Propositions from Euclid's *Elements of Geometry* Book III (T.L. Heath's Edition)

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March 29, 2023

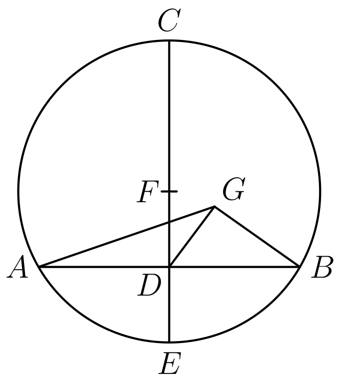
Definitions

1. **Equal circles** are those the diameters of which are equal, or the radii of which are equal.
2. A straight line is said to **touch a circle** which, meeting the circle and being produced, does not cut the circle.
3. **Circles** are said to **touch one another** which, meeting one another, do not cut one another.
4. In a circle straight lines are said **to be equally distant from the centre** when the perpendiculars drawn to them from the centre are equal.
5. And that straight line is said to be **at a greater distance** on which the greater perpendicular falls.
6. A **segment of a circle** is the figure contained by a straight line and a circumference of a circle.
7. An **angle of a segment** is that contained by a straight line and a circumference of a circle.
8. An **angle in a segment** is the angle which, when a point is taken on the circumference of the segment and straight lines are joined from it to the extremities of the straight line which is the **base of the segment**, is contained by the straight lines so joined.
9. And when the straight lines containing the angle cut off a circumference, the angle is said to **stand upon** that circumference.
10. A **sector of a circle** is the figure which, when an angle is constructed at the centre of the circle, is contained by the straight lines containing the angle and the circumference cut off by them.
11. **Similar segments of circles** are those which admit equal angles, or in which the angles are equal to one another.

Proposition 1

To find the centre of a given circle.

Let ABC be the given circle; thus it is required to find the centre of the circle ABC.



Let a straight line AB be drawn through it at random, and let it be bisected at the point D; from D let DC be drawn at right angles to AB and let it be drawn through to E; let CE be bisected at F; I say that F is the centre of the circle ABC.

For suppose it is not, but, if possible, let G be the centre, and let GA, GD, GB be joined.

Then, since AD is equal to DB, and DG is common, the two sides AD, DG are equal to the two sides BD, DG respectively; and the base GA is equal to the base GB, for they are radii; therefore the angle ADG is equal to the angle GDB [I. 8].

But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the the equal angles is right [I Def. 10]; therefore the angle GDB is right.

But the angle FDB is also right; Therefore the angle FDB is equal to the angle GDB, the greater to the less: which is impossible.

Therefore G is not the centre of the circle ABC.

Similarly we can prove that neither is any other point except F.

Therefore the point F is the centre of the circle ABC.

Porism. From this, it is manifest that, if in a circle a straight line cut a straight line into two equal parts and at right angles, the centre of the circle is on the cutting straight line.

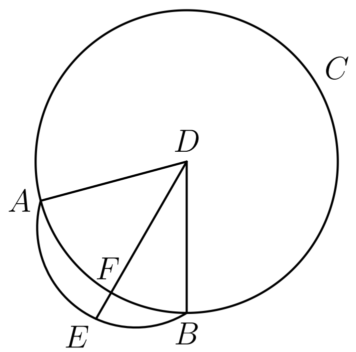
Q.E.F.

Proposition 2

If on the circumference of a given circle two points be taken at random, the straight line joining the points will fall within the circle.

Let ABC be a circle, and let two points A and B be taken at random on its circumference; I say that the straight line joined from A to B will fall within the circle.

For suppose it does not, but, if possible, let it fall outside, as AEB; let the centre of the circle ABC be taken [III. 1], and let it be D; let DA, DB be joined, and let DFE be drawn through.



Then since DA is equal to DB, the angle DAE is also equal to the angle DBE [I. 5]. And, since one side AEB of the triangle DAE is produced, the angle DEB is greater than the angle DAE [I. 16]. But the angle DAE is equal to the angle DBE; therefore the angle DEB is greater than the angle DBE. And the greater angle is subtended by the greater side [I. 19]; therefore DB is greater than DE.

But DB is equal to DF; therefore DF is greater than DE, the less than the greater: which is impossible.

Therefore the straight line joined from A to B will not fall outside the circle.

Similarly we can prove that neither will it fall on the circumference itself; therefore it will fall within. Therefore etc.

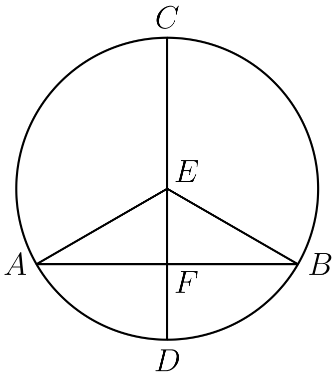
Q.E.D.

Proposition 3

If in a circle a straight line through the centre bisect a straight line not through the centre, it also cuts it at right angles; and if it cut it at right angles, it also bisects it.

Let ABC be a circle, and in it let a straight line CD through the centre bisect a straight line AB not through the centre at the point F; I say that it also cuts it at right angles.

For let the centre of the circle ABC be taken, and let it be E; let EA, EB be joined.



Then, since AF is equal to FB, and FE is common, two sides are equal to two sides; and the base EA is equal to the base EB; therefore the angle AFE is equal to the angle BFE [I. 8].

But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right; [I. Def. 10] therefore each of the angles AFE, BFE is right.

Therefore CD, which is through the centre, and bisects AB which is not through the centre, also cuts it at right angles.

Again, let CD cut AB at right angles; I say that it also bisects it, that is, that AF is equal to FB.

For, with the same construction, since EA is equal to EB, the angle EAF is also equal to the angle EBF [I. 5].

But the right angle AFE is equal to the right angle BFE, therefore EAF, EBF are two triangles having two angles equal to two angles and one side equal to one side, namely EF, which is common to them, and subtends one of the equal angles; therefore they will also have the remaining sides equal to the remaining sides [I. 26]; therefore AF is equal to FB.

Therefore etc.

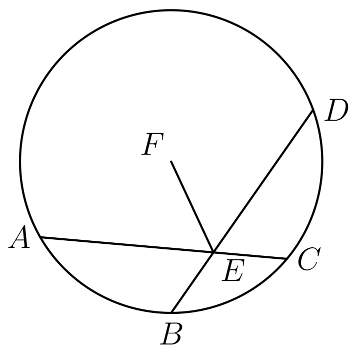
Q.E.D.

Proposition 4

If in a circle two straight lines cut one another which are not through the centre, they do not bisect one another.

Let $ABCD$ be a circle, and in it let the two straight lines AC, BD, which are not through the centre, cut one another in E; I say that they do not bisect one another.

For, if possible, let them bisect one another, so that AE is equal to EC, and BE to ED; let the centre of the circle $ABCD$ be taken [III. 1], and let it be F; let FE be joined.



Then, since a straight line FE through the centre bisects a straight line AC not through the centre;
it also cuts it at right angles [III. 3]; therefore the angle FEA is right.

Again, since a straight line FE bisects a straight line BD, it also cuts it at right angles [III. 3]; therefore the angle FEB is right.

But the angle FBA was also proved right; therefore the angle FEA is equal to the angle FEB, the less to the greater: which is impossible.

Therefore AC, BD do not bisect one another.

Therefore etc.

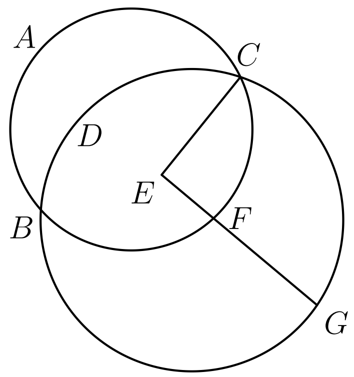
Q.E.D.

Proposition 5

If two circles cut one another, they will not have the same centre.

For let the circle ABC, CDG cut one another at the points B, C; I say that they will not have the same centre.

For, if possible, let it be E; let EC be joined, and let EFG be drawn through at random.



Then, since the point E is the centre of the circle ABC, EC is equal to EF [I. Def. 15].

Again, since the point E is the centre of the circle CDG, EC is equal to EG. But EC was proved equal to EF also; therefore EF is also equal to EG, the less to the greater: which is impossible.

Therefore the point E is not the centre of the circles ABC, CDG.

Therefore etc.

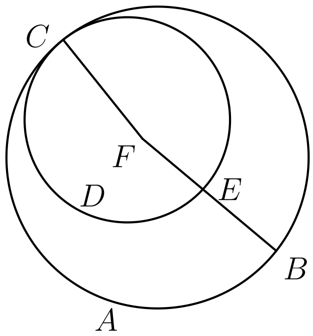
Q.E.D.

Proposition 6

If two circles touch one another, they will not have the same centre.

For let the two circles ABC, CDE touch one another at the point C; I say that they will not have the same centre.

For, if possible, let it be F; let FC be joined, and let FEB be drawn through at random.



Then, since the point F is the centre of the circle ABC, FC is equal to FB.

Again, since the point F is the centre of the circle CDE, FC is equal to FE.

But FC was proved equal to FB; therefore FE is also equal to FB, the less to the greater: which is impossible.

Therefore F is not the centre of the circles ABC, CDE.

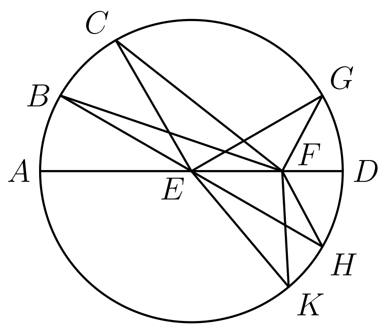
Therefore etc.

Q.E.D.

PROPOSITION 7

If on the diameter of a circle a point be taken which is not the centre of the circle, and from the point straight lines fall upon the circle, that will be greatest on which the centre is, the remainder of the same diameter will be least, and of the rest the nearer to the straight line through the centre is always greater than the more remote, and only two equal straight lines will fall from the point on the circle, one on each side of the least straight line.

Let $ABCD$ be a circle, and let AD be a diameter of it; on AD let a point F be taken which is not the centre of the circle, let E be the centre of the circle, and from F let straight lines FB, FC, FG fall upon the circle $ABCD$; I say that FA is greatest, FD the least, and of the rest FB is greater than FC, and FC than FG. For let BE, CE, GE be joined.



Then, since in any triangle, two sides are greater than the remaining one [I. 20], EB, EF are greater than BF. But AE is greater than BE; therefore AF is greater than BF.

Again, since BE is equal to CE, and FE is common, the two sides BE, EF are equal to the two sides CE, EF. But the angle BEF is also greater than the angle CEF; therefore the base BF is greater than the base CF [I. 24]. For the same reason CF is also greater than FG.

Again, since GF, FE are greater than EG, and EG is equal to ED, GF, FE are greater than ED. Let EF be subtracted from each; therefore the remainder GF is greater than the remainder FD.

Therefore FA is greatest, FD is least, and FB is greater than FC, and FC than FG.

I say also that from the point F only two equal straight lines will fall on the circle $ABCD$, one on each side of the least FD.

For on the straight line EF, and at the point E on it, let the angle FEH be constructed equal to the angle GEF [I. 23], and let FH be joined.

Then, since GE is equal to EH, and EF is common, the two sides GE, EF are equal to the two sides HE, EF; and the angle GEF is equal to the angle HEF; therefore the base FG is equal to the base FH [I. 4].

I say again that another straight line equal to FG will not fall on the circle from the point F.

For, if possible, let FK so fall.

Then, since FK is equal to FG, and FH to FG, FK is also equal to FH, the nearer to the straight line through the centre being thus equal to the more remote: which is impossible.

Therefore another straight line equal to GF will not fall from the point F upon the circle; therefore only one straight line will so fall.

Therefore etc.

Q.E.D.

Proposition 8

If a point be taken outside a circle and from the point straight lines be drawn through to the circle, one of which is through the centre and the others are drawn at random, then, of the straight lines which fall on the concave circumference, that through the centre is greatest, whil of the rest, the nearer to that through the centre is always greater than the more remote, but, of the straight lines falling on the convex circumference, that between the point and the diameter is least, while of the rest the nearer to the least is always less than the more remote, and only two equal straight lines will fall on the circle from the point, one on each side of the least.

Let ABC be a circle, and let a point D be taken outside ABC; let there by drawn through from it straight lines DA, DE, DF, DC, and let DA be through the centre; I say that, of the straight lines falling on the concave circumference $AEFC$, the straight line DA through the centre is greatest, while DE is greater than DF and DF than DC; but, of the straight lines falling on the convex circumference $HLKG$, the straight line DG between the point and the diameter AG is least; and the nearer to the least DG is always less than the more remote, namely DK than DL, and DL than DH.

For let the centre of the circle ABC be taken [III. 1], and let it be M; let ME, MF, MC, MK, ML, MH be joined.

Then, since AM is equal to EM, let MD be added to each; therefore AD is equal to EM, MD. But EM, MD are greater than ED; therefore AD is also greater than ED.

Again, since ME is equal to MF, and MD is common, therefore EM, MD are equal to FM, MD; and the angle EMD is greater than the angle FMD; therefore the base ED is greater than the base FD [I. 24].

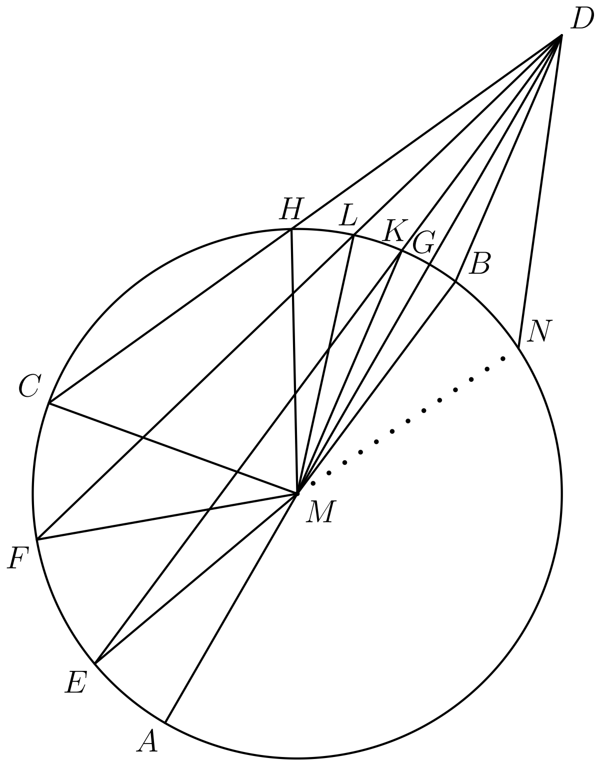
Similarly we can prove that FD is greater than CD; therefore DA is greatest, while DE is greater than DF, and DF than DC.

Next, since MK, KD are greater than MD [I. 20], and MG is equal to MK, therefore the remainder KD is greater than the remainder GD, so that GD is less than KD. And, since on MD, one of the sides of the triangle MLD, two straight lines MK, KD were constructed meeting within the triangle, therefore MK, KD are less than ML, LD [I 21]; and MK is equal to ML; therefore the remainder DK is less than the remainder DL.

Similarly we can prove that DL is also less than DH; therefore DG is least, while DK is less than DL, and DL than DH.

I say also that only two equal straight lines will fall from the point D on the circle, one on each side of the least DG.

On the straight line MD, and at the point M on it, let the angle DMB be constructed equal to the angle KMD, and let DB be joined.



Then, since MK is equal to MB, and MD is common, the two sides KM, MD are equal to the two sides BM, MD, respectively; and the angle KMD is equal to the angle BMD; therefore the base DK is equal to the angle DB [I, 4].

I say that no other straight line equal to the straight line DK will fall on the circle from D.

For, if possible, let a straight line so fall, and let it be DN.

Then, since DK is equal to DN, while DK is equal to DB, DB is also equal to DN, that is, the nearer to the least DG equal to the more remote: which was proved impossible.

Therefore no more than two equal straight lines will fall on the circle ABC from the point D, one on each side of DG the least.

Therefore etc.

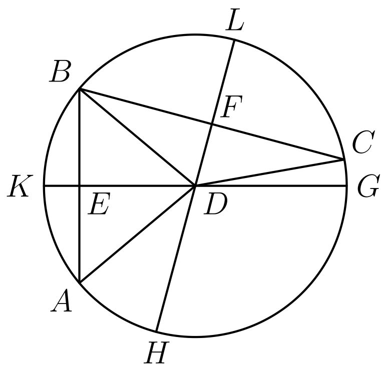
Q.E.D.

Proposition 9

If a point be taken within a circle, and more than two equal straight lines fall from the point on the circle, the point taken is the centre of the circle.

Let ABC be a circle and D a point within it, and from D let more than two equal straight lines, namely DA, DB, DC, fall on the circle ABC; I say that the point D is the centre of the circle ABC.

For let AB, BC be joined and bisected at the points E, F, and let ED, FD be joined and drawn through to the points G, K, H, L.



Then, since AE is equal to EB, and ED is common, the two sides AE, ED are equal to the two sides BE, ED; and the base DA is equal to the base DB; therefore the angle AED is equal to the angle BED [I. 8]. Therefore each of the angles AED, BED is right [I. Def. 10]; therefore GK cuts AB into two equal parts and at right angles.

And since, if in a circle a straight line cut a straight line into two equal parts and at right angles, the centre of the circle is on the cutting straight line [III. 1, Por.], the centre of the circle is on GK.

For the same reason the centre of the circle ABC is also on HL. And the straight lines GK, HL have no other point common but the point D; therefore the point D is the centre of the circle ABC.

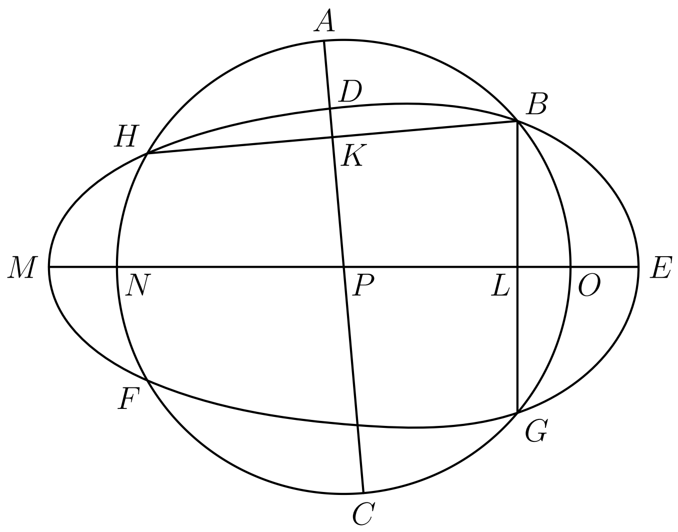
Therefore etc.

Q.E.D.

Proposition 10

A circle does not cut a circle at more points than two.

For, if possible, let the circle ABC cut the circle DEF at more points than two, namely B, G, F, H; let BH, BG be joined and bisected at the points K, L, and from K, L let KC, LM be drawn at right angles to BH, BG and carried through to the points A, E.



Then, since in the circle ABC a straight line AC cuts a straight line BH into two equal parts and at right angles, the centre of the circle ABC is on AC [III. 1, Por.].

Again, since in the same circle ABC a straight line NO cuts a straight line BG into two equal parts and at right angles, the centre of the circle ABC is on NO.

But it was also proved to be on AC, and the straight lines AC, NO meet at no point except at P; therefore the point P is the centre of the circle ABC.

Similarly we can prove that P is also the centre of the circle DEF; therefore the two circles ABC, DEF which cut one another have the same centre P: which is impossible. [III. 5].

Therefore etc.

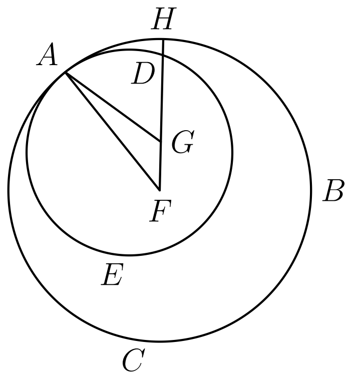
Q.E.D.

Proposition 11

If two circles touch one another internally, and their centres be taken, the straight line joining their centres, if it be also produced, will fall on the point of contact of the circles.

For let the two circles ABC, ADE touch one another internally at the point A, and let the centre F of the circle ABC, and the centre G of ADE, be taken; I say that the straight line joined from G to F and produced will fall on A.

For suppose it does not, but, if possible, let it fall as FGH, and let AF, AG be joined.



Then, since AG, GF are greater than FA, that is, than FH, let FG be subtracted from each; therefore the remainder AG is greater than the remainder GH.

But AG is equal to GD; therefore GD is also greater than GH, the less than the greater: which is impossible.

Therefore the straight line joined from F to G will not fall outside; ; therefore it will fall at A on the point of contact.

Therefore etc.

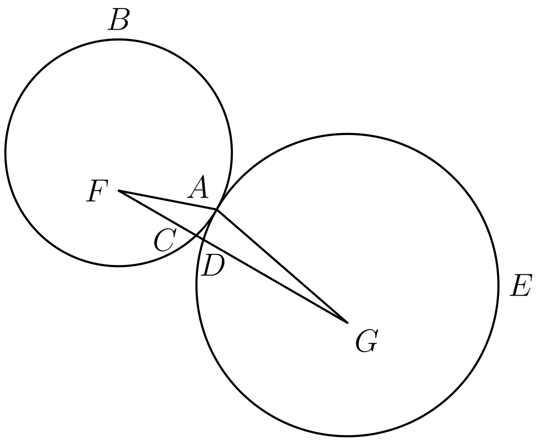
Q.E.D.

Proposition 12

If two circles touch one another externally, the straight line joining their centres will pass through the point of contact.

For let the two circles ABC, ADE touch one another externally at the point A, and let the centre F of ABC, and the centre G of ADE, be taken; I say that the straight line joined from F to G will pass through the point of contact at A.

For suppose it does not, but if possible, let it pass as $FCDG$, and let AF, AG be joined.



Then, since the point F is the centre of the circle ABC, FA is equal to FC.

Again, since the point G is the centre of the circle ADE, GA is equal to GD.

But FA was also proved equal to FC; therefore FA, AG are equal to FC, GD, brind so that the whole FG is greater than FA, AG; but it is also less [I. 20]: which is impossible.

Therefore the straight line joined from F to G will not fail to pass through the point of contact at A; therefore it will pass through it.

Therefore etc.

Q.E.D.

Proposition 13

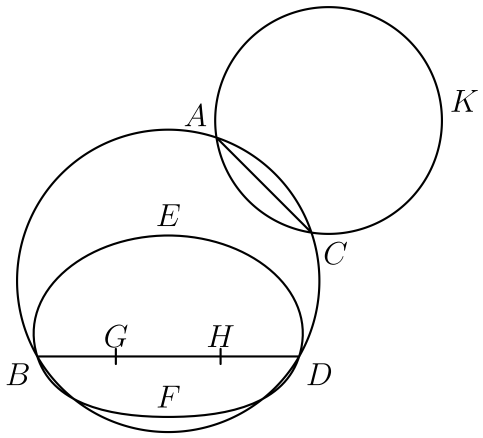
A circle does not touch a circle at more points than one, whether it touch it internally or externally.

For, if possible, let the circle $ABDC$ touch the circle $EBFD$, first internally, at more points than one, namely D, B.

Let the centre G of the circle $ABDC$, and the centre H of $EBFD$, be taken.

Therefore the straight line joined from G to H will fall on B, D [III. 11].

Let it so fall, as $BGHD$.



Then, since the point G is the centre of the circle $ABCD$, BG is equal to GD; therefore BG is greater than HD; therefore BH is much greater than HD.

Again, since the point H is the centre of the circle $EBFD$, BH is equal to HD; but it was also proved much greater than it: which is impossible.

Therefore a circle does not touch a circle internally at more points than one.

I say further that neither does it so touch it externally.

For, if possible, let the circle ACK touch the circle $ABDC$ at more points than one, namely A, C, and let AC be joined.

Then, since on the circumference of each of the circles $ABDC$, ACK two points A, C have been taken at random, the straight line joining the points will fall within each circle [IIII. 2]; but it fell within the circle $ABCD$ and outside ACK [III. Def. 3]: which is absurd.

Therefore a circle does not touch a circle externally at more points than one.

And it was proved that neither does it so touch it internally.

Therefore etc.

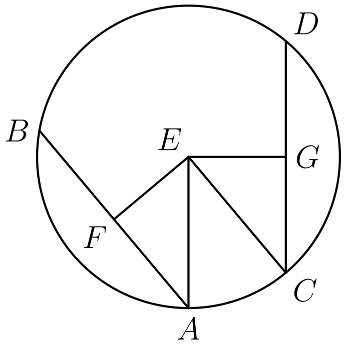
Q.E.D.

Proposition 14

In a circle equal straight lines are equally distant from the centre, and those which are equally distant from the centre are equal to one another.

Let $ABDC$ be a circle, and let AB, CD be equal straight lines in it; I say that AB, CD are equally distant from the centre.

For let the centre of the circle $ABDC$ be taken [III. 1], and let it be E; from E let EF, EG be drawn perpendicular to AB, CD, and let AE, EC be joined.



Then, since a straight line EF through the centre cuts a straight line AB not through the centre at right angles, it also bisects it [III. 3].

Therefore AF is equal to FB; therefore AB is the double of AF.

For the same reason CD is also the double of CG; and AB is equal to CD; therefore AF is also equal to CG.

And, since AE is equal to EC, the square on AE is also equal to the square on EC.

But the squares on AF, EF are equal to the square on AE, for the angle at F is right; and the squares on EG, GC are equal to the square on EC, for the angle at G is right [I. 47]; therefore the squares on AF, FE are equal to the squares on CG, GE, of which the square on AF is equal to the square on CG, for AF is equal to CG; therefore the square on FE which remains is equal to the square on EG, therefore EF is equal to EG.

But in a circle straight lines are said to be equally distant from the centre; that is, let EF be equal to EG.

I say that AB is also equal to CD.

For, with the same construction, we can prove, similarly, that AB is double of AF, and CD of CG.

And, since AE is equal to CE, the square on AE is equal to the square on CE. But the squares on EF, FA are equal to the square on AE, and the squares on EG, GC equal to the square on CE. [I. 47]

Therefore the squares on EF, FA are equal to the squares on EG, GC, of which the square on EF is equal to the square on EG, for EF is equal to EG; therefore the square on AF which remains is equal to the square on CG; therefore AF is equal to CG. And AB is double of AF, and CD double of CG; therefore AB is equal to CD.

Therefore etc.

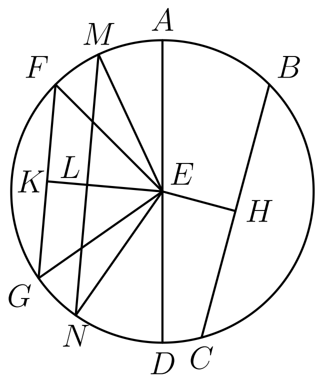
Q.E.D.

Proposition 15

Of straight lines in a circle the diameter is greatest, and of the rest the nearer to the centre is always greater than the more remote.

Let $ABCD$ be a circle, let AD be its diameter and E the centre; and let BC be nearer to the diameter AD, and FG more remote; I say that AD is the greatest and BC greater than FG.

For from the centre E let EH, EK be drawn perpendicular to BC, FG.



Then, since BC is nearer to the centre and FG more remote, EK is greater than EH [III. Def. 5].

Let EL be made equal to EH, through L let LM be drawn at right angles to EK and crried through to N, and let ME, EN, FE, EG be joined.

Then, since EH is equal to EL, BC is also equal to MN [III. 14].

Again, since AE is equal to EM, and ED to EN, AD is equal to ME, EN.

But ME, EN are greater than MN, and MN is equal to BC; therefore AD is greater than BC.

And since the two sides ME, EN are equal to the two sides FE, EG, and the angle MEN greater than the angle FEG, therefore the base MN is greater than the base FG [I. 24].

But MN was proved equal to BC.

Therefore the diameter AD is the greatest and BC greater than FG.

Therefore etc.

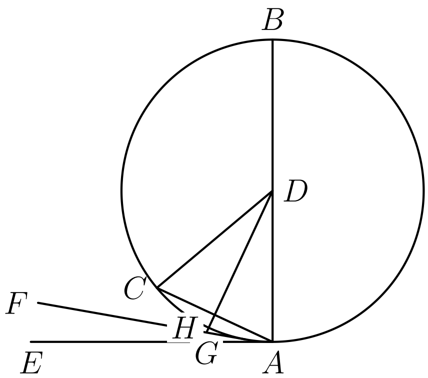
Q.E.D.

Proposition 16

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilinear angle.

Let ABC be a circle about D as centre and AB as diameter; I say that the straight line drawn from A at right angles to AB from its extremity will fall outside the circle.

For suppose it does not, but, if possible, let it fall within as CA, and let DC be joined.



Since DA is equal to DC, the angle DAC is also equal to the angle ACD [I. 5].

But the angle DAC is right; therefore the angle ACD is also right: thus, in the triangle ACD, the two angles DAC, ACD are equal to two right angles: which is impossible [I. 17].

Therefore the straight line drawn from the point A at right angles to BA will not fall within the circle.

Similarly we can prove that neither will it fall on the circumference; therefore it will fall outside.

Let it fall as AE; I say next that into the space between the straight line AE and the circumference CHA another straight line cannot be interposed.

For, if possible, let another straight line be so interposed, as FA, and let DG be drawn from the point D perpendicular to FA.

Then, since the angle AGD is right, and the angle DAG is less than a right angle, AD is greater than DG [I. 19].

But DA is equal to DH; therefore DH is greater than DG, the less than the greater, which is impossible.

Therefore another straight line cannot be interposed into the space between the straight line and the circumference.

I say further that the angle of the semicircle contained by the straight line BA and the circumference CHA is greater than any acute rectilineal angle, and the remaining angle contained by the circumference CHA and the straight line AE is less than any acute rectilinear angle.

For, if there is any rectilineal angle greater than the angle contained by the straight line BA and the circumference CHA, and any rectilineal angle less than the angle contained by the circumference CHA and the straight line AE, then into the space between the circumference and the straight line AE a straight line will be interposed such as will make an angle contined by straight lines which is greater than the angle contained by the straight line BA and the circumference CHA, and another angle contained by straight lines which is less than the angle contained by the circumference CHA and the straight line AE.

But such a straight line cannot be interposed; therefore there will not be any acute angle contained by straight lines which is greater than the angle contained by the straight line BA and the circumference CHA, nor yet any acute angle contained by straight lines which is less than the angle contained by the circumference CHA and the straight line AE.—

PORISM. From this it is manifest that the straight line drawn at right angles to the diameter of a circle from its extremity touches the circle.

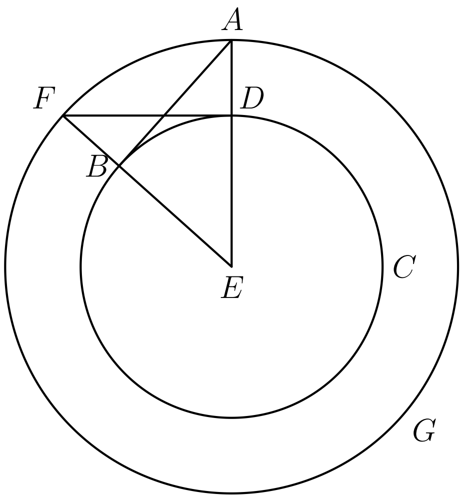
Q.E.D.

Proposition 17

From a given point to draw a straight line touching a given circle.

Let A be the given point, and BCD the given circle; thus it is required to draw from the point A a straight line touching the circle BCD.

For let the centre E of the circle be taken [III. 1]. let AE be joined, and with centre E and distance EA let the circle AFG be described; from D let DF be drawn at right angles to EA, and let EF, AB be joined; I say that AB has been drawn from the point A touching the circle BCD.



For, since E is the centre of the circles BCD, AFG, EA is equal to EF, and ED to EB; therefore the two sides AE, EB are equal to the two sides FE, ED: and they contain a common angle, the angle at E; therefore the base DF is equal to the base AB, and the triangle DEF is equal to the triangle BEA, and the remaining angles to the remaining angles [I. 4]; therefore the angle EDF is equal to the angle EBA.

But the angle EDF is right; therefore the angle EBA is also right.

Now EB is a radius; and the straight line drawn at right angles to the diameter of a circle, from its extremity, touches the circle; [III. 16, Por.] therefore AB touches the circle BCD.

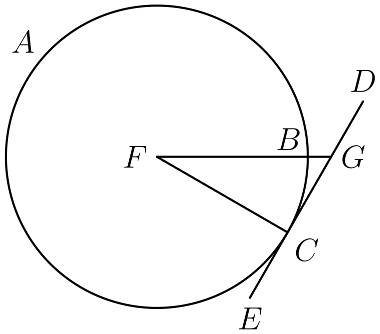
Therefore from the given point A the straight line AB has been drawn touching the circle BCD.

Proposition 18

If a straight line touch a circle, and a straight line be joined from the centre to the point of contact, the straight line so joined will be perpendicular to the tangent.

For let a straight line DE touch the circle ABC at the point C, let the centre F of the circle ABC be taken, and let FC be joined from F to C; I say that FC is perpendicular to DE.

For, if not, let FG be drawn from F perpendicular to DE.



Then, since the angle FGC is right, the angle FCG is acute [I. 17]; and the greater angle is subtended by the greater side; therefore FC is greater than FG.

But FC is equal to FB; therefore FB is also greater than FG, the less than the greater: which is impossible.

Therefore FG is not perpendicular to DE.

Similarly we can prove that neither is any other straight line except FC; therefore FC is perpendicular to DE. Therefore, etc.

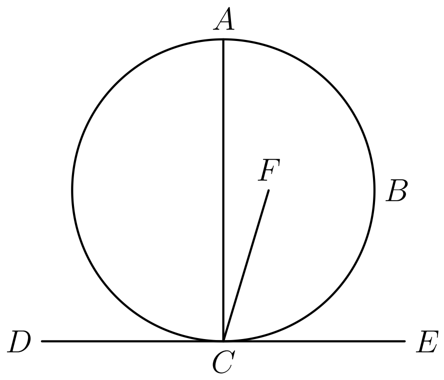
Q.E.D.

Proposition 19

If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the tangent, the centre of the circle will be on the straight line so drawn.

For let a straight line DE touch the circle ABC at the point C, and from C let CA be drawn at right angles to DE; I say that the centre of the circle is on AC.

For suppose it is not, but, if possible, let F be the centre, and let CF be joined.



Since a straight line DE touches the circle ABC, and FC has been joined from the point of contact, FC is perpendicular to DE [III. 18]; therefore the angle FCE is right.

But the angle ACE is also right; therefore the angle FCE is equal to the angle ACE, the less to the greater: which is impossible.

Therefore F is not the centre of the circle ABC.

Similarly we can prove that neither is any other point except a point on AC. Therefore, etc.

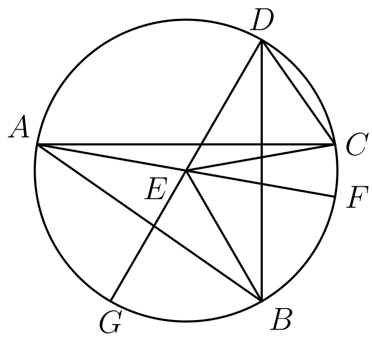
Q.E.D.

Proposition 20

In a circle the angle at the centre is double of the angle at the circumference, when the angles have the same circumference as base.

Let ABC be a circle, let the angle BEC be an angle at its centre, and the angle BAC an angle at the circumference, and let them have the same circumference BC as base; I say that the angle BEC is double of the angle BAC.

For let AE be joined and drawn through to F.



Then, since EA is equal to EB, the angle EAB is also equal to the angle EBA [I. 5]; therefore the angles EAB, EBA are double of the angle EAB.

But the angle BEF is equal to the angles EAB, EBA [I. 32]; therefore the angle BEF is also double of the angle EAB.

For the same reason the angle FEC is also double of the angle EAC.

Therefore the whole angle BEC is double of the whole angle BAC.

Again let another straight line be inflected, and let there be another angle BDC; let DE be joined and produced to G.

Similarly then we can prove that the angle GEC is double of the angle EDC, of which the angle GEB is double of the angle EDB; therefore the angle BEC which remains is double of the angle BDC. Therefore, etc.

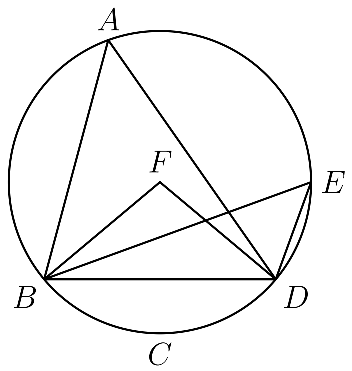
Q.E.D.

Proposition 21

In a circle the angles in the same segment are equal to one another.

Let $ABCD$ be a circle, and let the angles BAD, BED be angles in the same segment $BAED$; I say that the angles BAD, BED are equal to one another.

For let the centre of circle $ABCD$ be taken, and let it be F; let BF, FD be joined.



Now, since the angle BFD is at the centre, and the angle BAD at the circumference, and they have the same circumference BCD as base, therefore the angle BFD is double of the angle BAD [III. 20]

For the same reason the angle BFD is also double of the angle BED; therefore the angle BAD is equal to the angle BED.

Therefore, etc.

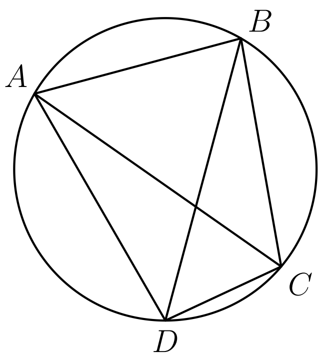
Q.E.D.

Proposition 22

The opposite angles of quadrilaterals in circles are equal to two right angles.

Let $ABCD$ be a circle, and let $ABCD$ be a quadrilateral in it; I say that the opposite angles are equal to two right angles.

Let AC, BD be joined.



Then, since in any triangle the three angles are equal to two right angles [I. 32], the three angles CAB, ABC, BCA of the triangle ABC are equal to two right angles.

But the angle CAB is equal to the angle BDC, for they are in the same segment $BADC$ [III. 21]; and the angle ACB is equal to the angle ADB, for they are in the same segment $ADCB$; therefore the whole angle ADC is equal to the angles BAC, ACB.

Let the angle ABC be added to each; therefore the angles ABC, BAC, ACB are equal to the angles ABC, ADC.

But the angles ABC, BAC, ACB are equal to two right angles; therefore the angles ABC, ADC are also equal to two right angles.

Similarly we can prove that the angles BAD, DCB are also equal to two right angles.

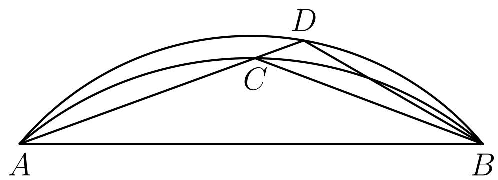
Therefore, etc.

Q.E.D.

Proposition 23

On the same straight line there cannot be constructed two similar and unequal segments of circles on the same side.

For, if possible, on the same straight line AB let two similar and unequal segments of circles ACB, ADB be constructed on the same side; let ACD be drawn through, and let CB, DB be joined.



Then, since the segment ACB is similar to the segment ADB, and similar segments of circles are those which admit equal angles [iii. Def. 11], the angle ACB is equal to the angle ADB, the exterior to the interior: which is impossible [i. 16]. Therefore, etc.

Q.E.D.

Proposition 24

Similar segments of circles on equal straight lines are equal to one another.

For let AEB, CFD be similar segments of circles on equal straight lines AB, CD; I say that the segment AEB is equal to the segment CFD.

For, if the segment AEB be applied to CFD, and if the point A be placed on C and the straight line AB on CD, the point B will also coincide with the point D, because AB is equal to CD; and, AB coinciding with CD, the segment AEB will also coincide with CFD.

For, if the straight line AB coincide with CD but the segment AEB do not coincide with CFD, it will either fall within it, or outside it; or it will fall awry, as CGD, and a circle cuts a circle at more points than two: which is impossible [III. 10].

Therefore, if the straight line AB be applied to CD, the segment AEB will not fail to coincide with CFF also; therefore it will coincide with it and will be equal to it.

Therefore, etc.

Q.E.D.

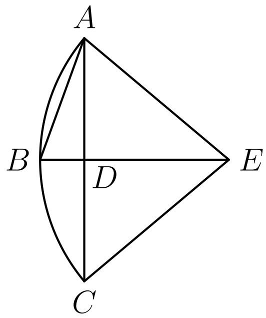
Proposition 25

Given a segment of a circle, to describe the complete circle of which it is a segment.

Let ABC be the given segment of a circle; thus it is required to describe the complete circle belonging to the segment ABC, that is, of which it is a segment.

For let AC be bisected at D, let DB be drawn from the point D at right angles to AC, and let AB be joined; the angle ABD is then greater than, equal to, or less than the angle BAD.

First let it be greater; and on the straight line BA, and at the point A on it, let the angle BAE be constructed equal to the angle ABD; let DB be drawn through to E, and let EC be joined.



Then, since the angle ABE is equal to the angle BAE, the straight line EB is also equal to EA [I. 6].

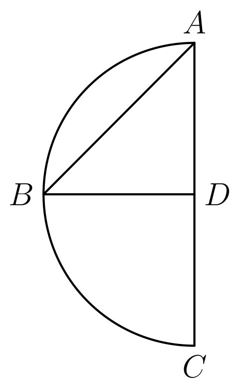
And, since AD is equal to DC, and DE is common, the two sides AD, DE are equal to the two sides CD, DE respectively; and the angle ADE is equal to the angle CD, for each is right; therefore the base AE is equal to the base CE; therefore the three straight lines AE, EB, EC are equal to one another.

Therefore the circle drawn with centre E and distance one of the straight line AE, EB, EC will also pass through the remaining points and will have been completed. [III. 9]

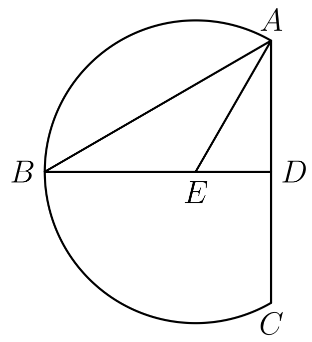
Therefore, given a segment of a circle, the complete circle has been described.

And it is manifest that the segment ABC is less than a semicircle, because the centre E happens to be outside it.

Similarly, even if the angle ABD be equal to the angle BAD, AD being equal to each of the two BD, DC, the three straight lines DA, DB, C will be equal to one another, D will be the centre of the completed circle, and ABC will clearly be a semicircle.



But, if the angle ABD be less than the angle BAD, and if we construct, on the straight line BA and at the point A on it, an angle equal to the angle ABD, the centre will fall on DB within the segment ABC, and the segment ABC will clearly be greater than a semicircle.



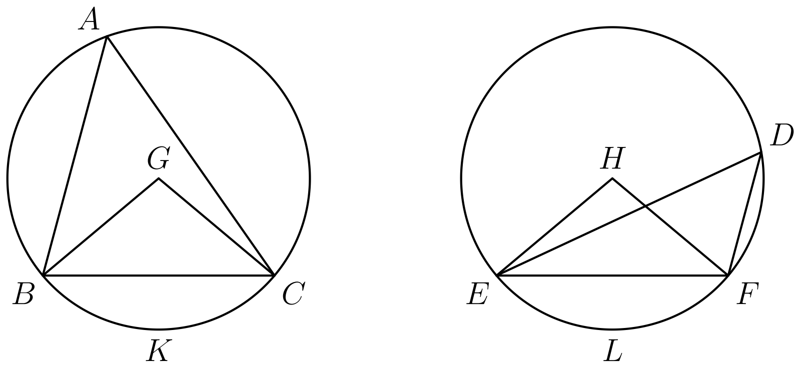
Therefore, given a segment of a circle, the complete circle has been described.

Q.E.F.

Proposition 26

In equal circles equal angles stand on equal circumferences, whether they stand at the centres or at the circumferences.

Let ABC, DEF be equal circles, and in them let there be equal angles, namely at the centres the angles BGC, EHF, and at the circumferences the angles BAC, EDF; I say that the circumference BKC is equal to the circumference ELF.



For let BC, EF be joined.

Now, since the circles ABC, DEF are equal, the radii are equal.

Thus the two straight lines BG, GC are equal to the two straight lines EH, HF; and the angle at G is equal to the angle at H; therefore the base BC is equal to the base EF [I. 4]. And, since the angle at A is equal to the angle at D, the segment BAC is similar to the segment EDF [III. Def. 11]; and there are upon equal straight lines.

But similar segments of circles on equal straight lines are equal to one another [III. 24]; therefore the segment BAC is equal to EDF.

But the whole circle ABC is also equal to the whole circle DEF; therefore the circumference BKC which remains is equal to the circumference ELF.

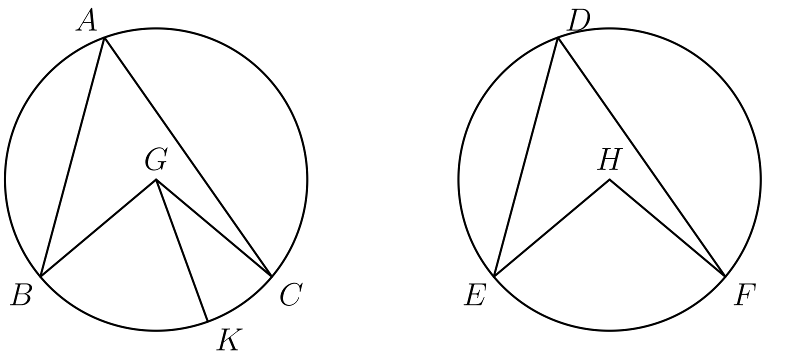
Therefore etc.

Q.E.D.

Proposition 27

In equal circles angles standing on equal circumferences are equal to one another, whether they stand at the centres or at the circumferences.

For in equal circles ABC, DEF, on equal circumferences BC, EF, let the angles BGC, EHF stand at the centres G, H, and the angles BAC, EDF at the circumferences; I say that the angle BGC is equal to the angle EHF, and the angle BAC is equal to the angle EDF.



For, if the angle BGC is unequal to the angle EHF, one of them is greater.

Let the angle BGC be greater: and on the straight line BG, and at the point G on it, let the angle BGK be constructed equal to the angle EHF.

Now equal angles stand on equal circumferences, when they are at the centres [III. 26]; therefore the circumference BK is equal to the circumference EF.

But EF is equal to BC; Therefore BK is also equal to BC, the less to the greater: which is impossible.

Therefore the angle BGC is not unequal to the angle EHF; therefore it is equal to it.

And the angle at A is half of the angle BGC, and the angle at D half of the angle EHF [III. 20]; therefore the angle at A is also equal to the angle at D.

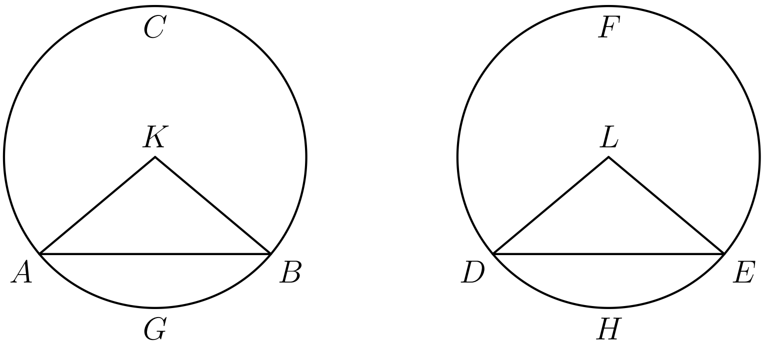
Therefore etc.

Q.E.D.

Proposition 28

In equal circles equal straight lines cut off equal circumferences, the greater equal to the greater and the less to the less.

Let ABC, DEF be equal circles, and in the circles let AB, DE be equal straight lines cutting off ACB, DFE as greater circumferences and AGB, DHE as lesser; I say that the greater circumference ACB is equal to the greater circumference DFE, and the less circumference AGB to DHE.



For let the centres K, L of the circles be taken, and let AK, KE, DL, LE be joined.

Now, since the circles are equal, the radii are also equal; therefore the two sides AK, KB are equal to the two sides DL, LE; and the base AB is equal to the base DE; therefore the angle AKB is equal to the angle DLE [I. 8].

But equal angles stand on equal circumferences, when they are at the centres [III. 26]; therefore the circumference AGB is equal to DHE.

And the whole circle ABC is also equal to the whole circle DEF; therefore the circumference ACB which remains is also equal to the circumference DFE which remains.

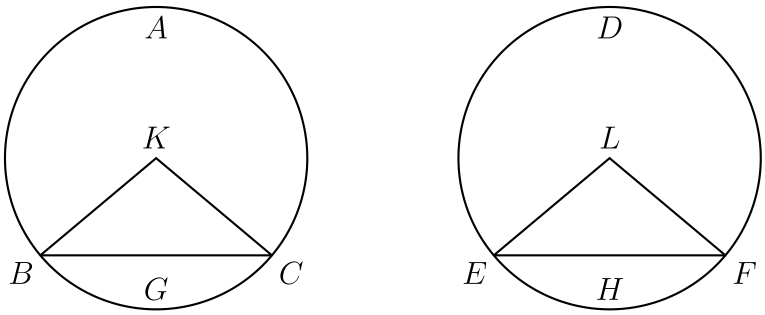
Therefore etc.

Q.E.D.

Proposition 29

In equal circles equal circumferences are subtended by equal straight lines.

Let ABC, DEF be equal circles, and in them let equal circumferences BGC, EHF be cut off; and let the straight lines BC, EF be joined; I say BC is equal to EF.



For let the centres of the circles be taken, and let them be K, L; let BK, KC, EL, LF be joined.

Now, since the circumference BGC is equal to the circumference EHF, the angle BKC is also equal to the angle ELF. III. 27

And, since the circles ABC, DEF are equal, the radii are also equal; therefore the two sides BK, KC are equal to the two sides EL, LF; and they contain equal angles; therefore the base BC is equal to the base EF [I. 4].

Therefore etc.

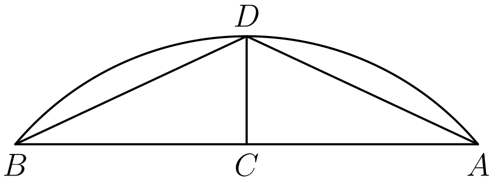
Q.E.D.

Proposition 30

To bisect a given circumference.

Let ADB be a given circumference; thus it is required to bisect the circumference ADB.

Let AB be joined and bisected at C; from the point C let CD be drawn at right angles to the straight line AB, and let AD, DB be joined.



Then, since AC is equal to CB, and CD is common, the two sides AC, CD are equal to the two sides BC, CD; and the angle ACD is equal to the angle BCD, for each is right; therefore the base AD is equal to the base DB [I. 4].

But equal straight lines cut off equal circumferences, the greater equal to the greater, and the less to the less [III. 28]; and each of the circumferences AD, DB is less than a semicircle; therefore the circumference AD is equal to the circumference DB.

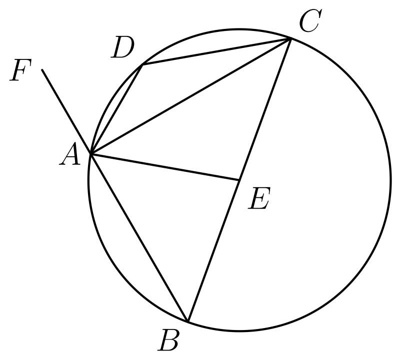
Therefore the given circumference has been bisected at the point D.

Q.E.F.

Proposition 31

In a circle the angle in the semicircle is right, that in a greater segment less than a right angle, and that in a less segment greater than a right angle; and further the angle of the greater segment is greater than a right angle, and the angle of the less segment is less than a right angle.

Let $ABCD$ be a circle, let BC be its diameter, and E its centre, and let BA, AC, AD, DC be joined; I say that the angle BAC in the semicircle is right, the angle in the segment ABC greater than the semicircle is less than a right angle, and the angle ADC in the segment ADC less than the semicircle is greater than a right angle.



Let AE be joined, and let BA be carried through to F.

Then, since BE is equal to EA, the angle ABE is aso equal to the angle BAE [I. 5]. Again, since CE is equal to EA, the angle ACE is also equal to the angle CAE [I. 5]. Therefore the whole angle BAC is equal to the two angles ABC, ACB. But the angle FAC exterior to the triangle ABC is also equal to the two angles ABC, ACB [I. 32]; therefore the angle BAC is also equal to the angle FAC; therefore each is right; therefore the angle BAC inthe semicircle BAC is right.

Next, since in the triangle ABC the two angles ABC, BAC are less than two right angles, and the angle BAC is a right angle, the angle ABC is less than a right angle; and its is the angle in the segment ABC greater than the semicircle.

Next, since $ABCD$ is a quadrilateral in a circle, and the opposite angles of quadrilaterals in circles are equal to two right angles [III, 22], while the angle ABC is less than a right angle, therefore the angle ADC which remains is greater than a right angle; and it is the angle in the segment ADC less than the semicircle.

I say further than the angle of the greater segment, namely that contained by the circumference ABC and the straight line AC, is greater than a right angle; and the angle of the less segment, namely that contained by the circumference ADC and the straight line AC, is less than a right angle.

This is at once manifest.

For, since the angle contained by the straight lines BA, AC is right, the angle contained by the circumference ABC and the straight line AC is greater than a right angle.

Again, since the angle contained by the straight lines AC, AF is right, the angle contained by the straight line CA and the circumference ADC is less than a right angle.

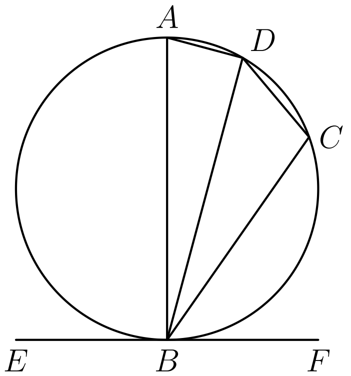
Therefore etc.

Q.E.D.

Proposition 32

If a straight line touch a circle, and from the point of contact there be drawn across, in the circle, a straight line cutting the circle, the angles which it makes with the tangent will be equal to the angles in the alternate segments of the circle.

For let a straight line EF touch the circle $ABCD$ at the point B, and from the point B let there be drawn across, in the circle $ABCD$, a straight line BD cutting it; I say that the angles which BD makes with the tangent EF will be equal to the angles in the alternate segments of the circle, that is, that the angle FBD is equal to the angle constructed in the segment BAD, and the angle EBD is equal to the angle constructed in the segment DCB.



For let BA be drawn from B at right angles to EF, let a point C be taken at random on the circumference BD, and let AD, DC, CB be joined.

Then, since a straight line EF touches the circle $ABCD$ at B, and BA has been drawn from the point of contact at right angles to the tangent, the centre of the circle $ABCD$ is on BA [iii. 19]. Therefore BA is a diameter of the circle $ABCD$; therefore the angle ADB, being an angle in a semicircle, is right. [iii. 31]. Therefore the remaining angles BAD, ABD, are equal to one right angle. [i. 32]. But the angle ABF is also right; therefore the angle ABF is equal to the angles BAD, ABD. Let the angle ABD be subtracted from each; therefore the angle DBF which remains is equal to the angle BAD in the alternate segment of the circle.

Next, since $ABCD$ is a quadrilateral in a circle, its opposite angles are equal to two right angles [iii. 22]. But the angles DBF, DBE are also equal to two right angles; therefore the angles DBF, DBE are equal to the angles BAD, BCD, of which the angle BAD was proved equal to the angle DBF; therefore the angle DBE which remains is equal to the angle DCB in the alternate segment DCB of the circle.

Therefore etc.

Q.E.D.

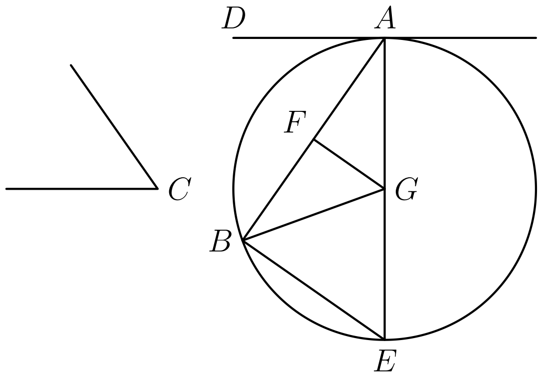
Proposition 33

On a given straight line to describe a segment of a circle admitting an angle equal to a given rectilinear angle.

Let AB be the given straight line, and the angle at C the given rectilineal angle; thus it is required to describe on the given straight line AB a segment of a circle admitting an angle equal to the angle at C.

The angle at C is then acute, or right, or obtuse.

First let it be acute, and, as in the first figure, on the straight line AB, and at the point A, let the angle BAD be constructed equal to the angle at C; therefore the angle BAD is also acute. Let AE be drawn at right angles to DA, let AB be bisected at F, let FG be drawn from the point F at right angles to AB, and let GB be joined.

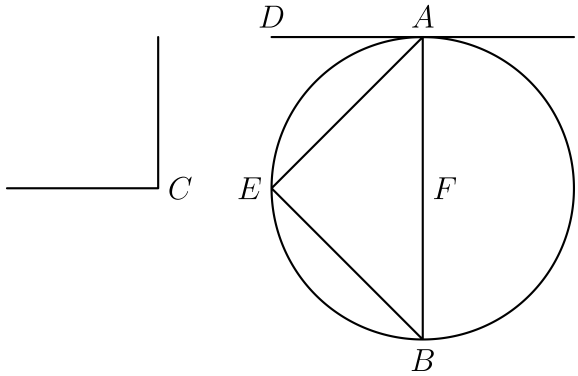


Then, since AF is equal to FB, and FG is common, the two sides AF, FG are equal to the two sides BF, FG; and the angle AFG is equal to the angle BFG; therefore the base AG is equal to the base BG [I. 4]. Therefore the circle described with centre G and distance GA will pass through B also. Let it be drawn, and let it be ABE; let EB be joined.

Now, since AD is drawn from A, the extremity of the diameter AE, at right angles to AE [III. 16, Por.]. Since then a straight line AD touches the circle ABE, and from the point of contact at A a straight line AB is drawn across in the circle ABE, the angle DAB is equal to the angle AEB in the alternate segment of the circle [III. 32]. But the angle DAB is equal to the angle at C; therefore the angle at C is also equal to the angle AEB.

Therefore on the given straight line AB the segment AEB of a circle has been described admitting the angle AEB equal to the given angle, the angle at C.

Next let the angle at C be right; and let it be again be required to describe on AB a segment of a circle admitting an angle equal to the right angle at C. Let the angle BAD be constructed equal to the right angle at C, as is the case in the second figure; Let AB be bisected at F, and with centre F and distance either FA or FB let the circle AEB be described.

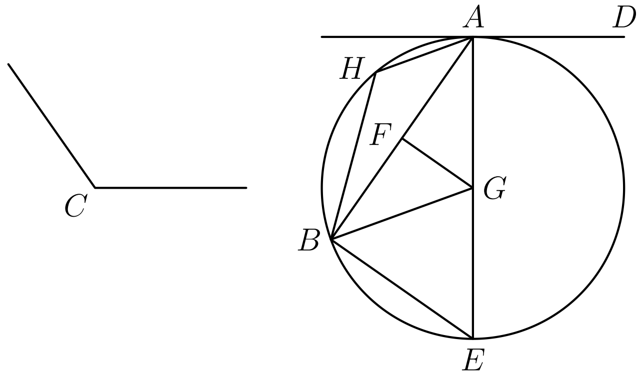


Therefore the straight line AD touches the circle ABE, because the angle at A is right [III. 16, Por]. And the angle BAD is equal to the angle in the segment AEB, for the latter too is itself a right angle, being an angle in a semicircle [III. 31]. But the angle BAD is also equal to the angle at C. Therefore the angle AEB is also equal to the angle at C.

Therefore again the segment AEB of a circle has been described on AB admitting an angle equal to the angle at C.

Next, let the angle at C be obtuse; and on the straight line AB, and at the point A, let the angle BAD be constructed equal to it, as in the case in the third figure; let AE be drawn at right angles to AD, let AB be again bisected at F, let FG be drawn at right angles to AB, and let GB be joined.

Then, since AF is again equal to FB; and FG is common, the two sides AF, FG are equal to the two sides BF, FG; and the angle AFG is equal to the angle BFG; therefore the base AG is equal to the base BG [I. 4]. Therefore the circle described with centre G and distance GA will pass through B also; let it so pass, as in AEB.



Now, since AD is drawn at right angles to the diameter AE from its extremity, AD touches the circle AEB [III. 16, Por.]. And AB has been drawn across from the point of contact at A; therefore the angle BAD is equal to the angle constructed in the alternate segment AHB of the circle [III. 32]. But the angle BAD is equal to the angle at C. Therefore the angle in the segment AHB is also equal to the angle at C.

Therefore on the given straight line AB, the segment AHB of a circle has been described admitting an angle equal to the angle at C.

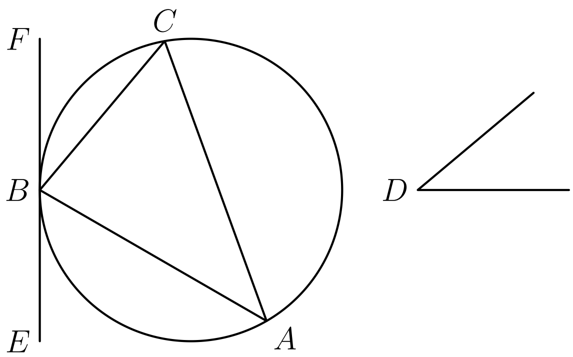
Q.E.F.

Proposition 34

From a given circle to cut off a segment admitting an angle equal to a given rectilineal angle.

Let ABC be the given circle, and the angle at D the given rectilineal angle; thus it is required to cut off from the circle ABC a segment admitting an angle equal to the given rectilineal angle, the angle at D.

Let EF be drawn touching ABC at the point B, and on the straight line FB, and at the point B on it, let the angle FBC be constructed equal to the angle at D [I. 23].



Then, since a straight line EF touches the circle ABC, and BC has been drawn across from the point of contact at B, the angle FBC is equal to the angle constructed in the alternate segment BAC [III. 32].

But the angle FBC is equal to the angle at D; therefore the angle in the segment BAC is equal to the angle at D.

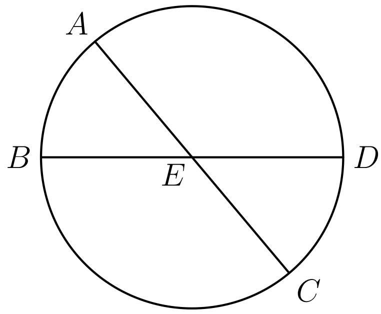
Therefore from the given circle ABC the segment ABC has been cut off admitting an angle equal to the given rectlineal angle, the angle at D.

Q.E.F.

Proposition 35

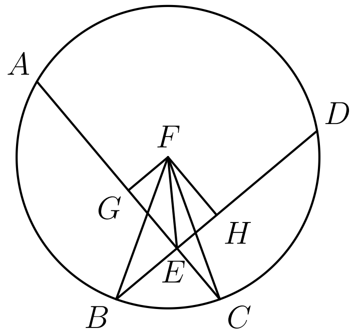
If in a circle two straight lines cut one another, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

For in the circle $ABCD$ let the two straight lines AC, BD cut one another at the point E; I say that the rectangle contained by AE, EC is equal to the rectangle contained by DE, EB.



If now AC, BD are through the centre, so that E is the centre of the circle $ABCD$, it is manifest that, AE, EC, DE, EB being equal, the rectangle contained by AE, EC is also equal to the rectangle contained by DE, EB.

Next let AC, DB not be through the centre; let the centre of $ABCD$ be taken, and let it be F; from F let FG, FH be drawn perpendicular to the straight lines AC, DB, and let FB, FC, FE be joined.



Then, since a straight line GF through the centre cuts a straight line AC not through the centre at right angles, it also bisects it [III. 3]; therefore AG is equal to GC. Since, then, the straight line AC has been cut into equal parts at G and into unequal parts at E, the rectangle contained by AE, EC together with the square on EG is equal to the square on GC [II. 5]. Let the square on GF be added; therefore the rectangle AE, EC together with the squares on GE, GF is equal to the squares on CG, GF.

But the square on FE is equal to the squares on EG, GF, and the square on FC is equal to the squares on CG, GF [I. 47]; therefore the rectangle AE, EC together with the square on FE is equal to the square on FC. And FC is equal to FB; therefore the rectangle AE, EC together with the square on EF is equal to the square on FB.

For the same reason, also, the rectangle DE, EB together with the square on FE is equal to the square on FB. But the rectangle AE, EC together with the square on FE was also proved equal to the square on FB; therefore the rectangle AE, EC together with the square on FE is equal to the rectangle DE, EB together with the square on FE. Let the square on FE be subtracted from each; therefore the rectangle contained by AE, EC which remains is equal to the rectangle contained by DE, EB.

Therefore etc.

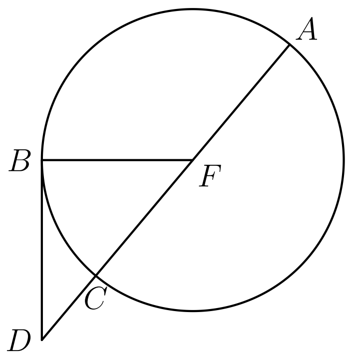
Q.E.D.

Proposition 36

If a point be taken outside a circle and from it there fall on the circle two straight lines, and if one of them cut the circle and the other touch it, the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference will be equal to the square on the tangent.

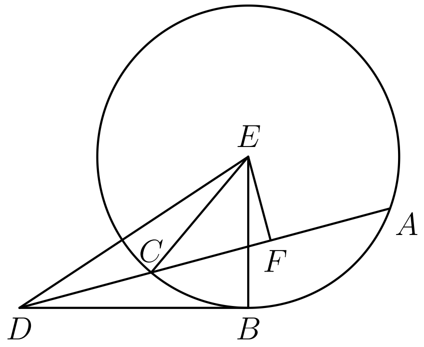
For let a point D be taken outside the circle ABC, and from D let the two straight lines DCA, DB fall on the circle ABC; let DCA cut the circle ABC and let BD touch it; I say that the rectangle contained by AD, DC is equal to the square on DB.

Then DCA is either through the centre or not through the centre.



First let it be through the centre, and let F be the centre of the circle ABC; let FB be joined; therefore the angle FBD is right [III. 18]. And, since AC has been bisected at F, and CD is added to it, the rectangle AD, DC together with the square on FC is equal to the square on FD [II. 6]. But FC is equal to FB; therefore the rectangle AD, DC together with the square on FB is equal to the square on FD. And the squares on FB, BD are equal to the square on FD [I. 47]; therefore the rectangle AC, DC together with the square on FB is equal to the squares on FB, BD. Let the square FB be subtracted from each; therefore the rectangle AD, DC which remains is equal to the square on the tangent DB.

Again, let DCA not be through the centre of the circle ABC; let the centre E be taken, and from E let EF be drawn perpendicular to AC; let EB, EC, ED be joined.



Then the angle EBD is right [III. 18]. And, since a straight line EF through the centre cuts a straight line AC not through the centre at right angles, it also bisects it [III. 3]; therefore AF is equal to FC.

Now, since the straight line AC has been bisected at the point F, and CD is added to it, the rectangle contained by AD, DC together with the square on FC is equal to the square on FD [II. 6]. Let the square on FE be added to each; therefore the rectangle AD, DC together with the squares on CF, FE is equal to the squares on FD, FE.

But the square on EC is equal to the squares on CF, FE, for the angle EFC is right [I. 47]; and the square on ED is equal to the squares on DF, FE; therefore the rectangle AD, DC together with the square on EC is equal to the square on ED. And EC is equal to EB; therefore the rectangle AD, DC together with the square on EB is equal to the square on ED. But the squares on EB, BD are equal to the square on ED, for the angle EBD is right [I. 47]; therefore the rectangle AD, DC together with the square on EB is equal to the squares on EB, BD. Let the square on EB be subtracted from each; therefore the rectangle AD, DC which remains is equal to the square on DB.

Therefore etc.

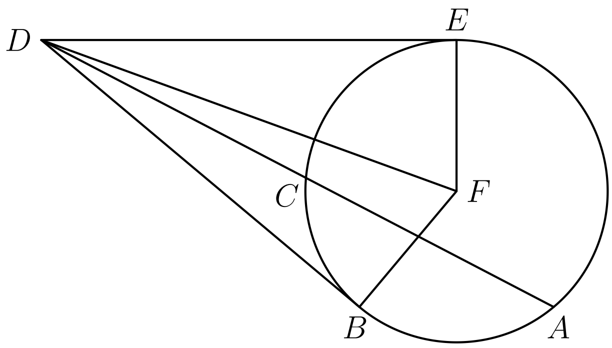
Q.E.D.

Proposition 37

If a point be taken outside a circle and from the point there fall on the circle two straight lines, if one of them cut the circle, and the other fall on it, and if further the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference be equal to the square on the straight line which falls on the circle, the straight line which fall on it will touch the circle.

For let a point D be taken outside the circle ABC, and from D let the two straight lines DCA, DB fall on the circle ACB; let DCA cut the circle and DB fall on it; and let the rectangle AD, DC be equal to the square on DB.

I say that DB touches the circle ABC.



For let DE be drawn touching ABC; let the centre of the circle ABC be taken, and let it be F; let FE, FB, FD be joined. Thus the angle FED is right [III. 18]. Now, since DE touches the circle ABC, and DCA cuts it, the rectangle AD, DC is equal to the square on DE [III. 36] But the rectangle AD, DC was also equal to the square on DB; therefore the square on DE is equal to the square on DB; therefore DE is equal to DB. And FE is equal to FB; therefore the two sides DE, EF are equal to the two sides DB, BF; and FD is the common base of the triangles; therefore the angle DEF is equal to the angle DBF [I. 8]. But the angle DEF is right; therefore the angle DBF is also right. And FB produced is a diameter; and the straight line drawn at right angles to the diameter of a circle, from its extremity, touches the circle [III. 16, Por]; therefore DB touches the circle.

Similarly this can be proved to be the case even if the centre be on AC.

Therefore etc.

Q.E.D.