[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 158–165 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Definition 2.]

Definition 2.

Γραμμὴ δὲ μῆκος ἀπλατές.

A line *is breadthless length.*

This definition may safely be attributed to the Platonic School, if not to Plato himself. Aristotle (*Topics* VI. 6, 143 b 11) speaks of it as open to objection because it "divides the genus by negation," length being necessarily either breadthless or possessed of breadth; it would seem however that the objection was only taken in order to score a point against the Platonists, since he says (*ibid.* 143 b 29) that the argument is of service *only* against those who assert that the genus [sc. length] is one numerically, that is, those who assume *ideas*," e.g. the idea of length (αὐτὸ μῆκος) which they regard as a genus: for if the genus, being one and self-existent, could be divided into two species, one of which asserts what the other denies, it would be self-contradictory (Waitz).

Proclus (pp. 96, 21–97, 3) observes that, whereas the definition of a point is merely negative, the line introduces the first "dimension," and so its definition is to this extent positive, while it has also a negative element which denies to it the other "dimensions" (διαστάσεις). The negation of both breadth and depth is involved in the single expression "breadthless" (ἀπλατές), since everything that is without breadth is also destitute of depth, though the converse is of course not true.

Alternative definitions.

The alternative definition alluded to by Proclus, μέγεθος ἐφ᾽ ἓν διαστατόν "magnitude in one dimension" or, better perhaps, "magnitude extended one way" (since διάστασις as used with reference to line, surface and solid scarcely corresponds to our use of "dimension" when we speak of "one," "two," or "three dimensions"), is attributed by an-Nairīzī to "Heromides," who must presumably be the same as "Herundes," to whom he attributes a certain definition of a point. It appears however in substance in Aristotle, though Aristotle does not use the adjective διαστατόν, nor does he apparently use διάσασις except of *body* as having *three* "dimensions" or "having dimension (or extension) *all* ways (πάντῃ)," the "dimensions" being in his view (1) up and down, (2) before and behind, and (3) right and left, and "up" being the principle or beginning of *length*, "right" of *breadth*, and "before" of *depth* (*De* *caelo* II. 2, 284 b 24). A line is, according to Aristotle, a magnitude "*divisible* in one way only" (μοναχῇ διαιρετόν), in contrast to a magnitude divisible in *two* ways (διχῇ διαιρετόν), or a surface, and a magnitude divisible "in all or in three ways" (πάντῃ καὶ τριχῇ διαιρετόν), or a body (*Metaph.* 1016 b 25–27); or it is a magnitude "*continuous* one way (or in one direction)," as compared with magnitudes continuous *two* ways or *three* ways, which curiously enough he describes as "breadth" and "depth" respectively (μέγεθος δὲ τὸ μὲν ἐφ᾽ ἓν συνεχὲς μῆκος, τὸ δ᾽ ἐπὶ δύο πλάτος, τὸ δ᾽ ἐπὶ τρία βάθος, *Metaph.* 1020 a 11), though he immediately adds that "length" means a line, "breadth" a surface, and "depth" a body.

Proclus gives another alternative definition as "*flux of a point*" (ῥύσις σημείου), i.e. the path of a point when moved. This idea is also alluded to in Aristotle (*De anima* I. 4, 409 a 4 above quoted): "they say that a line by its motion produces a surface, and a point by its motion a line." "This definition," says Proclus (p. 97, 8–13), "is a perfect one as showing the essence of the line: he who called it the flux of a point seems to define it from its genetic cause, and it is not every line that he sets before us, but only the immaterial line; for it is this that is produced by the point, which, though itself indivisible, is the cause of the existence of things divisible."

Proclus (p. 100, 5–19) adds the useful remark, which, he says, was current in the school of Apollonius, that we have the notion of a line when we ask for the length of a road or a wall measured merely as length; for in that case we mean something irrespective of breadth, viz. distance in one "dimension." Further we can obtain sensible perception of a line if we look at the division between the light and the dark when a shadow is thrown on the earth or the moon; for clearly the division is without breadth, but has length.

Species of "lines."

After defining the "line" Euclid only mentions *one* species of line, the straight line, although of course another species appears in the definition of a circle later. He doubtless omitted all *classification* of lines as unnecessary for his purpose, whereas, for example, Heron follows up his definition of a line by a division of lines into (1) those which are "straight" and (2) those which are not, and a further division of the latter into (*a*) "circular circumferences," (*b*) "spiral-shaped" (ἑλικοειδεῖς) lines and (*c*) "curved" (καμπύλαι) lines generally, and then explains the four terms. Aristotle tells us (*Metaph.* 986 a 25) that the Pythagoreans distinguished straight (εὐθύ) and curved (καμπύλον), and this distinction appears in Plato (cf. *Republic* X. 602 C) and in Aristotle (c.f. "to a line belong the attributes straight or curved," *Anal. post.* I. 4, 73 b 19; "as in mathematics it is useful to know what is meant by the terms straight and curved," *De anima* I. 1, 402 b 19). But from the class of "curved" lines Plato and Aristotle separate off the περιφερής or "circular" as a distinct species often similarly contrasted with straight. Aristotle seems to recognize broken lines forming an angle as one line: thus "a line, if it be bent (κεκαμμένη), but yet continuous, is called one" (*Metaph.* 1016 a 2); "the straight line is more one than the bent line" (*ibid.* 1016 a 12). Cf. Heron, Def. 12, "A broken line (κεκλασμένη γραμμή) so-called is a line which, when produced, does not meet *itself.*"

When Proclus says that both Plato and Aristotle divided lines into those which are "straight," "circular" (περιφερής) or "a mixture of the two," adding, as regards Plato, that he included in the last of these classes "those which are called helicoidal among plane (curves) and (curves) formed around solids, and such species of curved lines as arise from sections of solids" (p. 104, 1–5), he appears to be not quite exact. The reference as regards Plato seems to be to *Parmenides* 145 B: "At that rate it would seem that the one must have shape, either straight or round (στρογγύλου) or some combination of the two"; but this scarcely amounts to a formal classification of lines. As regards Aristotle, Proclus seems to have in mind the passage (*De caelo* I. 2, 268 b 17) where it is stated that "all *motion* in space, which we call translation (φορά), is (in) a straight line, a circle, or a combination of the two; for the first two are the only simple (*motions*)."

For completeness it is desirable to add the substance of Proclus' account of the classification of lines, for which he quotes Geminus as his authority.

Geminus' first classification of lines.

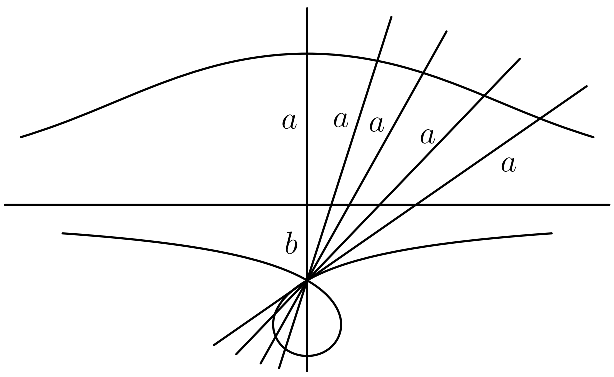
This begins (p. 111, 1–9) with a division of lines into *composite* (σύνθετος) and incomposite (ἀσύνθετος). The only illustration given of the *composite* class is the "broken line which forms an angle" (ἡ κεκλασμένη καὶ γωνίαν ποιοῦσα); the subdivision of the *incomposite* class then follows (in the text as it stands the word "composite" is clearly an error for "incomposite"). The subdivisions of the incomposite class are repeated in a later passage (pp. 176, 27–177, 23) with some additional details. The following diagram reproduces the effect of both versions as far as possible (all the illustrations mentioned by Proclus being shown in brackets).

lines

- composite (broken line forming an angle)
- incomposite
  - forming a figure σχηματοποιοῦσαι or determinate ὡρισμέναι (circle, ellipse, cissoid)
  - not forming a figure or indeterminate ἀόριστοι and extending without limit ἐπ′ ἄπειρον ἐκβαλλόμεναι (straight line, parabola, hyperbola, conchoid)

The additional details in the second version, which cannot easily be shown in the diagram, are as follows:

(1) Of the lines which extend without limit, some do not *form a figure* at all (viz. the straight line, the parabola and the hyperbola); but some first "come together and form a figure" (i.e. have a loop), "and, for the rest, extend without limit" (p. 177, 8).



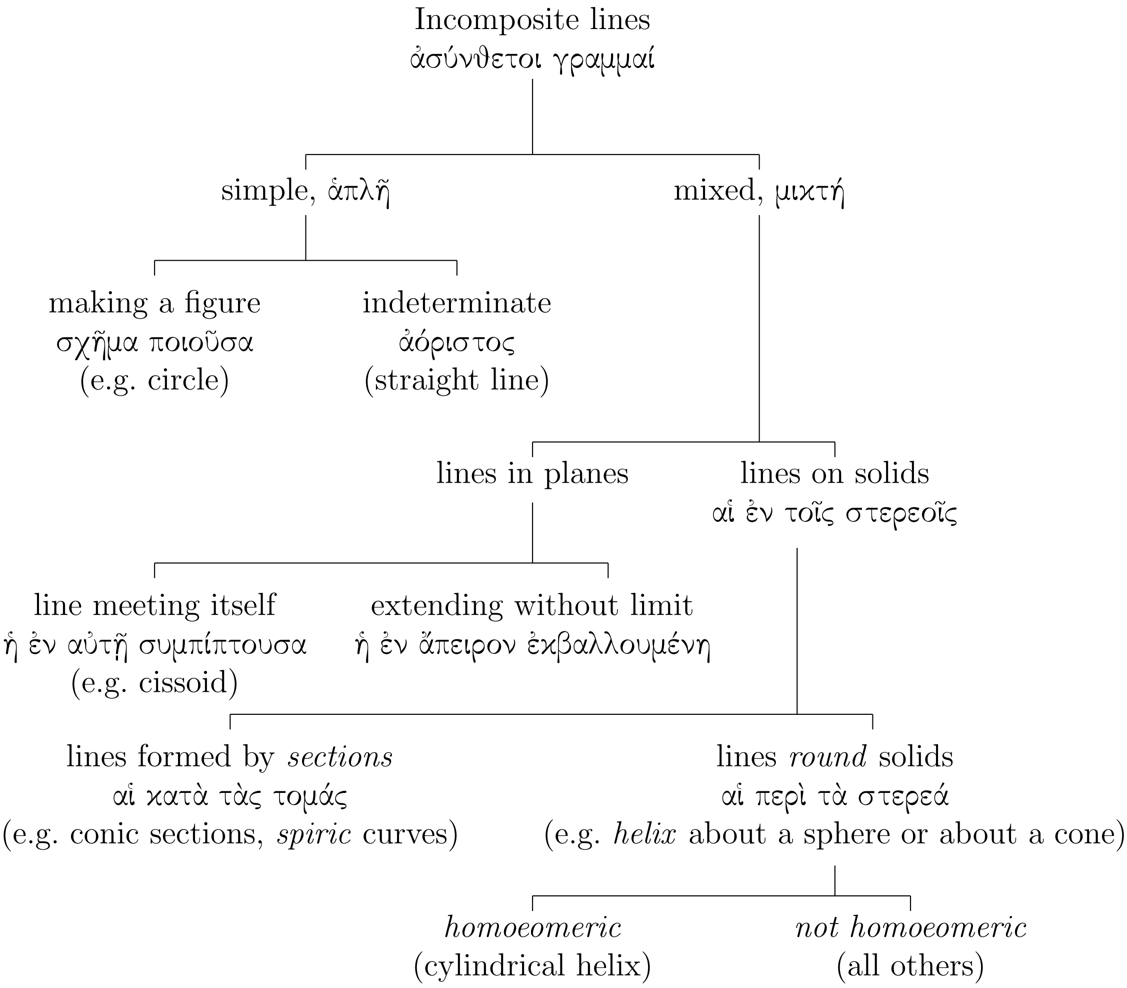
As the only other curve, besides the parabola and the hyperbola, which as been mentioned as proceeding to infinity is the *conchoid* (of Nicomedes), we can hardly avoid the conclusion of Tannery[1] that the curve which has a loop and proceeds to infinity is a variety of the *conchoid* itself. As is well known, the ordinary conchoid (which was used both for doubling the cube and for trisecting the angle) is obtained in this way. Suppose any number of rays passing through a fixed point (the *pole*) and intersecting a fixed straight line; and suppose that points are taken on the rays, beyond the fixed straight line, such that the portions of the rays intercepted between the fixed straight line and the point are equal to a constant *distance* (διάστημα), the locus of the points is a conchoid which has a fixed straight line for asymptote. If the "distance" a is measured from the intersection of the ray with the given straight line, not in the direction away from the pole, but towards the pole, we obtain three other curves according as a is less than, equal to, or greater than b, the distance of the pole from the fixed straight line, which is an asymptote in each case. The case in which $a > b$ gives a curve which forms a loop and then proceeds to infinity in the way Proclus describes. Now we know both from Eutocius (*Comm. on Archimedes*, ed. Heiberg, III. p. 98) and Proclus (p. 272, 3–7) that Nicomedes wrote on conchoides (in the plural), and Pappus (IV. p. 244, 18) says that besides the "first" (used as above stated) there were "the second, the third and the fourth which are useful for other theorems."

(2) Proclus next observes (p. 177, 9) that, of the line which extend without limit, some are "*asymptotic*" (ἀσύμπτωτοι), namely "those which never meet, however they are produced," some are "*symptotic*," namely "those which will meet sometime,"; and, of the "asymptotic class" class, some are in one plane, and others not. Lastly, of the "asymptotic" lines in one plane, some preserve always the same distance from one another, while others continually "lessen the distance, like the hyperbola with reference to the straight line, and the conchoid with reference to the straight line."

Geminus' second classification.

This (from Proclus, pp. 111, 9–20 and 122, 16–18) can be shown in a diagram thus:

[1] *Notes pour l'histoire des lignes et surfaces courbes dans l'antiquité* in *Bulletin des sciences mathém. et astronom.* 2 sér. VIII. (1884), pp. 108–9 (*Mémoires scientifiques*, II. p. 23).



Notes on classes of "lines" and on particular curves.

We will now add the most interesting notes found in Proclus with reference to the above classifications or the particular curves mentioned.

1. Homoeomeric lines.

By this term (ὁμοιομερεῖς) are meant lines which are alike in all parts, so that in any one such curve any part can be made to coincide with any other part. Proclus observes that these lines are only three in number, two being "simple" and in a plane (the straight line and the circle), and the third "mixed," (subsisting) "about a solid," namely the cylindrical helix. The latter curve was also called the *cochlias* or *cochlion*, and its *homoeomeric* property was proved by Apollonius in his work περὶ τοῦ κοχλίον (Proclus,

p. 105, 5). The fact that there are only three *homoeomeric* lines was proved by Geminus, "who proved, as a preliminary proposition, that, if from a point (ἀπό του σημείον, but on p. 251, 4 ἀφ᾽ ἑνὸς σημείον) two straight lines be drawn to a homoeomeric line making equal angles with it, the straight lines are equal" (pp. 112, 1–113, 3, cf. p. 251, 2–19).

1. Mixed lines.

It might be supposed, says Proclus (p. 105, 11), that the cylindrical helix, being *homoeomeric*, like the straight line and the circle, must like them be *simple*. He replies that it is not simple, but *mixed*, because it is generated by *two unlike* motions. Two *like* motions, said Geminus, e.g. two motions at the same speed in the directions of two adjoining sides of a square, produce a *simple* line, namely a straight line (the diagonal); and again, if a straight line moves with its extremities upon the two sides of a right angle respectively, this same motion gives a *simple* curve (a circle) for the locus of the middle point of the straight line, and a *mixed* curve (an ellipse) for the locus of any other point on it (p. 106, 3–15).

Geminus also explained that the term "mixed," as applied to curves, and as applied to surfaces, respectively, is used in different senses. As applied to curves, "mixing" neither means simple "putting together" (σύνθεσις) nor "blending" (κρᾶσις). Thus the helix (or spiral) is a "mixed" line, but (1) it is not "mixed" in the sense of "putting together," as it would be if, say, part of it were straight and part circular, and (2) it is not mixed in the sense of "blending," because, if it is cut in any way, it does not present the appearance of any *simple* lines (of which it might be supposed to be compounded, as it were). The "mixing" in the case of lines is rather that in which the constituents are destroyed so far as their own character is concerned, and are replaced, as it were, by a *chemical* combination (ἔστιν ἐν αὐτῇ συνεφθαρμένα τὰ ἄκρα καὶ συνκεχυμένα). On the other hand "mixed" surfaces are mixed in the sense of a sort of "blending" (κατά τινα κρᾶσιν). For take a cone generated by a straight line passing through a fixed point and passing always through the circumference of a circle: if you cut this by a plane parallel to that of the circle, you obtain a circular section, and if you cut it by a plane through the vertex, you obtain a triangle, the "mixed" surface of the cone being thus cut into *simple* lines (pp. 117, 22–118, 23).

3. Spiric curves.

These curves, classed with conics as being sections of solids, were discovered by Perseus, according to an epigram of Perseus' own quoted by Proclus (p. 112, 1), which says that Perseus found "three lines upon (or, perhaps, in addition to) five sections" (τρεῖς γραμμὰς ἐπὶ πέντε τομαῖς). Proclus throws some light on these in the following passages:

"Of the spiric sections, one is interlaced, resembling the horse-fetter (ἵππου πέδη); another is widened out in the middle and contracts on each side (of the middle), a third is elongated and is narrower in the middle, broadening out on each side of it" (p. 112, 4–8).

"This is the case with the *spiric surface*; for it is conceived as generated by the revolution of a circle remaining at right angles [to a plane] and turning about a point which is not its centre [in other words, generated by the revolution of a circle about a straight line in its plane not passing through the centre]. Hence the *spire* takes three forms, for the centre [of rotation] is either on the circumference, or within it, or without it. And if the centre of rotation is on the circumference, we have the *continuous* spire (συνεχής), if within, the *interlaced* (ἐμπεπλεγμένη), and if without, the *open* (διεχής). And the spiric sections are three according to these three differences" (p. 119, 8–17).

"When the *hippopede*, which is one of the spiric curves, forms an angle with itself, this angle also is contained by mixed lines" (p. 127, 1–3).

"Perseus showed for spirics what was their property (σύμπτωμα)" (p. 356, 12).

Thus the spiric surface was what we call a *tore*, or (when open) an *anchor-ring*. Heron (Def. 97) says it was called alternatively *spire* (σπεῖρα) or *ring* (κρίκος); he calls the variety in which "the circle cuts itself," not "interlaced," but "crossing-itself" (ἐπαλλάττουσα).

Tannery[2] has discussed these passages, as also did Schiaparelli[3]. It is clear that Proclus' remark that the difference in the three curves which he mentions corresponds to the difference between the three surfaces is a slip, due perhaps to too hurried transcribing from Geminus: all three arise from plane sections of the *open* anchor-ring. If r is the radius of the revolving circle, a the distance of its centre from the axis of rotation, d the distance of the plane section (suppposed to be parallel to the axis) from the axis, the three curves described in the first extract correspond to the following cases:

(1) $d = a - r$. In this case the curves is the *hippopede*, of which the lemniscate of Bernoulli is a particular case, namely that in which $a = 2r$.

The name *hippopede* was doubtless adopted for this one of Perseus' curves on the ground of its resemblance to the *hippopede* of Eudoxus, which seems to have been the curve of intersection of a sphere with a cylinder touching it

[2] *Pour l'histoire des lignes et surfaces courbes dans l'antiquité* in *Bulletin des sciences mathém. et astronom.* VIII. (1884), pp. 25–27 (*Mémoires scientifiques*, II. 24–28).

[3] *Die homocentrischen Sphären des Eudoxus, des Kallippus und des Aristoteles* (*Abhandlungen zur Gesch. der Math.* I. Heft, 1877, pp. 149–152).

internally.

(2) $a + r > d > a$. Here the curve is an oval.

(3) $a > d > a - r$. The curve is now narrowest in the middle.

Tannery explains the "three lines upon (in addition to) five sections" thus. He points out that with the *open tore* there are two other sections corresponding to

(4) $d = a$: transition from (2) to (3).

(5) $a - r > d > 0$, in which case the section corresponds to two symmetrical ovals.

He then shows that the sections of the *closed* or *continuous tore*, corresponding to $a = r$, give curves corresponding to (2), (3) and (4) only. Instead of (1) and (5) we have only a section consisting of two equal circles touching one another.

On the other hand, the *third* spire (the *interlaced variety*) gives three new *new* forms, which make a group of three in addition to the first group of *five* sections.

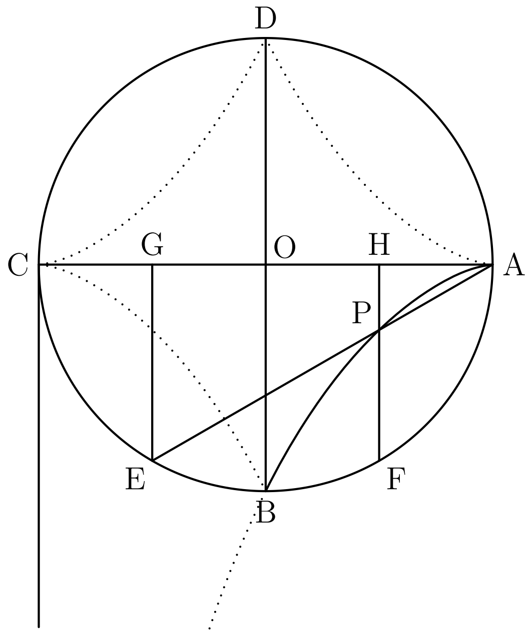
The difficulty which I see in this interpretation is the fact that, just after "three lines on five sections" are mentioned, Proclus describes three curves which were evidently the most important; but these three belong to three of the five sections of the open tore, and are not separated from them.

4. The cissoid.

This curve is assumed to be the same as that by means of which, according to Eutocius (*Comm. on Archimedes*, III. p. 66 sqq.), Diocles in his book περὶ πυρίων (*On burning-glasses*) solved the problem of doubling the cube. It is the locus of points which he found by the following construction. Let AC, BD be diameters at right angles in a circle with centre O.

Let E, F be points on the quadrants BC, BA respectively such that the arcs BE, BF are equal.

Draw EG, FH perpendicular to CA. Join AE, and let P be its intersection with FH.



The cissoid is the locus of all the points P corresponding to different positions of E on the quadrant BC and of F at an equal distance from B along the arc BA.

A is the point on the curve corresponding to the position C for the point E, and B the point on the curve corresponding to the position of E in which it coincides with B.

It is easy to see that the curve extends in the direction AB beyond B, and that CK drawn perpendicular to CA is an asymptote. It may be regarded also as having a branch AD symmetrical with AB, and, beyond D, approaching KC produced as asymptote.

If OA, OD are coordinate axes, the equation of the curve is obviously

$$y^2(a + x) = (a - x)^3,$$

where a is the radius of the circle.

There is a cusp at A, and it agrees with this that Proclus should say (p. 126, 24) that "cissoidal lines converging to one point like the leaves of ivy—for this is the origin of their name—form an angle." He makes the slight correction (p. 128, 5) that it is not two *parts* of a curve, but *one* curve, which in this case makes an angle.

But what is surprising is that Proclus seems to have no idea of the curve passing outside the circle and having an asymptote, for he several times speaks of it as a *closed* curve (forming a figure and including an area): cf.

p. 152, 7, "the plane (area) cut off by the cissoidal line has one bounding (line), but it has not in it a centre such that all (straight lines drawn to the curve) from it are equal." It would appear as if Proclus regarded the cissoid as formed by the *four* symmetrical cissoidal arcs shown in the figure.

Even more peculiar is Proclus' view of the

5. "Single-turn Spiral."

This is really the spiral of Archimedes traced by a point starting from the fixed extremity of a straight line and moving uniformly along it, while simultaneously the straight line itself moves uniformly in a plane about its fixed extremity. In Archimedes the spiral has of course any number of turns, the straight line making the same number of complete revolutions. Yet Proclus, while giving the same account of the generation of the spiral (p. 180, 8–12), regards the *single-turn spiral* as actually *stopping short* of the point reached after one complete revolution of the straight line: "it is necessary to know that extending without limit is not a property of all lines; for it, neither belongs to the circle nor to the cissoid, nor in general to lines which form figures. For even the single-turn spiral does not extend without limit—*for it is constructed between two points*—nor does any of the other lines so generated do so." (p. 187, 19–25). It is curious that Pappus (VIII. p. 1110 sqq.) uses the same term μονόστροφος ἕλιξ to denote one turn, not of the spiral, but of the *cylindrical helix*.