses in Aristotle, *Eth. Nic.* X. 8, 1178 a 25 τῶν μὲν γὰρ ἀναγκαίων χρεία καὶ ἐξ ἴσου ἔστω, "both need the necessaries of life *to the same extent*, let us say"; *Topics* IX. 15, 174 a 32 ἐξ ἴσου ποιοῦντα τὴν ἐρώτησιν, "asking the question indifferently" (i.e. without showing any expectation of one answer being given rather than another). The natural meaning would therefore appear to be "evenly placed" (or balanced), "in equal measure," "indifferently" or "without bias" one way or the other. Next, is the dative τοῖς ἐφ᾿ ἑαυτῆς συμείοις constructed with ἐξ ἴσου or with κεῖται? In the first case the phrase must mean "that which lies *evenly with* (or in respect to) the points on it," in the second apparently "that which, in (or by) the points on it, lies (or is placed) evenly (or uniformly)." Max Simon takes the first construction to give the sense "die Gerade liegt in gleicher Weise wie ihre Punkte." If the last words mean "in the same way as (or in like manner as) its points," I cannot see that they tell us anything, although Simon attaches to the words the notion of *distance* (Abstand) like Proclus. The second construction he takes as giving "die Gerade liegt für (durch) ihre Punkte gleichmässig," "the straight line lies symmetrically for (or through) its points"; or, if *keĩtai* is taken as the passive of τίθημι, "die Gerade ist durch ihre Punkte gleichmässig gegeben worden," "the straight line is symmetrically determined by its points." He adds that the idea is here *direction*, and that both *direction* and *distance* (as between two different given points simply) would be to Euclid, as later to Bolzano (*Betrachtungen über einige Gegenstände der Elementargeometrie*, 1804, quoted by Schotten, *Inhalt und Methode des planimetrischen Unterrichts*, II. p. 16), primary irreducible notions.

While the language is thus seen to be hopelessly obscure, we can safely say that the sort of idea which Euclid wished to express was that of a line which presents the same shape at and relatively to all points on it, without any irregular or unsymmetrical feature distinguishing one part or side of it from another. Any such irregularity could, as Saccheri points out (Engel and Stäckel, *Die Theorie der Parallellinien von Euklid bis Gauss*, 1895, p. 109), be at once made perceptible by keeping the ends fixed and turning the line about them right round; if any two positions were distinguishable, e.g. one being to the left or right relatively to another, "it would not lie in a uniform manner between its points."

A conjecture as to its origin and meaning.

The question arises, what was the origin of Euclid's definition, or, how was it suggested to him? It seems to me that the basis of it was really Plato's definition of a straight line as "that line the middle of which covers the ends." Euclid was a Platonist, and what more natural than that he should have adopted Plato's definition in substance, while regarding it as essential to change the form of words in order to make it independent of any implied appeal to vision, which, as a physical fact, could not properly find a place in a purely geometrical definition? I believe therefore that Euclid's definition is simply an attempt (albeit unsuccessful, from the nature of the case) to express, in terms to which a geometer could not object as not being part of geometrical subject-matter, the same thing as the Platonic definition.

The truth is that Euclid was attempting the impossible. As Pfleiderer says (Scholia to Euclid), "It seems as though the notion of a *straight line*, owing to its simplicity, cannot be explained by any regular definition which does not introduce words already containing in themselves, by implication, the notion to be defined (such e.g. are direction, equality, uniformity or evenness of position, unswerving course), and as though it were impossible, if a person who does not already know what the term *straight* here means, to teach it to him unless by putting before him in some way a picture or drawing of it." This is accordingly done in such books as Veronese's *Elementi di geometria* (Part 1., 1904, p. 10): "A stretched string, e.g. a plummet, a ray of light entering by a small hole into a dark room, are *rectilineal* objects. The image of them gives us the abstract idea of the limited line which is called a *rectilineal segment*."

Other definitions.

We will conclude this note with ome other famous definitions of a straight line. The following are given by Proclus (p. 110, 18–23).

1. *A line stretched to the utmost*, ἐπ᾽ ἄκρον τεταμένη γραμμή. This appears in Heron also, with the words "towards the ends" (ἐπὶ τὰ πέρατα) added. (Heron, Def. 4).

2. *Part of it cannot be in the assumed plane while part is in one higher up* (ἐν μετεωροτέρῳ). This is a *proposition* in Euclid (XI. 1).

3. *All its parts fit on all* (*other parts*) *alike*, πάντα αὐτῆς τὰ μέρη πᾶσιν ὁμοίως ἐφαρμόζει). Heron has this too (Def. 4), but instead of "alike" he says παντοίως, "in all ways," which is better as indicating that the applied part may be applied one way or the *reverse* way, with the same result.

4. *That line which, when its ends remain fixed, itself remains fixed*, ἡ τῶν περάτων μενόντων καὶ αὐτὴ μένουσα. Heron's addition to this, "*when it is, as it were, turned round in the same plane*" (οἷον ἐν τῷ αὐτῷ ἐπιπέδῳ στρεφομένη), and his next variation, "and about the same ends having always the same position," show that the definition of a straight line as "that which does not change its position when it is turned about its extremities (or any two points on it) as poles" was no original discovery of Leibniz, or Saccheri, or Krafft, or Gauss, but goes back at least to the beginning of the Christian era. Gauss' form of this definition was "The line in which all points that, during the revolution of a body (a part of space) about two fixed points, maintain their position unchanged is called a straight line." Schotten (I. p. 315) maintains that the notion of a straight line and its property of being determined by two points are unconsciously assumed in this definition which is therefore a logical "circle."

5. *That line which with one other of the same species connot complete a figure*, ἡ μετὰ τῆς ὁμειοδοῦς μιᾶς σχῆμα μὴ ἀποτελοῦσα. This is an obvious ὕστερον-πρότερον, since it assumes the notion of a *figure*.

Lastly Leibniz' definition should be mentioned: *A straight line is one which divides a plane into two halves identical in all but position.* Apart from the fact that this definition introduces the plane, it does not seem to have any advantages over the definition last but one referred to.

Legendre uses the Archimedean property of a straight line as *the shortest distance between two points.* Van Swinden observes (*Elemente der Geometrie*, 1834, p. 4), that to take this as the definition involves *assuming* the proposition that any two sides of a triangle are greater than the third and *proving* that straight lines which have two points in common coincide throughout their length (cf. Legendre *Éléments de Geométrie*, I. 3, 8).

The above definitions all illustrate the observation of Unger (*Die Geometrie des Euklid*, 1833): "*Straight* is a simple notion, and hence all definitions of it must fail.... But if the proper idea of a straight line has once been grasped, it will be recognised in all the various definitions usually given of it; all the definitions must therefore be regarded as *explanations*, and among them that one is the best from which further inferences can immediately be drawn as to the essence of the straight line."