Selected Propositions from Euclid's *Elements of Geometry*
Books II, III and IV (T.L. Heath's Edition)

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SELECTED PROPOSITIONS FROM EUCLID'S *ELEMENTS*, BOOK II

Definitions

1. Any rectangular parallelogram is said to be **contained** by the two straight lines containing the right angle.

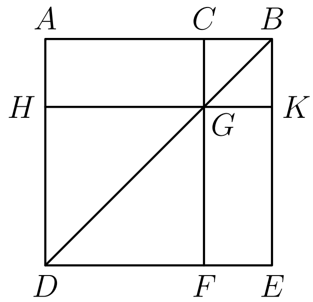
2. And in any parallelogrammic area let any one whatever of the parallelograms about its diameter with the two complements be called a **gnomon**.

BOOK II, PROPOSITION 4

If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.

For let the straight line AB be cut at random at C; I say that the square on AB is equal to the squares on AC, CB and twice the rectangle contained by AC, CB.

For let the square $ADEB$ be described on AB [I. 46], let BD be joined; through C let CF be drawn parallel to either AD or EB, and through G let HK be drawn parallel to either AB or DE [I. 31].



Then, since CF is parallel to AD, and BD has fallen on them, the exterior angle CGB is equal to the interior and opposite angle ADB [I. 29]. But the angle ADB is equal to the angle ABD, since the side BA is also equal to AD [I. 5.]; therefore the angle CGB is also equal to the angle GBC, so that the side BC is also equal to the side CG [I. 6]. But CB is equal to GK, and CG to KB [I. 34] therefore GK is also equal to KB; therefore $CGKB$ is equilateral.

I say next that it is also right-angled. For, since CG is parallel to BK, the angles KBC, GCB are equal to two right angles [I. 29]. But the angle KBC is right; therefore the angle BCG is also right, so that the opposite angles CGK, GKB are also right [I. 34]. Therefore $CGKB$ is right-angled; and it was also proved equilateral; therefore it is a square; and it is described on CB.

For the same reason HF is also a square; and it is described on HG, that is AC [I. 34]. Therefore the squares HF, KC are the squares on AC, CB.

Now, since AG is equal to GE, and AG is the rectangle AC, CB, for GC is equal to CB, therefore GE is also equal to the rectangle AC, CB. Therefore AG, GE are equal to twice the rectangle AC, CB.

But the squares HF, CK are also the squares on AC, CB; therefore the four areas HF, CK, AG, GE are equal to the squares on AC, CB and twice the rectangle contained by AC, CB. But HF, CK, AG, GE are the whole

$ADEB$, which is the square on AB. Therefore the square on AB is equal to the squares on AC, CB and twice the rectangle contained by AC, CB.

Therefore etc.

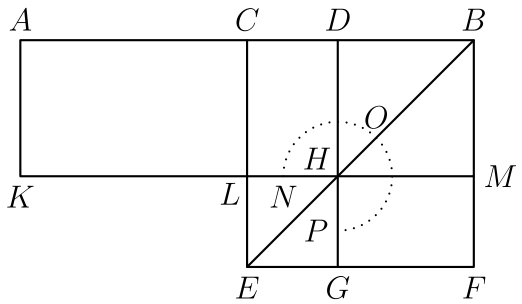
Q.E.D.

Book II, Proposition 5

If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half.

For let a straight line AB be cut into equal segments at C and into unequal segments at D; I say that the rectangle contained by AD, DB together with the square on CD is equal to the square on CB.

For let the square $CEFB$ be described on CB [I. 46], and let BE be joined; through D let DG be drawn parallel to either CE or BF, through H again let KM be drawn parallel to either AB or EF, and again through A let AK be drawn parallel to either CL or BM [I. 31].



Then, since the complement CH is equal to the complement HF [I. 43], Let DM be added to each; therefore the whole CM is equal to the whole DF. But CM is equal to AL, since AC is also equal to CB [I. 36]; therefore AL is also equal to DF. Let CH be added to each; therefore the whole AH is equal to the gnomon NOP. But AH is the rectangle AD, DB, for DH is equal to DB, therefore the gnomon NOP is also equal to the rectangle AD, DB. Let LG, which is equal to the square on CD, be added to each; therefore the gnomon NOP and LG are equal to the rectangle contained by AD, DB and the square on CD. But the gnomon NOP and LG are the whole square $CEFB$, which is described on CB; therefore the rectangle contained by AD, DB together with the square on CD is equal to the square on CB.

Therefore etc.

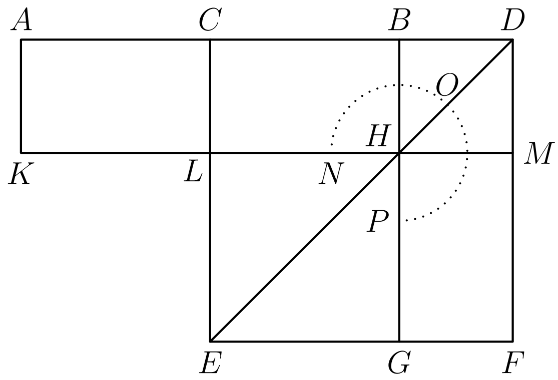
Q.E.D.

Book II, Proposition 6

If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.

For let a straight line AB be bisected at the point C, and let a straight line BD be added to it in a straight line; I say that the rectangle contained by AD, DB together with the square on CB is equal to the square on CD.

For let the square $CEFD$ be described on CD [I. 46], and let DE be joined; through the point B let BG be drawn parallel to either EC or DF, through the point H let KM be drawn parallel to either AB or EF, and further through A let AK be drawn parallel to either CL or DM [I. 31].



Then, since AC is equal to CB, AL is also equal to CH [I. 36]. But CH is equal to HF [I. 43]. Therefore AL is also equal to HF. Let CM be added to each; therefore the whole AM is equal to the gnomon NOP. But AM is the rectangle AD, DB, for DM is equal to DB, therefore the gnomon NOP is also equal to the rectangle AD, DB. Let LG, which is equal to the square on BC, be added to each; therefore the rectangle contained by AD, DB together with the square on CB is equal to the gnomon NOP and LG. But the gnomon NOP and LG are the whole square $CEFD$, which is described on CD; therefore the rectangle contained by AD, DB together with the square on CB is equal to the square on CD.

Therefore etc.

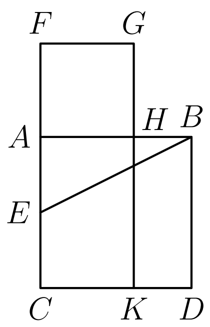
Q.E.D.

Book II, Proposition 11

To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.

Let AB be the given straight line; thus it is required to cut AB so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.

For let the square $ABDC$ be described on AB; let AC be bisected at the point E, and let BE be joined; let CA be drawn through to F, and let EF be made equal to BE; let the square FH be described on AF, and let GH be drawn through to K. I say that AB has been cut at H so as to make the rectangle contained by AB, BH equal to the square on AH.



For, since the straight line AC has been bisected at E, and FA added to it, the rectangle contained by CF, FA together with the square on AE is equal to the square on EF [II. 6]. But EF is equal to EB; therefore the rectangle CF, FA together with the square on AE is equal to the square on EB. But the squares on BA, AE are equal to the square on EB, for the angle at A is right [I. 47]: therefore the rectangle CF, FA together with the square on AE is equal to the squares on BA, AE. Let the square on AE be subtracted from each; therefore the rectangle CF, FA which remains is equal to the square on AB.

Now the rectangle CF, FA is FK, for AF is equal to FG; and the square on AB is AD; therefore FK is equal to AD. Let AK be subtracted from each; therefore FH which remains is equal to HD. And HD is the rectangle AB, BH, for AB is equal to BD; and FH is the square on AH; therefore the rectangle contained by AB, BH is equal to the square on HA. Therefore the given straight line AB has been cut at H so as to make the rectangle contained by AB, BH equal to the square on HA.

Q.E.F.

Book II, Proposition 14

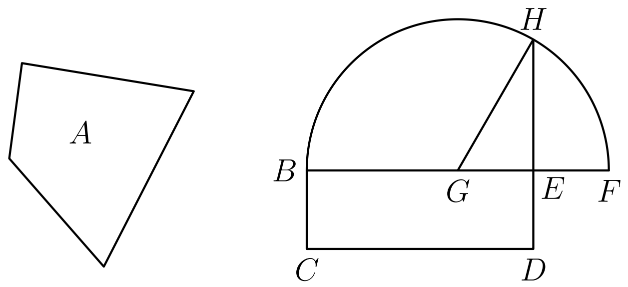
To construct a square equal to a given rectilineal figure.

Let A be the given rectilineal figure; thus it is required to construct a square equal to the rectilineal figure A.

For let there be constructed the rectangular parallelogram BD equal to the rectilineal figure A [I, 45]. Then, if BE is equal to ED, that which was enjoined will have been done; for a square BD has been constructed equal to the rectilineal figure A.

But, if not, one of the straight lines BE, ED is greater.

Let BE be greater, and let it be produced to F; let EF be made equal to ED, and let BF be bisected at G. With centre G and distance one of the straight lines GB, GF let the semicircle BHF be described; let DE be produced to H, and let GH be joined.



Then, since the straight line BF has been cut into equal segments at G, and into unequal segments at E, the rectangle contained by BE, EF together with the square on EG is equal to the square on GF [II. 5]. But GF is equal to GH; therefore the rectangle BE, EF together with the square on GE is equal to the square on GH. But the squares on HE, EG are equal to the square on GH [I. 47]; therefore the rectangle BE, EF together with the square on GE is equal to the squares on HE, EG. Let the square on GE be subtracted from each; therefore the rectangle contained by BE, EF which remains is equal to the square on EH. But the rectangle BE, EF is BD, for EF is equal to ED; therefore the parallelogram BD is equal to the square on HE. And BD is equal to the rectilineal figure A. Therefore the rectilineal figure A is also equal to the square which can be described on EH.

Therefore a square, namely that which can be described on EH, has been constructed equal to the given rectilineal figure A.

Q.E.F.

SELECTED PROPOSITIONS FROM EUCLID'S *ELEMENTS*, BOOK III

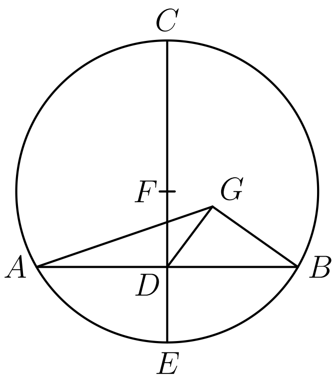
Definitions

1. **Equal circles** are those the diameters of which are equal, or the radii of which are equal.
2. A straight line is said to **touch a circle** which, meeting the circle and being produced, does not cut the circle.
3. **Circles** are said to **touch one another** which, meeting one another, do not cut one another.
4. In a circle straight lines are said **to be equally distant from the centre** when the perpendiculars drawn to them from the centre are equal.
5. And that straight line is said to be **at a greater distance** on which the greater perpendicular falls.
6. A **segment of a circle** is the figure contained by a straight line and a circumference of a circle.
7. An **angle of a segment** is that contained by a straight line and a circumference of a circle.
8. An **angle in a segment** is the angle which, when a point is taken on the circumference of the segment and straight lines are joined from it to the extremities of the straight line which is the **base of the segment**, is contained by the straight lines so joined.
9. And when the straight lines containing the angle cut off a circumference, the angle is said to **stand upon** that circumference.
10. A **sector of a circle** is the figure which, when an angle is constructed at the centre of the circle, is contained by the straight lines containing the angle and the circumference cut off by them.
11. **Similar segments of circles** are those which admit equal angles, or in which the angles are equal to one another.

Book III, Proposition 1

To find the centre of a given circle.

Let ABC be the given circle; thus it is required to find the centre of the circle ABC.



Let a straight line AB be drawn through it at random, and let it be bisected at the point D; from D let DC be drawn at right angles to AB and let it be drawn through to E; let CE be bisected at F; I say that F is the centre of the circle ABC.

For suppose it is not, but, if possible, let G be the centre, and let GA, GD, GB be joined.

Then, since AD is equal to DB, and DG is common, the two sides AD, DG are equal to the two sides BD, DG respectively; and the base GA is equal to the base GB, for they are radii; therefore the angle ADG is equal to the angle GDB [I. 8].

But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the the equal angles is right [I Def. 10]; therefore the angle GDB is right.

But the angle FDB is also right; Therefore the angle FDB is equal to the angle GDB, the greater to the less: which is impossible.

Therefore G is not the centre of the circle ABC.

Similarly we can prove that neither is any other point except F.

Therefore the point F is the centre of the circle ABC.

PORISM. From this, it is manifest that, if in a circle a straight line cut a straight line into two equal parts and at right angles, the centre of the circle is on the cutting straight line.

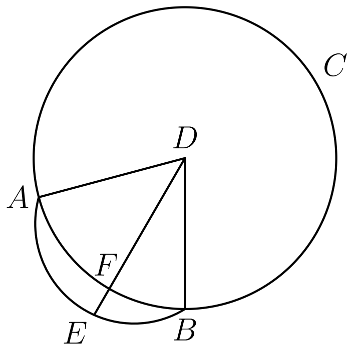
Q.E.F.

Book III, Proposition 2

If on the circumference of a given circle two points be taken at random, the straight line joining the points will fall within the circle.

Let ABC be a circle, and let two points A and B be taken at random on its circumference; I say that the straight line joined from A to B will fall within the circle.

For suppose it does not, but, if possible, let it fall outside, as AEB; let the centre of the circle ABC be taken [III. 1], and let it be D; let DA, DB be joined, and let DFE be drawn through.



Then since DA is equal to DB, the angle DAE is also equal to the angle DBE [I. 5]. And, since one side AEB of the triangle DAE is produced, the angle DEB is greater than the angle DAE [I. 16]. But the angle DAE is equal to the angle DBE; therefore the angle DEB is greater than the angle DBE. And the greater angle is subtended by the greater side [I. 19]; therefore DB is greater than DE.

But DB is equal to DF; therefore DF is greater than DE, the less than the greater: which is impossible.

Therefore the straight line joined from A to B will not fall outside the circle.

Similarly we can prove that neither will it fall on the circumference itself; therefore it will fall within. Therefore etc.

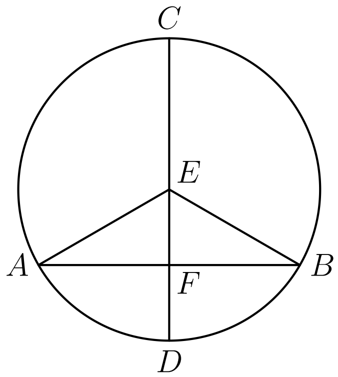
Q.E.D.

Book III, Proposition 3

If in a circle a straight line through the centre bisect a straight line not through the centre, it also cuts it at right angles; and if it cut it at right angles, it also bisects it.

Let ABC be a circle, and in it let a straight line CD through the centre bisect a straight line AB not through the centre at the point F; I say that it also cuts it at right angles.

For let the centre of the circle ABC be taken, and let it be E; let EA, EB be joined.



Then, since AF is equal to FB, and FE is common, two sides are equal to two sides; and the base EA is equal to the base EB; therefore the angle AFE is equal to the angle BFE [I. 8].

But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right; [I. Def. 10] therefore each of the angles AFE, BFE is right.

Therefore CD, which is through the centre, and bisects AB which is not through the centre, also cuts it at right angles.

Again, let CD cut AB at right angles; I say that it also bisects it, that is, that AF is equal to FB.

For, with the same construction, since EA is equal to EB, the angle EAF is also equal to the angle EBF [I. 5].

But the right angle AFE is equal to the right angle BFE, therefore EAF, EBF are two triangles having two angles equal to two angles and one side equal to one side, namely EF, which is common to them, and subtends one of the equal angles; therefore they will also have the remaining sides equal to the remaining sides [I. 26]; therefore AF is equal to FB.

Therefore etc.

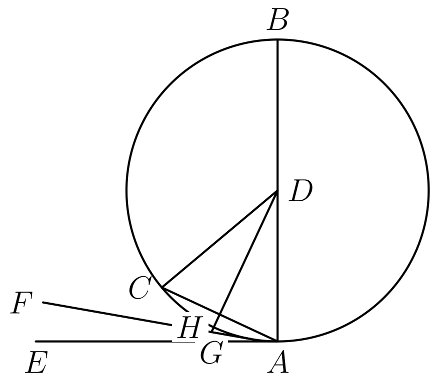
Q.E.D.

Book III, Proposition 16

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilinear angle.

Let ABC be a circle about D as centre and AB as diameter; I say that the straight line drawn from A at right angles to AB from its extremity will fall outside the circle.

For suppose it does not, but, if possible, let it fall within as CA, and let DC be joined.



Since DA is equal to DC, the angle DAC is also equal to the angle ACD [I. 5].

But the angle DAC is right; therefore the angle ACD is also right: thus, in the triangle ACD, the two angles DAC, ACD are equal to two right angles: which is impossible [I. 17].

Therefore the straight line drawn from the point A at right angles to BA will not fall within the circle.

Similarly we can prove that neither will it fall on the circumference; therefore it will fall outside.

Let it fall as AE; I say next that into the space between the straight line AE and the circumference CHA another straight line cannot be interposed.

For, if possible, let another straight line be so interposed, as FA, and let DG be drawn from the point D perpendicular to FA.

Then, since the angle AGD is right, and the angle DAG is less than a right angle, AD is greater than DG [I. 19].

But DA is equal to DH; therefore DH is greater than DG, the less than the greater, which is impossible.

Therefore another straight line cannot be interposed into the space between the straight line and the circumference.

I say further that the angle of the semicircle contained by the straight line BA and the circumference CHA is greater than any acute rectilineal angle, and the remaining angle contained by the circumference CHA and the straight line AE is less than any acute rectilinear angle.

For, if there is any rectilineal angle greater than the angle contained by the straight line BA and the circumference CHA, and any rectilineal angle less than the angle contained by the circumference CHA and the straight line AE, then into the space between the circumference and the straight line AE a straight line will be interposed such as will make an angle contined by straight lines which is greater than the angle contained by the straight line BA and the circumference CHA, and another angle contained by straight lines which is less than the angle contained by the circumference CHA and the straight line AE.

But such a straight line cannot be interposed; therefore there will not be any acute angle contained by straight lines which is greater than the angle contained by the straight line BA and the circumference CHA, nor yet any acute angle contained by straight lines which is less than the angle contained by the circumference CHA and the straight line AE.—

Porism. From this it is manifest that the straight line drawn at right angles to the diameter of a circle from its extremity touches the circle.

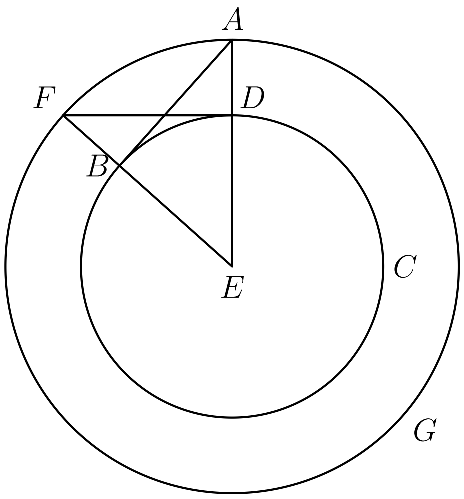
Q.E.D.

Book III, Proposition 17

From a given point to draw a straight line touching a given circle.

Let A be the given point, and BCD the given circle; thus it is required to draw from the point A a straight line touching the circle BCD.

For let the centre E of the circle be taken [III. 1]. let AE be joined, and with centre E and distance EA let the circle AFG be described; from D let DF be drawn at right angles to EA, and let EF, AB be joined; I say that AB has been drawn from the point A touching the circle BCD.



For, since E is the centre of the circles BCD, AFG, EA is equal to EF, and ED to EB; therefore the two sides AE, EB are equal to the two sides FE, ED: and they contain a common angle, the angle at E; therefore the base DF is equal to the base AB, and the triangle DEF is equal to the triangle BEA, and the remaining angles to the remaining angles [1. 4]; therefore the angle EDF is equal to the angle EBA.

But the angle EDF is right; therefore the angle EBA is also right.

Now EB is a radius; and the straight line drawn at right angles to the diameter of a circle, from its extremity, touches the circle; [III. 16, Por.] therefore AB touches the circle BCD.

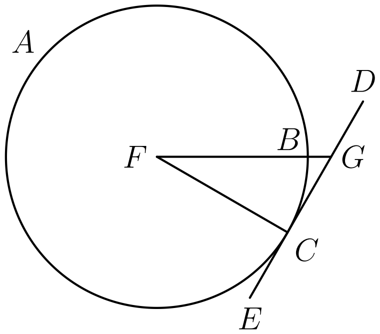
Therefore from the given point A the straight line AB has been drawn touching the circle BCD.

Book III, Proposition 18

If a straight line touch a circle, and a straight line be joined from the centre to the point of contact, the straight line so joined will be perpendicular to the tangent.

For let a straight line DE touch the circle ABC at the point C, let the centre F of the circle ABC be taken, and let FC be joined from F to C; I say that FC is perpendicular to DE.

For, if not, let FG be drawn from F perpendicular to DE.



Then, since the angle FGC is right, the angle FCG is acute [I. 17]; and the greater angle is subtended by the greater side; therefore FC is greater than FG.

But FC is equal to FB; therefore FB is also greater than FG, the less than the greater: which is impossible.

Therefore FG is not perpendicular to DE.

Similarly we can prove that neither is any other straight line except FC; therefore FC is perpendicular to DE. Therefore, etc.

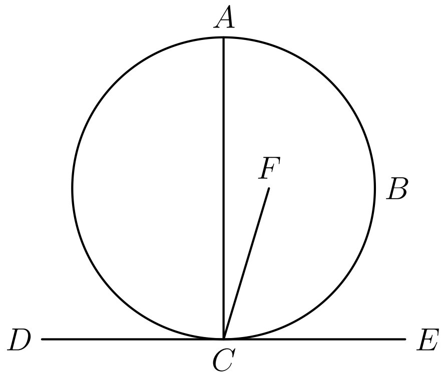
Q.E.D.

Book III, Proposition 19

If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the tangent, the centre of the circle will be on the straight line so drawn.

For let a straight line DE touch the circle ABC at the point C, and from C let CA be drawn at right angles to DE; I say that the centre of the circle is on AC.

For suppose it is not, but, if possible, let F be the centre, and let CF be joined.



Since a straight line DE touches the circle ABC, and FC has been joined from the point of contact, FC is perpendicular to DE [III. 18]; therefore the angle FCE is right.

But the angle ACE is also right; therefore the angle FCE is equal to the angle ACE, the less to the greater: which is impossible.

Therefore F is not the centre of the circle ABC.

Similarly we can prove that neither is any other point except a point on AC. Therefore, etc.

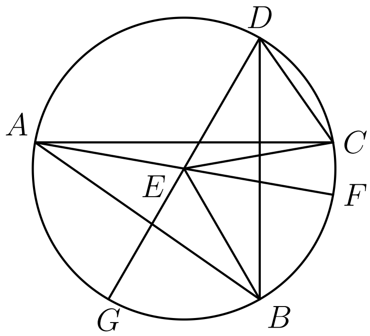
Q.E.D.

Book III, Proposition 20

In a circle the angle at the centre is double of the angle at the circumference, when the angles have the same circumference as base.

Let ABC be a circle, let the angle BEC be an angle at its centre, and the angle BAC an angle at the circumference, and let them have the same circumference BC as base; I say that the angle BEC is double of the angle BAC.

For let AE be joined and drawn through to F.



Then, since EA is equal to EB, the angle EAB is also equal to the angle EBA [I. 5]; therefore the angles EAB, EBA are double of the angle EAB.

But the angle BEF is equal to the angles EAB, EBA [I. 32]; therefore the angle BEF is also double of the angle EAB.

For the same reason the angle FEC is also double of the angle EAC.

Therefore the whole angle BEC is double of the whole angle BAC.

Again let another straight line be inflected, and let there be another angle BDC; let DE be joined and produced to G.

Similarly then we can prove that the angle GEC is double of the angle EDC, of which the angle GEB is double of the angle EDB; therefore the angle BEC which remains is double of the angle BDC. Therefore, etc.

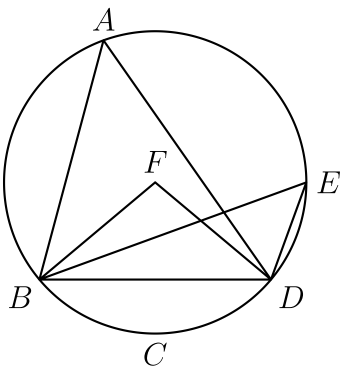
Q.E.D.

Book III, Proposition 21

In a circle the angles in the same segment are equal to one another.

Let $ABCD$ be a circle, and let the angles BAD, BED be angles in the same segment $BAED$; I say that the angles BAD, BED are equal to one another.

For let the centre of circle $ABCD$ be taken, and let it be F; let BF, FD be joined.



Now, since the angle BFD is at the centre, and the angle BAD at the circumference, and they have the same circumference BCD as base, therefore the angle BFD is double of the angle BAD [III. 20]

For the same reason the angle BFD is also double of the angle BED; therefore the angle BAD is equal to the angle BED.

Therefore, etc.

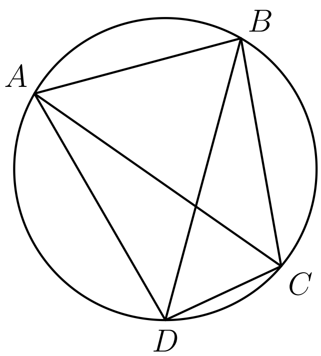
Q.E.D.

Book III, Proposition 22

The opposite angles of quadrilaterals in circles are equal to two right angles.

Let $ABCD$ be a circle, and let $ABCD$ be a quadrilateral in it; I say that the opposite angles are equal to two right angles.

Let AC, BD be joined.



Then, since in any triangle the three angles are equal to two right angles [I. 32], the three angles CAB, ABC, BCA of the triangle ABC are equal to two right angles.

But the angle CAB is equal to the angle BDC, for they are in the same segment $BADC$ [III. 21]; and the angle ACB is equal to the angle ADB, for they are in the same segment $ADCB$; therefore the whole angle ADC is equal to the angles BAC, ACB.

Let the angle ABC be added to each; therefore the angles ABC, BAC, ACB are equal to the angles ABC, ADC.

But the angles ABC, BAC, ACB are equal to two right angles; therefore the angles ABC, ADC are also equal to two right angles.

Similarly we can prove that the angles BAD, DCB are also equal to two right angles.

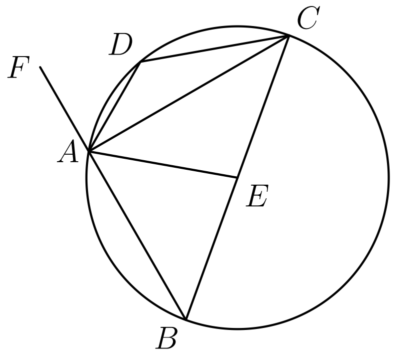
Therefore, etc.

Q.E.D.

Book III, Proposition 31

In a circle the angle in the semicircle is right, that in a greater segment less than a right angle, and that in a less segment greater than a right angle; and further the angle of the greater segment is greater than a right angle, and the angle of the less segment is less than a right angle.

Let $ABCD$ be a circle, let BC be its diameter, and E its centre, and let BA, AC, AD, DC be joined; I say that the angle BAC in the semicircle is right, the angle in the segment ABC greater than the semicircle is less than a right angle, and the angle ADC in the segment ADC less than the semicircle is greater than a right angle.



Let AE be joined, and let BA be carried through to F.

Then, since BE is equal to EA, the angle ABE is aso equal to the angle BAE [I. 5]. Again, since CE is equal to EA, the angle ACE is also equal to the angle CAE [I. 5]. Therefore the whole angle BAC is equal to the two angles ABC, ACB. But the angle FAC exterior to the triangle ABC is also equal to the two angles ABC, ACB [I. 32]; therefore the angle BAC is also equal to the angle FAC; therefore each is right; therefore the angle BAC inthe semicircle BAC is right.

Next, since in the triangle ABC the two angles ABC, BAC are less than two right angles, and the angle BAC is a right angle, the angle ABC is less than a right angle; and its is the angle in the segment ABC greater than the semicircle.

Next, since $ABCD$ is a quadrilateral in a circle, and the opposite angles of quadrilaterals in circles are equal to two right angles [III, 22], while the angle ABC is less than a right angle, therefore the angle ADC which remains is greater than a right angle; and it is the angle in the segment ADC less than the semicircle.

I say further than the angle of the greater segment, namely that contained by the circumference ABC and the straight line AC, is greater than a right angle; and the angle of the less segment, namely that contained by the circumference ADC and the straight line AC, is less than a right angle.

This is at once manifest.

For, since the angle contained by the straight lines BA, AC is right, the angle contained by the circumference ABC and the straight line AC is greater than a right angle.

Again, since the angle contained by the straight lines AC, AF is right, the angle contained by the straight line CA and the circumference ADC is less than a right angle.

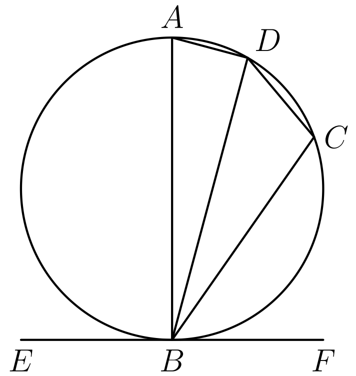
Therefore etc.

Q.E.D.

Book III, Proposition 32

If a straight line touch a circle, and from the point of contact there be drawn across, in the circle, a straight line cutting the circle, the angles which it makes with the tangent will be equal to the angles in the alternate segments of the circle.

For let a straight line EF touch the circle $ABCD$ at the point B, and from the point B let there be drawn across, in the circle $ABCD$, a straight line BD cutting it; I say that the angles which BD makes with the tangent EF will be equal to the angles in the alternate segments of the circle, that is, that the angle FBD is equal to the angle constructed in the segment BAD, and the angle EBD is equal to the angle constructed in the segment DCB.



For let BA be drawn from B at right angles to EF, let a point C be taken at random on the circumference BD, and let AD, DC, CB be joined.

Then, since a straight line EF touches the circle $ABCD$ at B, and BA has been drawn from the point of contact at right angles to the tangent, the centre of the circle $ABCD$ is on BA [III. 19]. Therefore BA is a diameter of the circle $ABCD$; therefore the angle ADB, being an angle in a semicircle, is right. [III. 31]. Therefore the remaining angles BAD, ABD, are equal to one right angle. [I. 32]. But the angle ABF is also right; therefore the angle ABF is equal to the angles BAD, ABD. Let the angle ABD be subtracted from each; therefore the angle DBF which remains is equal to the angle BAD in the alternate segment of the circle.

Next, since $ABCD$ is a quadrilateral in a circle, its opposite angles are equal to two right angles [III. 22]. But the angles DBF, DBE are also equal to two right angles; therefore the angles DBF, DBE are equal to the angles BAD, BCD, of which the angle BAD was proved equal to the angle DBF; therefore the angle DBE which remains is equal to the angle DCB in the alternate segment DCB of the circle.

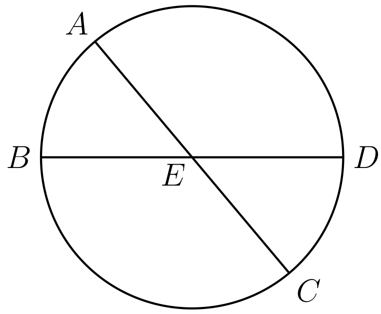
Therefore etc.

Q.E.D.

Book III, Proposition 35

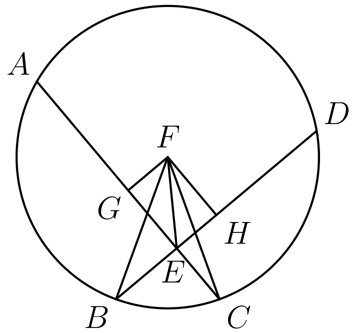
If in a circle two straight lines cut one another, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

For in the circle $ABCD$ let the two straight lines AC, BD cut one another at the point E; I say that the rectangle contained by AE, EC is equal to the rectangle contained by DE, EB.



If now AC, BD are through the centre, so that E is the centre of the circle $ABCD$, it is manifest that, AE, EC, DE, EB being equal, the rectangle contained by AE, EC is also equal to the rectangle contained by DE, EB.

Next let AC, DB not be through the centre; let the centre of $ABCD$ be taken, and let it be F; from F let FG, FH be drawn perpendicular to the straight lines AC, DB, and let FB, FC, FE be joined.



Then, since a straight line GF through the centre cuts a straight line AC not through the centre at right angles, it also bisects it [III. 3]; therefore AG is equal to GC. Since, then, the straight line AC has been cut into equal parts at G and into unequal parts at E, the rectangle contained by AE, EC together with the square on EG is equal to the square on GC [II. 5]. Let the square on GF be added; therefore the rectangle AE, EC together with the squares on GE, GF is equal to the squares on CG, GF.

But the square on FE is equal to the squares on EG, GF, and the square on FC is equal to the squares on CG, GF [I. 47]; therefore the rectangle AE, EC together with the square on FE is equal to the square on FC. And FC is equal to FB; therefore the rectangle AE, EC together with the square on EF is equal to the square on FB.

For the same reason, also, the rectangle DE, EB together with the square on FE is equal to the square on FB. But the rectangle AE, EC together with the square on FE was also proved equal to the square on FB; therefore the rectangle AE, EC together with the square on FE is equal to the rectangle DE, EB together with the square on FE. Let the square on FE be subtracted from each; therefore the rectangle contained by AE, EC which remains is equal to the rectangle contained by DE, EB.

Therefore etc.

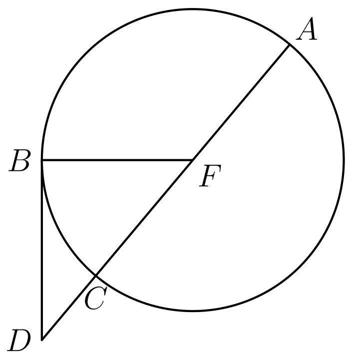
Q.E.D.

Book III, Proposition 36

If a point be taken outside a circle and from it there fall on the circle two straight lines, and if one of them cut the circle and the other touch it, the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference will be equal to the square on the tangent.

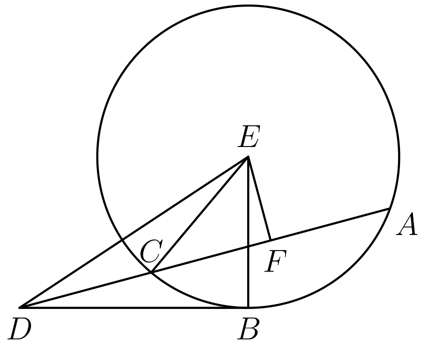
For let a point D be taken outside the circle ABC, and from D let the two straight lines DCA, DB fall on the circle ABC; let DCA cut the circle ABC and let BD touch it; I say that the rectangle contained by AD, DC is equal to the square on DB.

Then DCA is either through the centre or not through the centre.



First let it be through the centre, and let F be the centre of the circle ABC; let FB be joined; therefore the angle FBD is right [III. 18]. And, since AC has been bisected at F, and CD is added to it, the rectangle AD, DC together with the square on FC is equal to the square on FD [II. 6]. But FC is equal to FB; therefore the rectangle AD, DC together with the square on FB is equal to the square on FD. And the squares on FB, BD are equal to the square on FD [I. 47]; therefore the rectangle AC, DC together with the square on FB is equal to the squares on FB, BD. Let the square FB be subtracted from each; therefore the rectangle AD, DC which remains is equal to the square on the tangent DB.

Again, let DCA not be through the centre of the circle ABC; let the centre E be taken, and from E let EF be drawn perpendicular to AC; let EB, EC, ED be joined.



Then the angle EBD is right [III. 18]. And, since a straight line EF through the centre cuts a straight line AC not through the centre at right angles, it also bisects it [III. 3]; therefore AF is equal to FC.

Now, since the straight line AC has been bisected at the point F, and CD is added to it, the rectangle contained by AD, DC together with the square on FC is equal to the square on FD [II. 6]. Let the square on FE be added to each; therefore the rectangle AD, DC together with the squares on CF, FE is equal to the squares on FD, FE.

But the square on EC is equal to the squares on CF, FE, for the angle EFC is right [I. 47]; and the square on ED is equal to the squares on DF, FE; therefore the rectangle AD, DC together with the square on EC is equal to the square on ED. And EC is equal to EB; therefore the rectangle AD, DC together with the square on EB is equal to the square on ED. But the squares on EB, BD are equal to the square on ED, for the angle EBD is right [I. 47]; therefore the rectangle AD, DC together with the square on EB is equal to the squares on EB, BD. Let the square on EB be subtracted from each; therefore the rectangle AD, DC which remains is equal to the square on DB.

Therefore etc.

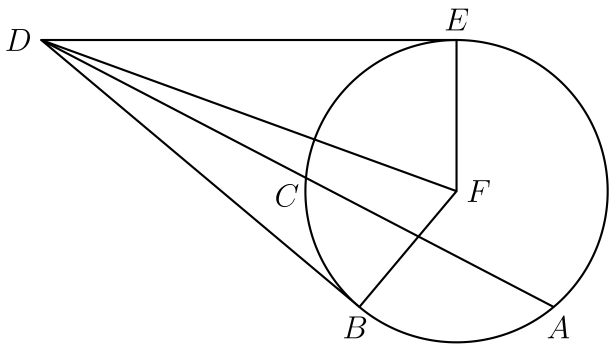
Q.E.D.

Book III, Proposition 37

If a point be taken outside a circle and from the point there fall on the circle two straight lines, if one of them cut the circle, and the other fall on it, and if further the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference be equal to the square on the straight line which falls on the circle, the straight line which fall on it will touch the circle.

For let a point D be taken outside the circle ABC, and from D let the two straight lines DCA, DB fall on the circle ACB; let DCA cut the circle and DB fall on it; and let the rectangle AD, DC be equal to the square on DB.

I say that DB touches the circle ABC.



For let DE be drawn touching ABC; let the centre of the circle ABC be taken, and let it be F; let FE, FB, FD be joined. Thus the angle FED is right [III. 18]. Now, since DE touches the circle ABC, and DCA cuts it, the rectangle AD, DC is equal to the square on DE [III. 36] But the rectangle AD, DC was also equal to the square on DB; therefore the square on DE is equal to the square on DB; therefore DE is equal to DB. And FE is equal to FB; therefore the two sides DE, EF are equal to the two sides DB, BF; and FD is the common base of the triangles; therefore the angle DEF is equal to the angle DBF [I. 8]. But the angle DEF is right; therefore the angle DBF is also right. And FB produced is a diameter; and the straight line drawn at right angles to the diameter of a circle, from its extremity, touches the circle [III. 16, Por]; therefore DB touches the circle.

Similarly this can be proved to be the case even if the centre be on AC.

Therefore etc.

Q.E.D.

SELECTED PROPOSITIONS FROM EUCLID'S *ELEMENTS*, BOOK IV

Definitions

1. A rectilineal figure is said to be **inscribed in a rectilineal figure** when the respective angles of the inscribed figure lie on the respective sides of that in which it is inscribed.

2. Similarly a figure is said to be **circumscribed about a figure** when the respective sides of the circumscribed figure pass through the respective angles of that about which it is circumscribed.

3. A rectilineal figure is said to be **inscribed in a circle** when each angle of the inscribed figure lies on the circumference of the circle.

4. A rectilineal figure is said to be **circumscribed about a circle**, when each side of the circumscribed figure touches the circumference of the circle.

5. Similarly a circle is said to be **inscribed in a figure** when the circumference of the circle touches each side of the figure in which it is circumscribed.

6. A circle is said to be **circumscribed about a figure** when the circumference of the circle passes through each angle of the figure about which it is circumscribed.

7. A straight line is said to be **fitted into a circle** when its extremities are on the circumference of the circle.

Book IV, Proposition 1

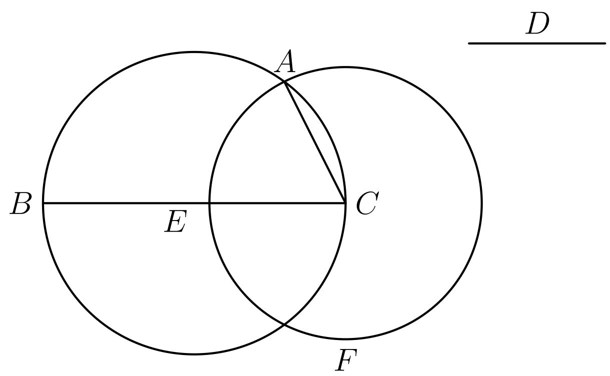
Into a given circle to fit a straight line equal to a given straight line which is not greater than the diameter of the circle.

Let ABC be the given circle, and D the given straight line not greater than the diameter of the circle; thus it is required to fit into the circle ABC a straight line equal to the straight line D.

Let a diameter BC of the circle ABC be drawn.

Then, if BC is equal to D, that which was enjoined will have been done; for BC has been fitted into the circle ABC equal to the straight line D.

But, if BC is greater than D, let CE be made equal to D, and with centre C and distance CE let the circle EF be described; let CA be joined.



Then, since the point C is the centre of the circle EAF, CA is equal to CE. But CE is equal to D; therefore D is also equal to CA.

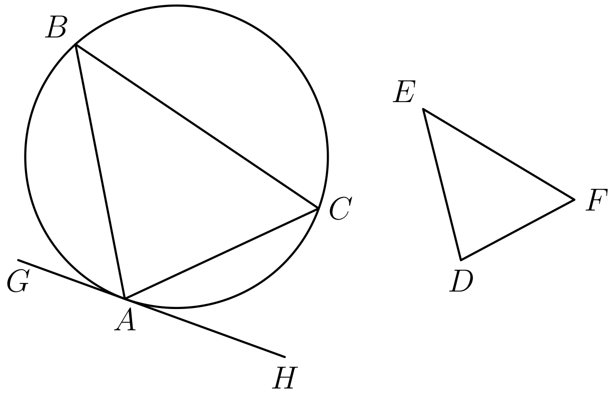
Therefore into the given circle ABC there has been fitted CA equal to the given straight line D.

Book IV, Proposition 2

In a given circle to inscribe a triangle equiangular with a given triangle.

Let ABC be the given circle, and DEF the given triangle; thus it is required to inscribe in the circle ABC a triangle equiangular with the triangle DEF.

Let GH be drawn touching the circle ABC at A [III. 16, Por.]; on the straight line AH, and at the point A on it, let the angle HAC be constructed equal to the angle DEF, and on the straight line AG, and at the point A on it, let the angle GAB be constructed equal to the angle DFE [I. 23]; let BC be joined.



Then, since a straight line AH touches the circle ABC, and from the point of contact at A the straight line AC is drawn across in the circle, therefore the angle HAC is equal to the angle ABC in the alternate segment of the circle [III. 32]. But the angle HAC is equal to the angle DEF; therefore the angle ABC is also equal to the angle DEF. For the same reason the angle ACB is also equal to the angle DFE; therefore the remaining angle BAC is also equal to the remaining angle EDF.

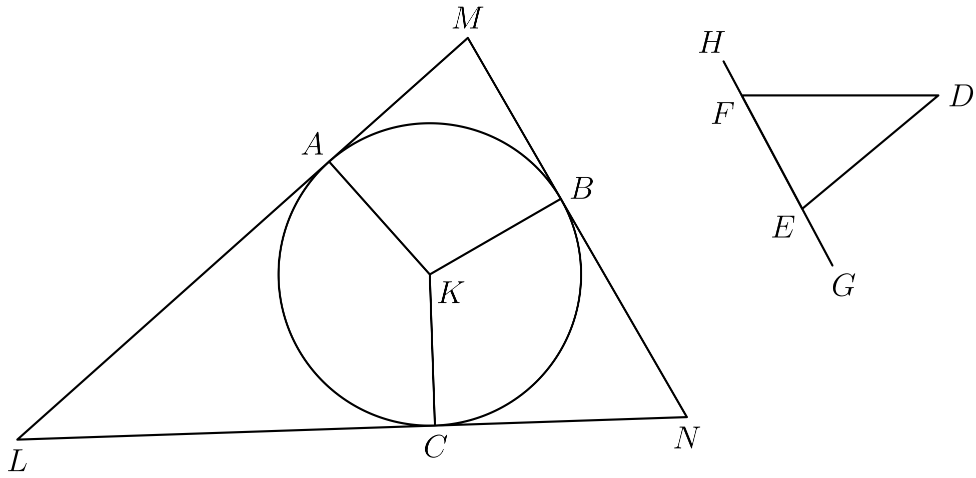
Therefore in the given circle there has been inscribed a triangle equiangular with the given triangle.

Q.E.F.

About a given circle to circumscribe a triangle equiangular with a given triangle.

Let ABC be the given circle, and DEF the given triangle; thus it is required to circumscribe about the circle ABC a triangle equiangular with the triangle DEF.

Let EF be produced in both directions to the points G, H, let the centre K of the circle ABC be taken [iii. 1], and let the straight line KB be drawn across at random; on the straight line KB, and at the point K on it, let the angle BKA be constructed equal to the angle DEG, and the angle BKC equal to the angle DFH; and through the points A, B, C let LAM, MBN, NCL be drawn touching the circle ABC iii. 16, Por..



Now, since LM, MN, NL touch the circle ABC at the points A, B, C, and KA, KB, KC have been joined from the centre K to the points A, B, C, therefore the angles at the points A, B, C are right [iii. 18]. And, since the four angles of the quadrilateral $AMBK$ are equal to four right angles, inasmuch as $AMBK$ is in fact divisible into two triangles, and the angles KAM, KBM are right; therefore the remaining angles AKB, AMB are equal to two right angles. But the angles DEG, DEF are also equal to two right angles. [i. 13]; therefore the angles AKB, AMB are equal to the angles DEG, DEF, of which the angle AKB is equal to the angle DEG; therefore the angle AMB which remains is equal to the angle DEF which remains.

Similarly it can be proved that the angle LNB is also equal to the angle DFE; therefore the remaining angle MLN is equal to the angle EDF.

Therefore the triangle LMN is equiangular with the triangle DEF; and it has been circumscribed about the circle ABC.

Therefore about a given circle there has been circumscribed a triangle equiangular with the given triangle.

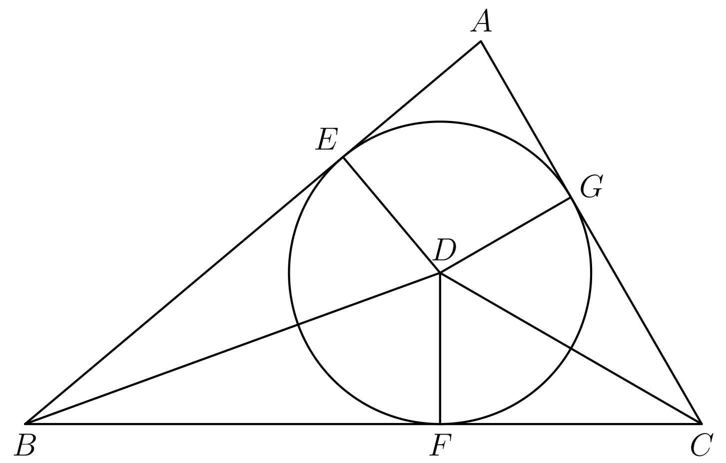
Q.E.F.

Book IV, Proposition 4

In a given triangle to inscribe a circle.

Let ABC be the given triangle; thus it is required to inscribe a circle in the triangle ABC.

Let the angles ABC, ACB be bisected by the straight lines BD, CD [I. 9], and let these meet one another at the point D; from D let DE, DF, DG be drawn perpendicular to the straight lines AB, BC, CA.



Now, since the angle ABD is equal to the angle CBD, and the right angle BED is also equal to the right angle BFD, EBD, FBD are two triangles having the two angles equal to two angles and one side equal to one side, namely that subtending one of the equal angles, which is BD common to the triangles; therefore they will also have the remaining sides equal to the remaining sides; therefore DE is equal to DF.

For the same reason DG is also equal to DF. Therefore the three straight lines DE, DF, DG are equal to one another; therefore the circle described with centre D and distance one of the straight lines DE, DF, DG will pass also through the remaining points, and will touch the straight lines AB, BC, CA, because the angles at the points E, F, G are right.

For if it cuts them, the straight line drawn at right angles to the diameter of the circle from its extremity will be found to fall within the circle: which was proved absurd [III. 16]; therefore the circle described with centre D and distance one of the straight lines DE, DF, DG will not cut the straight lines AB, BC, CA; therefore it will touch them, and will be the circle inscribed in the triangle ABC [IV. Def. 5]. Let it be inscribed, as FGE.

Therefore, in the given triangle ABC the circle EFG has been inscribed.

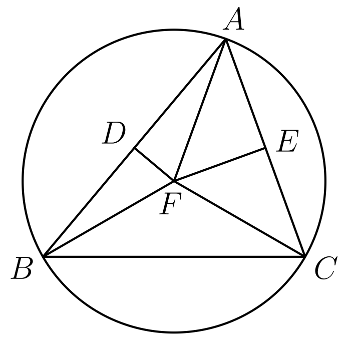
Q.E.F.

About a given triangle to circumscribe a circle.

Let ABC be the given triangle; thus it is required to circumscribe a circle about the given triangle ABC.

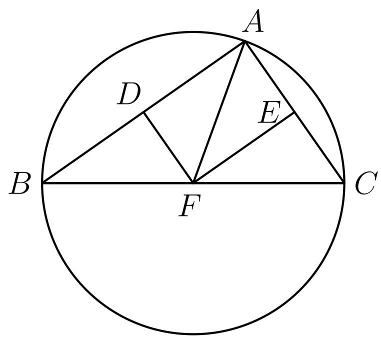
Let the straight lines AB, AC be bisected at the points D, E [I. 10], and from the points D, E let DF, EF be drawn at right angles to AB, AC; they will then meet within the triangle ABC, or on the straight line BC, or outside BC.

First let them meet within at F, and let FB, FC, FA be joined.



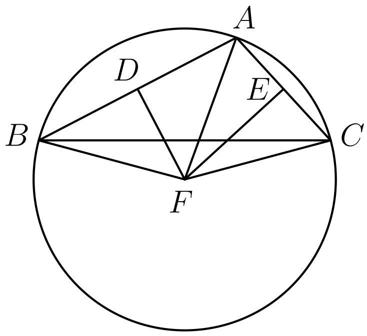
Then, since AD is equal to DB, and DF is common and at right angles, therefore the base AF is equal to the base FB [I. 4]. Similarly we can prove that CF is also equal to AF; so that FB is also equal to FC; therefore the three straight lines FA, FB, FC are equal to one another, Therefore the circle described with centre F and distance one of the straight lines FA, FB, FC will pass also through the remaining points, and the circle will have been circumscribed about the triangle ABC. Let it be circumscribed, as ABC.

Next, let DE, EF meet on the straight line BC at F, as is the case in the second figure; and let AF be joined.



Then, similarly, we shall prove that the point F is the centre of the circle circumscribed about the triangle ABC.

Again, let DF, EF meet outside the triangle ABC at F, as is the case in the third figure, and let AF, BF, CF be joined.



Then again, since AD is equal to DB, and DF is common and at right angles, therefore the base AF is equal to the base BF [I. 4]. Similarly we can prove that CF is also equal to AF; so that BF is also equal to FC; therefore the circle described with centre F and distance on of the straight lines FA, FB, FC will pass also through the remaining points, and will have been circumscribed about the triangle ABC.

Therefore about the given triangle a circle has been circumscribed.

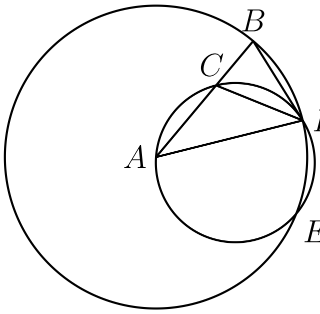
Q.E.F.

And it is manifest that, when the centre of the circle falls within the triangle, the angle BAC, being in a segment greater than the semicircle, is less than a right angle; when the centre falls on the straight line BC, the angle BAC, being in a semicircle, is right; and when the centre of the circle falls outside the triangle, the angle BAC, being in a segment less than a semicircle, is greater than a right angle [III. 31].

Book IV, Proposition 10

To construct an isosceles triangle having each of the angles at the base double of the remaining one.

Let any straight line AB be set out, and let it be cut at the point C so that the rectangle contained by AB, BC is equal to the square on CA [II. 11]; with centre A and distance AB let the circle BDE be described, and let there be fitted in the circle BDE the straight line BD equal to the straight line AC which is not greater than the diameter of the circle BDE [IV. 1]. Let AD, DC be joined, and let the circle ACD be circumscribed about the triangle ACD [IV. 5].



Then, since the rectangle AB, BC is equal to the square on AC, and AC is equal to BD, Therefore the rectangle AB, BC is equal to the square on BD.

And, since a point B has been taken outside the circle ACD, and from B the two straight lines BA, BD have fallen on the circle ACD, and one of them cuts it, while the other falls on it, and the rectangle AB, BC is equal to the square on BD, therefore BD touches the circle ACD [III. 37].

Since, then, BD touches it, and DC is drawn across from the contact at D, therefore the angle BDC is equal to the angle DAC in the alternate segment of the circle [III. 32].

Since, then, the angle BDC is equal to the angle DAC, let the angle CDA be added to each; therefore the whole angle BDA is equal to the two angles CDA, DAC.

But the exterior angle BCD is equal to the angles CDA, DAC; therefore the angle BDA is also equal to the angle BCD.

But the angle BDA is equal to the angle CBD, since the side AD is also equal to AB [I. 5]; so that the angle DBA is also equal to the angle BCD.

Therefore the three angles BDA, DBA, BCD are equal to one another.

And, since the angle DBC is equal to the angle BCD, the side BD is also equal to the side DC [I. 6].

But BD is by hypothesis equal to CA; therefore CA is also equal to CD, so that the angle CDA is also equal to the angle DAC [I. 5]; therefore the angles CDA, DAC are double of the angle DAC.

But the angle BCD is equal to the angles CDA, DAC; therefore the angle BCD is also double of the angle CAD.

But the angle BCD is equal to each of the angles BDA, DBA; therefore each of the angles BDA, DBA is also double of the angle DAB.

Therefore the isosceles triangle ABD has been constructed having each of the angles at the base DB double of the remaining one.

Q.E.F.

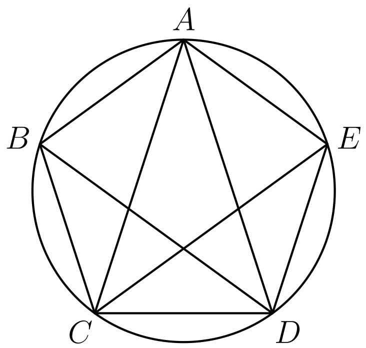
Book IV, Proposition 11

In a given circle to inscribe an equilateral and equiangular pentagon.

Let $ABCDE$ be the given circle; thus it is required to inscribe in the circle $ABCDE$ an equilateral and equiangular pentagon.

Let the isosceles triangle FGH be set out having each of the angles at G, H double of the angle at F [IV. 10]; let there be inscribed in the circle $ABCDE$ the triangle ACD equiangular with the triangle FGH, so that the angle CAD is equal to the angle at F, and the angles at G, H respectively equal to the angles ACD, CDA [IV. 2]; therefore each of the angles ACD, CDA is also double of the angle CAD.

Now let the angles ACD, CDA be bisected respectively by the straight lines CE, DB [I. 9], and let AB, BC, DE, EA be joined.



Then, since each of the angles ACD, CDA is double of the angle CAD, and they have been bisected by the straight lines CE, DB, therefore the five angles DAC, ACE, ECD, CDB, BDA are equal to one another. But equal angles stand on equal circumferences [III. 26]; therefore the five circumferences AB, BC, CD, DE, EA are equal to one another. But equal circumferences are subtended by equal straight lines [III. 29]; therefore the five straight lines AB, BC, CD, DE, EA are equal to one another; therefore the pentagon $ABCDE$ is equilateral.

I say next that it is also equiangular.

For since the circumference AB is equal to the circumference DE, let BCD be adde to each; therefore the whole circumference $ABCD$ is equal to the whole circumference $EDCB$.

And the angle AED stands on the circumference $ABCD$, and the angle BAE on the circumference $EDCB$; therefore the angle BAE is also equal to the angle AED [III. 27]. For the same reason each of the angles ABC, BCD, CDE is also equal to each of the angles BAE, AED; therefore the pentagon

$ABCDE$ is equiangular. But it was also proved equilateral; therefore in a given circle an equilateral and equiangular pentagon has been inscribed.

Q.E.F.