[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 155–158 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Definition 1.]

Definition 1.

Σημεῖόν ἐστιν, οὗ μέρος οὐθέν.

A point *is that which has no part.*

An exactly parallel use of μέρος (ἐστί) in the singular is found in Aristotle, *Metaph.* 1035 b 32 μέρος μὲν οὖν ἐστὶ καὶ τοῦ εἴδους, literally "There is a *part* even of the form"; Bonitz translates as if the plural were use, "Theile giebt es," and the meaning is simply "even the form is *divisible* (into parts)." Accordingly it would be quite justifiable to translated in this case "A point is that which is *indivisible into parts*."

Martianus Capella (5th c. A.D.) alone or almost alone translated differently, "Punctum est cuius pars *nihil* est," "a point is that a part of which is *nothing*." Notwithstanding that Max Simon (*Euclid und die sechs planimetrischen Bücher*, 1901) has adopted this translation (on grounds which I shall presently mention), I cannot think that it gives any sense. If a part of a point is *nothing*, Euclid might as well have said that a point is *itself* "nothing," which of course he does not do.

Pre-Euclidean definitions.

It would appear that this was not the definition given in earlier textbooks; for Aristotle (*Topics* vi. 4, 141 b 20), in speaking of "*the* definitions" of point, line and surface, says that they *all* define the prior by means of the posterior, a point as an extremity of a line, a line of a surface, and a surface of a solid.

The first definition of a point which we hear is that given by the Pythagoreans (cf. Proclus, p. 95, 25), who defined it as a "monad having position" or "with position added" (μονὰς προσλαβοῦσα θέσιν). It is frequently used by Aristotle, either in this exact form (cf. *De anima* i. 4, 409 a 6) or its equivalent: e.g. in *Metaph.* 1016 b 24 he says that that which is indivisible every way with respect to magnitude and *quâ* magnitude but has not position is a *monad*, while that which is similarly indivisible and has position is a *point*.

Plato appears to have objected to this definition. Aristotle says (*Metaph.* 992 a 20) that he objected "to this genus [that of points] as being a geometrical fiction (γεωμετρικὸν δόγμα), and called a point the beginning of a line (ἀρχὴ γραμμῆς), while again he frequently spoke of 'indivisible lines.'" To which Aristotle replies that even "indivisible lines" must have extremities, so that the same argument which proves the existence of *lines* can be used to prove that *points* exist. It would appear therefore that, when Aristotle objects to the definition of a point as the extremity of a line (πέρας γραμμῆς) as unscientific (*Topics* VI. 4, 141 b 21), he is aiming at Plato. Heiberg conjectures (*Mathematisches zu Aristoteles*, p. 8) that it was due to Plato's influence that the word for "point" generally used by Aristotle (στιγμή) was replaced by σημεῖον (the regular term used by Euclid, Archimedes and later writers), the latter term (= *nota*, a conventional mark) probably being considered more suitable than στιγμή (a *puncture*) which might appear to claim greater *reality* for a point.

Aristotle's conception of a point as that which is indivisible and has position is further illustrated by such observations as that a point is not a *body* (*De caelo* II. 13, 296 a 17) and has no *weight* (*ibid.* III. 1, 299 a 30); again we can make no distinction between a point and the *plane* (τόπος) where it is (*Physics* IV. 1, 209 a 11). He finds the usual difficulty in accounting for the transition from the indivisible, or infinitely small, to the finite or divisible magnitude. A point being *indivisible*, no accumulation of points, however far it may be carried, can give us anything divisible, whereas of course a line is a divisible magnitude. Hence he holds that points cannot make up anything continuous like a line, point cannot be continuous with point (οὐ γάρ ἐστιν ἐχόμενον σημεῖον σημείου ἢ στιγμὴ στιγμῆς, *De gen. et corr.* I. 2, 217 a 10), and a line is not *made up* of points (οὐ σύγκειται ἐκ στιγμῶν), *Physics* IV. 8, 215 b 19). A point, he says, is like the *now* in time: *now* is indivisible and is not a *part* of time, it is only the beginning or end, or a division, of time, and similarly a point may be an extremity, beginning or division of a line, but is not *part* of it or of magnitude (cf. *De caelo* III. 1, 300 a 14, *Physics* IV. 11, 220 a 1–21, VI. 1, 231 b 6 sqq.). It is only by *motion* that a point can generate a line (*De anima* I. 4, 409 a 4) and thus be the origin or magnitude.

Other ancient definitions.

According to an-Nairīzī (ed. Curtze, p. 3) one "Herundes" (not so far identified) defined a point as "the invisible beginning of all magnitudes," and Posidonius as "an extremity which has no dimension, or an extremity of a line."

Criticisms by commentators.

Euclid's definition itself is of course practically the same as that which Aristotle's frequent allusions show to have been then current, except that it omits to say that the point must have position. It is then sufficient, seeing that there are other things which are without parts or indivisible, e.g. the *now* in time, and the *unit* in number? Proclus answers (p. 93, 18) that the point is the only thing *in the subject-matter of geometry* that is indivisible.

Relatively therefore to the particular science the definition is sufficient. Secondly, the definition has been over and over again criticised because it is purely negative. Proclus' answer to this is (p. 94, 10) that negative descriptions are appropriate to first principles, and he quotes Parmenides as having described his first and last cause by means of negations merely. Aristotle too admits that it may sometimes be necessary for one framing a definition to use negations, e.g. in defining privative terms such as "blind"; and he seems to accept as proper the negative element in the definition of a point, since he says (*De anima* III. 6, 430 b 20) that "the point and every division [e.g. in a length or in a period of time], and that which is indivisible in this sense is exhibited as privation (δηλοῦται ὡς στέρησις)."

Simplicius (quoted by an-Nairīzi) says that "a point is the beginning of magnitudes and that from which they grow; it is also the only thing which, having position, is not divisible." He, like Aristotle, adds that it is by its *motion* that a point can generate a magnitude: the particular magnitude can only be "of one dimension," viz. a line, since the point does not "spread itself" (dimittat). Simplicius further observes that Euclid defined a point negatively because it was arrived at by detaching surface from body, line from surface, and finally point from line. "Since then body has three dimensions it follows that a point [arrived at after successively eliminating all three dimensions] has *none of the dimensions* and has no part." This of course reappears in modern treatises (cf. Rausenberger, *Elementar-geometrie des Punktes, der Geraden und der Ebene*, 1887, p. 7).

An-Nairīzī adds an interesting observation. "If any one seeks to know the essence of a point, a thing more simple than a line, let him, in the sensible world, think of the centre of the universe and the *poles*." But there is nothing new under the sun: the same idea is mentioned, in an Aristotelian treatise, in controverting those who imagine that the poles have some influence in the motion of the sphere, "when the poles have no magnitude but are extremities and points" (*De motu animalium* 3, 699 a 21).

Modern views.

In the new geometry represented by the excellent treatises which start from new systems of postulates or axioms, the result of the profound study of the fundamental principles of geometry during recent years (I need only mention the names of Pasch, Veronese, Enriques and Hilbert), points come before lines, but the vain effort to define them *a priori* is not made; instead of this, the nearest material things in nature are mentioned as illustrations, with the remark that it is from them that we can get the abstract idea. Cf. the full statement as regards the notion of a point in Weber and Wellstein, *Encyclopädie der elementaren Mathematik*, II., 1905, p. 9. "This notion is evolved from the notion of the real or supposed *material* point by the process of limits, i.e. by an act of the mind which sets a term to a series of presentations in itself unlimited. Suppose a grain of sand, or a mote in a sunbeam which continually becomes smaller and smaller. In this way vanishes more and more the possibility of determining still smaller atoms in the grain of sand, and there is evolved, so we say, with growing certainty, the presentation of the point as a definite position in space which is one and is incapable of further division. But this view is untenable; we have, it is true, some idea how the grain of sand gets smaller and smaller, but only so long as it remains just visible; after that we are completely in the dark, and we cannot see or imagine the further diminution. That this procedure comes to an end is unthinkable; that nevertheless there exists a term beyond which it cannot go, we must believe or postulate without ever reaching it. . . . It is a pure act of *will*, not of the understanding." Max Simon observes similarly (*Euclid*, p. 25) "The notion 'point' belongs to the limit-notions (Grenzbegriffe), the necessary conclusions of continued, and in themselves unlimited, series of presentations." He adds, "The point is the limit of localisation; if this is presentations." He adds, "The point is the limit of localisation; if this is more and more energetically continued, it leads to the limit-notion 'point,' better 'position,' which at the same time involves a change of notion. Content of *space* vanishes, relative *position* remains. 'Point' then, according to our interpretation of Euclid, is the extremest limit of that which we can still think of (not observe) as a *spatial* presentation, and if we go further than that, not only does extension cease but even relative *place*, and in this sense the 'part' is *nothing*." I confess I think that even the meaning which Simon intends to convey is better expressed by "it has *no* part" than by "the part is nothing," since to take a "part" of a thing in Euclid's sense of the result of a simple division, corresponding to an arithmetical fraction, would not be to change the *notion* from that of the thing divided into an entirely different one.