Propositions from Euclid's *Elements of Geometry* Book II (T.L. Heath's Edition)

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Definitions

1. Any rectangular parallelogram is said to be **contained** by the two straight lines containing the right angle.

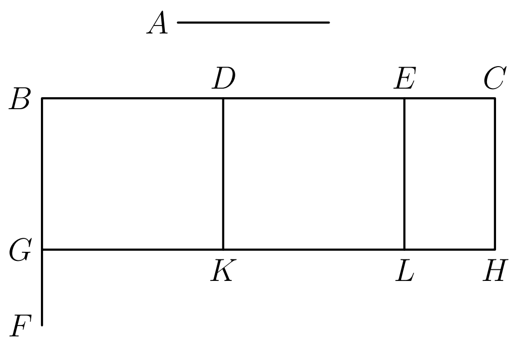
2. And in any parallelogrammic area let any one whatever of the parallelograms about its diameter with the two complements be called a **gnomon**.

Proposition 1

If there be two straight lines, and one of them be cut into any number of segments whatever, the rectangle contained by the two straight lines is equal to the rectangle contained by the uncut straight line and each of the segments.

Let A, BC be two straight lines, and let BC be cut at random at the points D, E; I say that the rectangle contained by A, BC is equal to the rectangle contained by A, BD, that contained by A, DE, and that contained by A, EC.

For let BF be drawn from B at right angles to BC [I. 11]; let BG be made equal to A [I. 3]; through G let GH be drawn parallel to BC [I. 31], and through D, E, C let DK, EL, CH be drawn parallel to BG. Then BH is equal to BK, DL, EH.



Now BH is the rectangle A, BC, for it is contained by GB, BC, and BG is equal to A; BK is the rectangle A, BD, for it is contained by GB, BD, and BG is equal to A; and DL is the rectangle A, DE, for DK, that is BG [I. 34], is equal to A. Similarly also EH is the rectangle A, EC. Therefore the rectangle A, BC is equal to the rectangle A, BD, the rectangle A, DE and the rectangle A, EC.

Therefore etc.

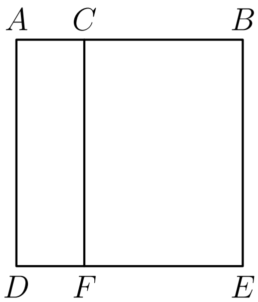
Q.E.D.

Proposition 2

If a straight line be cut at random, the rectangle contained by the whole and both of the segments is equal to the square on the whole.

For let the straight line AB be cut at random at the point C; I say that the rectangle contained by AB, BC together with the rectangle contained by BA, AC is equal to the square on AB.

For let the square $ADEB$ be described on AB [I. 46], and let CF be drawn through C parallel to either AD or BE. Then AE is equal to AF, CE.



Now AE is equal to the square on AB; AF is the rectangle contained by BA, AC, for it is contained by DA, AC, and AD is equal to AB; and CE is the rectangle AB, BC, for BE is equal to AB. Therefore the rectangle BA, AC together with the rectangle AB, BC is equal to the square on AB.

Therefore etc.

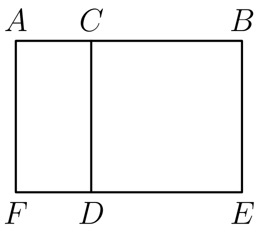
Q.E.D.

PROPOSITION 3

If a straight line be cut at random, the rectangle contained by the whole and one of the segments is equal to the rectangle contained by the segments and the square on the aforesaid segment.

For let the straight line AB be cut at random at C; I say that the rectangle contained by AB, BC is equal to the rectangle contained by AC, CB together with the square on BC.

For let the square $CDEB$ be described on CB [I. 46]; let ED be drawn through to F, and through A let AF be drawn parallel to either CD or BE [I. 31]. Then AE is equal to AD, CE.



Now AE is the rectangle contained by AB, BC, for it is contained by AB, BE, and BE is equal to BC; AD is the rectangle AC, CB, for DC is equal to CB; and DB is the square on CB. Therefore the rectangle contained by AB, BC is equal to the rectangle contained by AC, CB together with the square on BC.

Therefore etc.

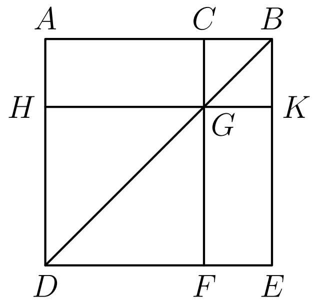
Q.E.D.

Proposition 4

If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.

For let the straight line AB be cut at random at C; I say that the square on AB is equal to the squares on AC, CB and twice the rectangle contained by AC, CB.

For let the square $ADEB$ be described on AB [I. 46], let BD be joined; through C let CF be drawn parallel to either AD or EB, and through G let HK be drawn parallel to either AB or DE [I. 31].



Then, since CF is parallel to AD, and BD has fallen on them, the exterior angle CGB is equal to the interior and opposite angle ADB [I. 29]. But the angle ADB is equal to the angle ABD, since the side BA is also equal to AD [I. 5.]; therefore the angle CGB is also equal to the angle GBC, so that the side BC is also equal to the side CG [I. 6]. But CB is equal to GK, and CG to KB [I. 34] therefore GK is also equal to KB; therefore $CGKB$ is equilateral.

I say next that it is also right-angled. For, since CG is parallel to BK, the angles KBC, GCB are equal to two right angles [I. 29]. But the angle KBC is right; therefore the angle BCG is also right, so that the opposite angles CGK, GKB are also right [I. 34]. Therefore $CGKB$ is right-angled; and it was also proved equilateral; therefore it is a square; and it is described on CB.

For the same reason HF is also a square; and it is described on HG, that is AC [I. 34]. Therefore the squares HF, KC are the squares on AC, CB.

Now, since AG is equal to GE, and AG is the rectangle AC, CB, for GC is equal to CB, therefore GE is also equal to the rectangle AC, CB. Therefore AG, GE are equal to twice the rectangle AC, CB.

But the squares HF, CK are also the squares on AC, CB; therefore the four areas HF, CK, AG, GE are equal to the squares on AC, CB and twice the rectangle contained by AC, CB. But HF, CK, AG, GE are the whole

$ADEB$, which is the square on AB. Therefore the square on AB is equal to the squares on AC, CB and twice the rectangle contained by AC, CB.

Therefore etc.

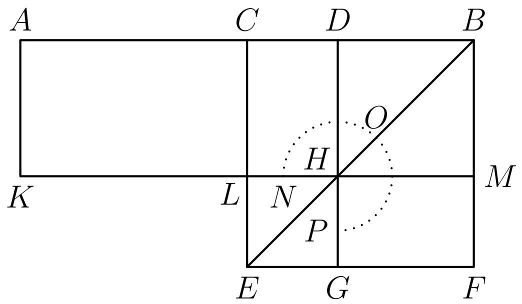
Q.E.D.

Proposition 5

If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half.

For let a straight line AB be cut into equal segments at C and into unequal segments at D; I say that the rectangle contained by AD, DB together with the square on CD is equal to the square on CB.

For let the square $CEFB$ be described on CB [I. 46], and let BE be joined; through D let DG be drawn parallel to either CE or BF, through H again let KM be drawn parallel to either AB or EF, and again through A let AK be drawn parallel to either CL or BM [I. 31].



Then, since the complement CH is equal to the complement HF [I. 43], Let DM be added to each; therefore the whole CM is equal to the whole DF. But CM is equal to AL, since AC is also equal to CB [I. 36]; therefore AL is also equal to DF. Let CH be added to each; therefore the whole AH is equal to the gnomon NOP. But AH is the rectangle AD, DB, for DH is equal to DB, therefore the gnomon NOP is also equal to the rectangle AD, DB. Let LG, which is equal to the square on CD, be added to each; therefore the gnomon NOP and LG are equal to the rectangle contained by AD, DB and the square on CD. But the gnomon NOP and LG are the whole square $CEFB$, which is described on CB; therefore the rectangle contained by AD, DB together with the square on CD is equal to the square on CB.

Therefore etc.

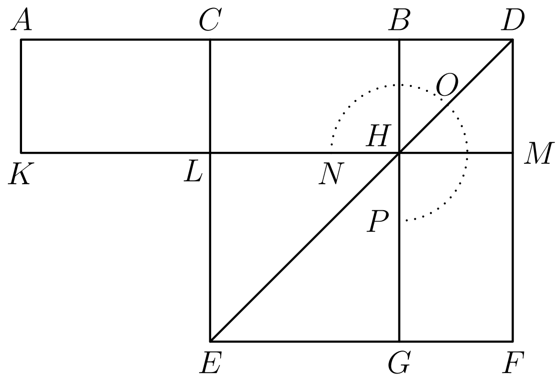
Q.E.D.

Proposition 6

If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.

For let a straight line AB be bisected at the point C, and let a straight line BD be added to it in a straight line; I say that the rectangle contained by AD, DB together with the square on CB is equal to the square on CD.

For let the square $CEFD$ be described on CD [I. 46], and let DE be joined; through the point B let BG be drawn parallel to either EC or DF, through the point H let KM be drawn parallel to either AB or EF, and further through A let AK be drawn parallel to either CL or DM [I. 31].



Then, since AC is equal to CB, AL is also equal to CH [I. 36]. But CH is equal to HF [I. 43]. Therefore AL is also equal to HF. Let CM be added to each; therefore the whole AM is equal to the gnomon NOP. But AM is the rectangle AD, DB, for DM is equal to DB, therefore the gnomon NOP is also equal to the rectangle AD, DB. Let LG, which is equal to the square on BC, be added to each; therefore the rectangle contained by AD, DB together with the square on CB is equal to the gnomon NOP and LG. But the gnomon NOP and LG are the whole square $CEFD$, which is described on CD; therefore the rectangle contained by AD, DB together with the square on CB is equal to the square on CD.

Therefore etc.

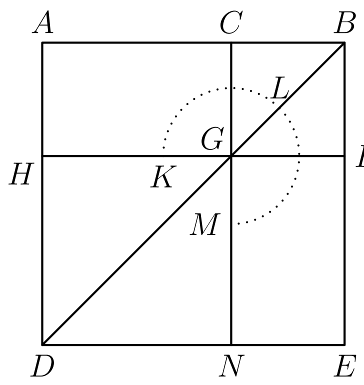
Q.E.D.

PROPOSITION 7

If a straight line be cut at random, the square on the whole and that on one of the segments both together are equal to twice the rectangle contained by the whole and the said segment and the square on the remaining segment.

For let a straight line AB be cut at random at the point C; I say that the squares on AB, BC are equal to twice the rectangle contained by AB, BC and the square on CA.

For let the square $ADEB$ be described on AB [I. 46], and let the figure be drawn.



Then, since AG is equal to GE [I. 43], let CF be added to each; therefore the whole AF is equal to the whole CE. Therefore AF, CE are double of AF. But AF, CE are the gnomon KLM and the square CF; therefore the gnomon KLM and the square CF are double of AF. But twice the rectangle AB, BC is also double of AF; for BF is equal to BC; therefore the gnomon KLM and the square CF are equal to twice the rectangle AB, BC. Let DG, which is the square on AC, be added to each; therefore the gnomon KLM and the squares BG, GD are equal to twice the rectangle contained by AB, BC and the square on AC. But the gnomon KLM and the squares BG, GD are the whole $ADEB$ and CF, which are the squares described on AB, BC; therefore the squares on AB, BC are equal to twice the rectangle contained by AB, BC together with the square on AC.

Therefore etc.

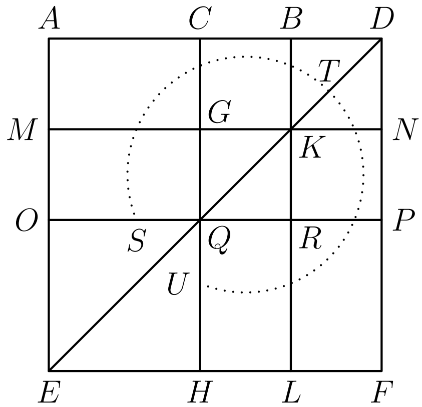
Q.E.D.

Proposition 8

If a straight line be cut at random, four times the rectangle contained by the whole and one of the segments together with the square on the remaining segment is equal to the square described on the whole and the aforesaid segment as on one straight line.

For let a straight line AB be cut at random at the point C; I say that four times the rectangle contained by AB, BC together with the square on AC is equal to the square described on AB, BC as on one straight line.

For let [the straight line] BD be produced in a straight line [with AB], and let BD be made equal to CB; let the square $AEFD$ be described on AD, and let the figure be drawn double.



Then, since CB is equal to BD, while CB is equal to GK, and BD to KN, therefore GK is also equal to KN.

For the same reason, QR is also equal to RP.

And, since BC is equal to BD, and GK to KN, therefore CK is also equal to KD, and GR to RN [I. 36]. But CK is equal to RN, for they are complements of the parallelogram CP [I. 43]; therefore KD is also equal to GR; therefore the four areas DK, CK, GR, RN are equal to one another. Therefore the four are quadruple of CK.

Again, since CB is equal to BD, while BD is equal to BK, that is CG, and CB is equal to GK, that is GQ, therefore CG is also equal to GQ. And, since CG is equal to GQ, and QR to RP, AG is also equal to MQ, and QL to RF [I. 36]. But MQ is equal to QL, for they are complements of the parallelogram ML [I. 43]; therefore AG is also equal to RF; therefore the four areas AG, MQ, QL, RF are equal to one another. Therefore the four are quadruple of AG. But the four areas CK, KD, GR, RN were proved to be quadruple of CK; therefore the eight areas, which contain the gnomon STU, are quadruple of AK.

Now, since AK is the rectangle AB, BD, for BK is equal to BD, therefore four times the rectangle AB, BD is quadruple of AK. But the gnomon STU was also proved to be quadruple of AK; therefore four times the rectangle AB, BD is equal to the gnomon STU. Let OH, which is equal to the square on AC, be added to each; therefore four times the rectangle AB, BD together with the square on AC is equal to the gnomon STU and OH. But the gnomon STU and OH are the whole square $AEFD$, which is described on AD; therefore four times the rectangle AB, BD together with the square on AC is equal to the square on AD. But BD is equal to BC; therefore four times the rectangle contained by AB, BC together with the square on AC is equal to the square on AD, that is to the square described on AB and BC as on one straight line.

Therefore etc.

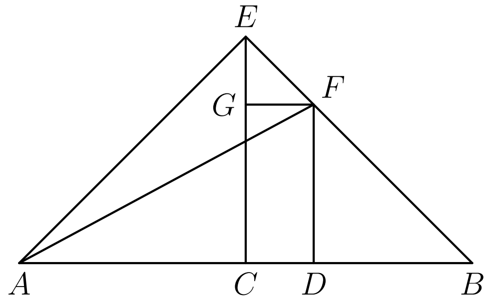
Q.E.D.

Proposition 9

If a straight line be cut into equal and unequal segments, the squares on the unequal segments of the whole are double of the square on the half and of the square on the straight line between the points of section.

For let a straight line AB be cut into equal segments at C, and into unequal segments at D. I say that the squares on AD, DB are double of the squares on AC, CD.

For let CE be drawn from C at right angless to AB, and let it be made equal to either AC or CB; let EA, EB be joined, let DF be drawn through D parallel to EC, and FG through F parallel to AB, and let AF be joined.



Then, since AC ie equal to CE, the angle EAC is also equal to the angle AEC. And, since the angle at C is right, the remaining angles EAC, AEC are equal to one right angle [I. 32]. And they are equal; therefore each of the angles CEA, CEB is half a right angle.

For the same reason each of the angles CEB, EBC is also half a right angle; therefore the whole angle AEB is right. And, since the angle GEF is half a right angle, and the angle EGF is right, for it is equal to the interior and opposite angle ECB [I. 29], the remaining angle EFG is half a right angle [I. 32]; therefore the angle GEF is equal to the angle EFG, so that the side EG is also equal to GF [I. 6]. Again, since the angle at B is half a right angle, and the angle FDB is right, for it is again equal to the interior and opposite angle ECB [I. 29], the remaining angle BFD is half a right angle [I. 32]; therefore the angle at B is equal to the angle DFB, so that the side FD is also equal to the side DB [I. 6].

Now, since AC is equal to CE, the square on AC is also equal to the square on CE; therefore the squares on AC, CE are double of the square on AC. But the square on EA is equal to the squares on AC, CE, for the angle ACE is right [I. 47]; therefore the square on EA is double of the square on AC. Again, since EG is equal to GF, the square on EG is also equal to the square on GF; therefore the squares on EG, GF are double of the square on

GF. But the square on EF is equal to the squares on EG, GF; therefore the square on EF is double of the square on GF. But GF is equal to CD [I. 34]; therefore the square on EF is double of the square on CD. But the square on EA is also double of the square on AC; therefore the squares on AE, EF are double of the squares on AC, CD. And the square on AF is equal to the squares on AE, EF, for the angle AEF is right; [I. 47] therefore the square on AF is double of the squares on AC, CD. But the squares on AD, DF are equal to the square on AF, for the angle at D is right [I. 47]; therefore the squares on AD, DF are double of the squares on AC, CD. And DF is equal to DB; therefore the squares on AD, DB are double of the squares on AC, CD.

Therefore etc.

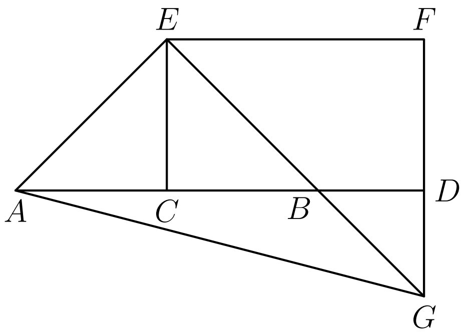
Q.E.D.

Proposition 10

If a straight line be bisected, and a straight line be added to it in a straight line, the square on the whole with the added straight line and the square on the added straight line both together are double of the square on the half and of the square described on the straight line made up of the half and the added straight line as on one straight line.

For let a straight line AB be bisected at C, and let a straight line BD be added to it in a straight line; I say that the squares on AD, DB are double of the squares on AC, CD.

For let CE be drawn from the point C at right angless to AB [I. 11], and let it be made equal to either AC or CB [I. 3]; let EA, EB be joined; through E let EF be drawn parallel to AD, and through D let FD be drawn parallel to CE [I. 31].



Then, since a straight line EF falls on the parallel straight lines EC, FD, the angles CEF, EFD are equal to two right angles [I. 29]; therefore the angles FEB, EFD are less than two right angles. But straight lines produced from angles less than two right angles meet [I. Post. 5]; therefore EB, FD, if produced in the direction B, D, will meet. Let them be produced and meet at G, and let AG be joined. Then, since AC is equal to CE, the angle EAC is also equal to the angle AEC [I 5]; and the angle at C is right; therefore each of the angles EAC, AEC is half a right angle [I. 32]. For the same reason each of the angles CEB, EBC is also half a right angle; therefore the angle AEB is right. And, since the angle EBC is half a right angle, the angle DBG is also half a right angle [I. 15]. But the angle BDG is also right, for it is equal to the angle DCE, they being alternate [I. 29]; therefore the remaining angle DGB is half a right angle [I. 32]; therefore the angle DGB is equal to the angle DBG, so that the side BD is also equal to the side GD [I. 6]. Again, since the angle EGF is half a right angle, and the angle at F is right, for it is equal to the opposite ange, the angle at C [I. 34], the remaining angle FEG is half a right angle [I. 32]; therefore the angle EGF is equal to the angle FEG, so that the side GF is also equal to the side EF [I. 6].

Now, since the square on EA is equal to the square on CA, the squares on EC, CA are double of the square on CA. But the square on EA is squal to the squares on EC, CA [I. 47]; therefore the square on EA is double of the square on AC [C. N. 1]. Again, since FG is equal to EF, the square on FG is also equal to the square on FE; therefore the squares on GF, FE are double of the squares on EF. But the square on EG is equal to the squares on GF, FE [I. 47]; therefore the square on EG is double of the square on EF. And EF is equal to CD [I. 34]; therefore the square on EG is double of the square on CD. But the square on EA was also proved double of the square on AC; therefore the squares on AE, EG are double of the squares on AC, CD. And the square on AG is equal to the squares on AE, EG [I. 47]; therefore the square on AG is double of the squares on AC, CD. But the squares on AD, DG are equal to the square on AG [I. 47]; therefore the squares on AD, DG are double of the squares on AC, CD. And DG is equal to DB; therefore the squares on AD, DB are double of the squares on AC, CD.

Therefore etc.

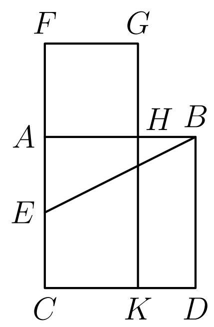
Q.E.D.

Proposition 11

To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.

Let AB be the given straight line; thus it is required to cut AB so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.

For let the square $ABDC$ be described on AB; let AC be bisected at the point E, and let BE be joined; let CA be drawn through to F, and let EF be made equal to BE; let the square FH be described on AF, and let GH be drawn through to K. I say that AB has been cut at H so as to make the rectangle contained by AB, BH equal to the square on AH.



For, since the straight line AC has been bisected at E, and FA added to it, the rectangle contained by CF, FA together with the square on AE is equal to the square on EF [II. 6]. But EF is equal to EB; therefore the rectangle CF, FA together with the square on AE is equal to the square on EB. But the squares on BA, AE are equal to the square on EB, for the angle at A is right [I. 47]: therefore the rectangle CF, FA together with the square on AE is equal to the squares on BA, AE. Let the square on AE be subtracted from each; therefore the rectangle CF, FA which remains is equal to the square on AB.

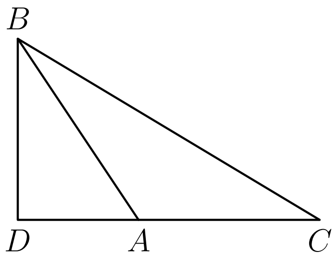
Now the rectangle CF, FA is FK, for AF is equal to FG; and the square on AB is AD; therefore FK is equal to AD. Let AK be subtracted from each; therefore FH which remains is equal to HD. And HD is the rectangle AB, BH, for AB is equal to BD; and FH is the square on AH; therefore the rectangle contained by AB, BH is equal to the square on HA. Therefore the given straight line AB has been cut at H so as to make the rectangle contained by AB, BH equal to the square on HA.

Q.E.F.

Proposition 12

In obtuse-angled triangles the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle by twice the rectangle contained by one of the sides about the obtuse angle, namely that on which the perpendicular falls, and the straight line cut off outside by the perpendicular towards the obtuse angle.

Let ABC be an obtuse-angled triangle having the angle BAC obtuse, and let BD be drawn from the point B perpendicular to CA produced; I say that the square on BC is greater than the squares on BA, AC by twice the rectangle contained by CA, AD.



For, since the straight line CD has been cut at random at the point A, the square on DC is equal to the squares on CA, AD and twice the rectangle contained by CA, AD [II. 4]. Let the square on DB be added to each; therefore the squares on CD, DB are equal to the squares on CA, AD, DB and twice the rectangle CA, AD. But the square on CB is equal to the squares on CD, DB, for the angle at D is right [I. 47]; and the square on AB is equal to the squares on AD, DB; therefore the square on CB is equal to the squares on CA, AB and twice the rectangle contained by CA, AD; so that the square on CB is greater than the squares on CA, AB by twice the rectangle contained by CA, AD.

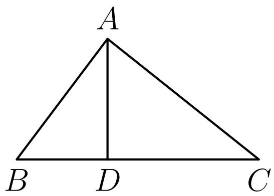
Therefore, etc.

Q.E.D.

Proposition 13

In acute-angled triangles the square on the side subtending the acute angle is less than the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acute angle.

Let ABC be an acute-angled triangle having the angle at B acute, and let AD be drawn from the point A perpendicular to BC; I say that the square on AC is less than the squares on CB, BA by twice the rectangle contained by CB, BD.



For, since the straight line CB has been cut at random at D, the squares on CB, BD are equal to twice the rectangle contained by CB, BD and the square on DC [II. 7]. Let the square on DA be added to each; therefore the squares on CB, BD, DA are equal to twice the rectangle contained by CB, BD and the squares on AD, DC. But the square on AB is equal to the squares on BD, DA, for the angle at D is right [I. 47]; and the square on AC is equal to the squares on AD, DC; therefore the squares on CB, BA are equal to the square on AC and twice the rectangle CB, BD, so that the square on AC alone is less than the squares on CB, BA by twice the rectangle contained by CB, BD.

Therefore, etc.

Q.E.D.

Proposition 14

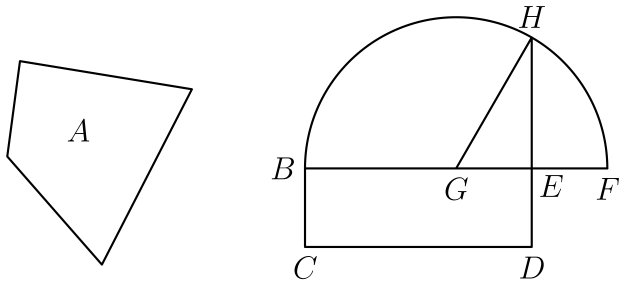
To construct a square equal to a given rectilineal figure.

Let A be the given rectilineal figure; thus it is required to construct a square equal to the rectilineal figure A.

For let there be constructed the rectangular parallelogram BD equal to the rectilineal figure A [I, 45]. Then, if BE is equal to ED, that which was enjoined will have been done; for a square BD has been constructed equal to the rectilineal figure A.

But, if not, one of the straight lines BE, ED is greater.

Let BE be greater, and let it be produced to F; let EF be made equal to ED, and let BF be bisected at G. With centre G and distance one of the straight lines GB, GF let the semicircle BHF be described; let DE be produced to H, and let GH be joined.



Then, since the straight line BF has been cut into equal segments at G, and into unequal segments at E, the rectangle contained by BE, EF together with the square on EG is equal to the square on GF [II. 5]. But GF is equal to GH; therefore the rectangle BE, EF together with the square on GE is equal to the square on GH. But the squares on HE, EG are equal to the square on GH [I. 47]; therefore the rectangle BE, EF together with the square on GE is equal to the squares on HE, EG. Let the square on GE be subtracted from each; therefore the rectangle contained by BE, EF which remains is equal to the square on EH. But the rectangle BE, EF is BD, for EF is equal to ED; therefore the parallelogram BD is equal to the square on HE. And BD is equal to the rectilineal figure A. Therefore the rectilineal figure A is also equal to the square which can be described on EH.

Therefore a square, namely that which can be described on EH, has been constructed equal to the given rectilineal figure A.

Q.E.F.