MAU34804—Fixed Point Theorems and Economic Equilibria
School of Mathematics, Trinity College
Hilary Term 2024
Section 3: Simplices and Convexity

David R. Wilkins

3. Simplices and Convexity

3.1. Affine Independence

Definition

Points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ in some Euclidean space $\mathbb{R}^k$ are said to be *affinely independent* (or *geometrically independent*) if the only solution of the linear system

$$\begin{cases} \sum_{j=0}^{q} s_j \mathbf{v}_j &= \mathbf{0}, \\ \sum_{j=0}^{q} s_j &= 0 \end{cases}$$

is the trivial solution $s_0 = s_1 = \cdots = s_q = 0$.

Lemma 3.1

Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be points of Euclidean space $\mathbb{R}^k$ of dimension k. Then the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent if and only if the displacement vectors $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$ are linearly independent.

Proof

Suppose that the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent. Let $s_1, s_2, \ldots, s_q$ be real numbers which satisfy the equation

$$\sum_{j=1}^{q} s_j(\mathbf{v}_j - \mathbf{v}_0) = \mathbf{0}.$$

Then $\sum_{j=0}^{q} s_j \mathbf{v}_j = \mathbf{0}$ and $\sum_{j=0}^{q} s_j = 0$, where $s_0 = -\sum_{j=1}^{q} s_j$, and therefore

$$s_0 = s_1 = \cdots = s_q = 0.$$

It follows that the displacement vectors $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$ are linearly independent.

Conversely, suppose that these displacement vectors are linearly independent. Let $s_0, s_1, s_2, \ldots, s_q$ be real numbers which satisfy the equations $\sum_{j=0}^{q} s_j \mathbf{v}_j = \mathbf{0}$ and $\sum_{j=0}^{q} s_j = 0$. Then $s_0 = -\sum_{j=1}^{q} s_j$, and therefore

$$\mathbf{0} = \sum_{j=0}^{q} s_j \mathbf{v}_j = s_0 \mathbf{v}_0 + \sum_{j=1}^{q} s_j \mathbf{v}_j = \sum_{j=1}^{q} s_j (\mathbf{v}_j - \mathbf{v}_0).$$

It follows from the linear independence of the displacement vectors $\mathbf{v}_j - \mathbf{v}_0$ for $j = 1, 2, \ldots, q$ that

$$s_1 = s_2 = \cdots = s_q = 0.$$

But then $s_0 = 0$ also, because $s_0 = -\sum_{j=1}^{q} s_j$. It follows that the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent, as required. ∎

It follows from Lemma 3.1 that any set of affinely independent points in $\mathbb{R}^k$ has at most $k+1$ elements. Moreover if a set consists of affinely independent points in $\mathbb{R}^k$, then so does every subset of that set.

3.2. Simplices in Euclidean Spaces

Definition

A *q-simplex* in $\mathbb{R}^k$ is defined to be a set of the form

$$\left\{ \sum_{j=0}^{q} t_j \mathbf{v}_j : 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q \text{ and } \sum_{j=0}^{q} t_j = 1 \right\},$$

where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent points of $\mathbb{R}^k$. These points are referred to as the *vertices* of the simplex. The non-negative integer q is referred to as the *dimension* of the simplex. (Thus a simplex of dimension q has $q + 1$ vertices.)

Example

A 0-simplex in a Euclidean space $\mathbb{R}^k$ is a single point of that space.

Example

A 1-simplex in a Euclidean space $\mathbb{R}^k$ of dimension at least one is a line segment in that space. Indeed let λ be a 1-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}$ and $\mathbf{w}$. Then

$$\begin{aligned} \lambda &= \{s\,\mathbf{v} + t\,\mathbf{w} : 0 \leq s \leq 1,\ 0 \leq t \leq 1 \text{ and } s + t = 1\} \\ &= \{(1 - t)\mathbf{v} + t\,\mathbf{w} : 0 \leq t \leq 1\}, \end{aligned}$$

and thus λ is a line segment in $\mathbb{R}^k$ with endpoints $\mathbf{v}$ and $\mathbf{w}$.

Example

A 2-simplex in a Euclidean space $\mathbb{R}^k$ of dimension at least two is a triangle in that space. Indeed let τ be a 2-simplex in $\mathbb{R}^k$ with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Then

$$\tau = \{r\,\mathbf{u} + s\,\mathbf{v} + t\,\mathbf{w} : 0 \leq r, s, t \leq 1 \text{ and } r + s + t = 1\}.$$

Let $\mathbf{x} \in \tau$. Then there exist $r, s, t \in [0, 1]$ such that $\mathbf{x} = r\,\mathbf{u} + s\,\mathbf{v} + t\,\mathbf{w}$ and $r + s + t = 1$. If $r = 1$ then $\mathbf{x} = \mathbf{u}$. Suppose that $r < 1$. Then

$$\mathbf{x} = r\,\mathbf{u} + (1 - r)\Big((1 - p)\mathbf{v} + p\mathbf{w}\Big)$$

where $p = \dfrac{t}{1 - r}$. Moreover $0 \leq r < 1$ and $0 \leq p \leq 1$. Also the above formula determines a point of the 2-simplex τ for each pair of real numbers r and p satisfying $0 \leq r \leq 1$ and $0 \leq p \leq 1$.

Thus

$$\tau = \left\{ r\,\mathbf{u} + (1-r)\Big((1-p)\mathbf{v} + p\mathbf{w}\Big) : 0 \leq p, r \leq 1. \right\}.$$

Now the point $(1-p)\mathbf{v} + p\mathbf{w}$ traverses the line segment $\mathbf{v}\,\mathbf{w}$ from $\mathbf{v}$ to $\mathbf{w}$ as p increases from 0 to 1. It follows that τ is the set of points that lie on line segments with one endpoint at $\mathbf{u}$ and the other at some point of the line segment $\mathbf{v}\,\mathbf{w}$. This set of points is thus a triangle with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

Example

A 3-simplex in a Euclidean space $\mathbb{R}^k$ of dimension at least three is a tetrahedron on that space. Indeed let $\mathbf{x}$ be a point of a 3-simplex σ in $\mathbb{R}^3$ with vertices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$. Then there exist non-negative real numbers s, t, u and v such that

$$\mathbf{x} = s\,\mathbf{a} + t\,\mathbf{b} + u\,\mathbf{c} + v\,\mathbf{d},$$

and $s + t + u + v = 1$. These real numbers s, t, u and v all have values between 0 and 1, and moreover $0 \leq t \leq 1 - s$, $0 \leq u \leq 1 - s$ and $0 \leq v \leq 1 - s$. Suppose that $\mathbf{x} \neq \mathbf{a}$. Then $0 \leq s < 1$ and $\mathbf{x} = s\,\mathbf{a} + (1 - s)\mathbf{y}$, where

$$\mathbf{y} = \frac{t}{1-s}\,\mathbf{b} + \frac{u}{1-s}\,\mathbf{c} + \frac{v}{1-s}\,\mathbf{d}.$$

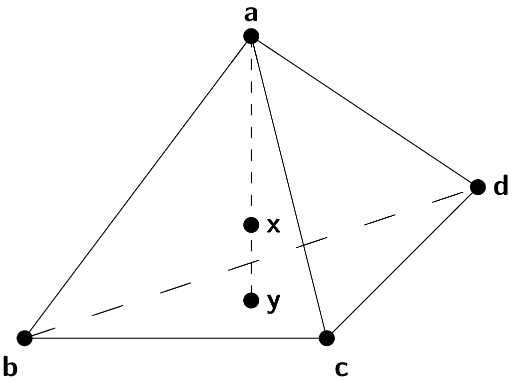
Moreover **y** is a point of the triangle **b c d**, because

$$0 \leq \frac{t}{1-s} \leq 1, \quad 0 \leq \frac{u}{1-s} \leq 1, \quad 0 \leq \frac{v}{1-s} \leq 1$$

and

$$\frac{t}{1-s} + \frac{u}{1-s} + \frac{v}{1-s} = 1.$$

It follows that the point **x** lies on a line segment with one endpoint at the vertex **a** of the 3-simplex and the other at some point **y** of the triangle **b c d**. Thus the 3-simplex σ has the form of a tetrahedron (i.e., it has the form of a pyramid on a triangular base **b c d** with apex **a**).


a
d
x
y
b
c

A simplex of dimension q in $\mathbb{R}^k$ determines a subset of $\mathbb{R}^k$ that is a translate of a q-dimensional vector subspace of $\mathbb{R}^k$. Indeed let the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of a q-dimensional simplex σ in $\mathbb{R}^k$. Then these points are affinely independent. It follows from Lemma 3.1 that the displacement vectors

$$\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$$

are linearly independent. These vectors therefore span a q-dimensional vector subspace V of $\mathbb{R}^k$. Now, given any point $\mathbf{x}$ of σ, there exist real numbers $t_0, t_1, \ldots, t_q$ such that $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$, $\sum_{j=0}^{q} t_j = 1$ and $\mathbf{x} = \sum_{j=0}^{q} t_j \mathbf{v}_j$. Then

$$\mathbf{x} = \left(\sum_{j=0}^{q} t_j \right) \mathbf{v}_0 + \sum_{j=1}^{q} t_j (\mathbf{v}_j - \mathbf{v}_0) = \mathbf{v}_0 + \sum_{j=1}^{q} t_j (\mathbf{v}_j - \mathbf{v}_0).$$

It follows that

$$\sigma = \left\{ \mathbf{v}_0 + \sum_{j=1}^{q} t_j(\mathbf{v}_j - \mathbf{v}_0) : 0 \leq t_j \leq 1 \text{ for } j = 1, 2, \ldots, q \right. \\ \left. \text{and } \sum_{j=1}^{q} t_j \leq 1 \right\},$$

and therefore $\sigma \subset \mathbf{v_0} + V$. Moreover the q-dimensional vector subspace V of $\mathbb{R}^k$ is the unique q-dimensional vector subspace of $\mathbb{R}^k$ that contains the displacement vectors between each pair of points belonging to the simplex σ.

3.3. Faces of Simplices

Definition

Let σ and τ be simplices in $\mathbb{R}^k$. We say that τ is a *face* of σ if the set of vertices of τ is a subset of the set of vertices of σ. A face of σ is said to be a *proper face* if it is not equal to σ itself. An r-dimensional face of σ is referred to as an *r-face* of σ. A 1-dimensional face of σ is referred to as an *edge* of σ.

Note that any simplex is a face of itself. Also the vertices and edges of any simplex are by definition faces of the simplex.

3.4. Barycentric Coordinates on a Simplex

Let σ be a q-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. If $\mathbf{x}$ is a point of σ then there exist real numbers $t_0, t_1, \ldots, t_q$ such that

$$\sum_{j=0}^{q} t_j \mathbf{v}_j = \mathbf{x}, \quad \sum_{j=0}^{q} t_j = 1 \text{ and } 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q.$$

Moreover $t_0, t_1, \ldots, t_q$ are uniquely determined: if $\sum_{j=0}^{q} s_j \mathbf{v}_j = \sum_{j=0}^{q} t_j \mathbf{v}_j$ and $\sum_{j=0}^{q} s_j = \sum_{j=0}^{q} t_j = 1$, then $\sum_{j=0}^{q} (t_j - s_j)\mathbf{v}_j = \mathbf{0}$ and $\sum_{j=0}^{q} (t_j - s_j) = 0$, and therefore $t_j - s_j = 0$ for $j = 0, 1, \ldots, q$, because the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent.

Definition

Let σ be a q-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$, and let $\mathbf{x} \in \sigma$. The *barycentric coordinates* of the point $\mathbf{x}$ (with respect to the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$) are the unique real numbers $t_0, t_1, \ldots, t_q$ for which

$$\sum_{j=0}^{q} t_j \mathbf{v}_j = \mathbf{x} \quad \text{and} \quad \sum_{j=0}^{q} t_j = 1.$$

The barycentric coordinates $t_0, t_1, \ldots, t_q$ of a point of a q-simplex satisfy the inequalities $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$.

Example

Consider the triangle τ in $\mathbb{R}^3$ with vertices at **i**, **j** and **k**, where

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0) \quad \text{and} \quad \mathbf{k} = (0, 0, 1).$$

Then

$$\tau = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x, y, z \leq 1 \text{ and } x + y + z = 1\}.$$

The barycentric coordinates on this triangle τ then coincide with the Cartesian coordinates x, y and z, because

$$(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\mathbf{k}$$

for all $(x, y, z) \in \tau$.

Example

Consider the triangle in $\mathbb{R}^2$ with vertices at $(0,0)$, $(1,0)$ and $(0,1)$. This triangle is the set

$$\{(x,y) \in \mathbb{R}^2 : x \geq 0, \;\; y \geq 0 \text{ and } x + y \leq 1.\}.$$

The barycentric coordinates of a point (x,y) of this triangle are t_0, t_1 and t_2, where

$$t_0 = 1 - x - y, \quad t_1 = x \quad \text{and} \quad t_2 = y.$$

Example
Consider the triangle in $\mathbb{R}^2$ with vertices at $(1,2)$, $(3,3)$ and $(4,5)$. Let t_0, t_1 and t_2 be the barycentric coordinates of a point (x,y) of this triangle. Then t_0, t_1, t_2 are non-negative real numbers, and $t_0 + t_1 + t_2 = 1$. Moreover

$$(x,y) = (1 - t_1 - t_2)(1,2) + t_1(3,3) + t_2(4,5),$$

and thus

$$x = 1 + 2t_1 + 3t_2 \quad \text{and} \quad y = 2 + t_1 + 3t_2.$$

It follows that

$$t_1 = x - y + 1 \quad \text{and} \quad t_2 = \tfrac{1}{3}(x - 1 - 2t_1) = \tfrac{2}{3}y - \tfrac{1}{3}x - 1,$$

and therefore

$$t_0 = 1 - t_1 - t_2 = \tfrac{1}{3}y - \tfrac{2}{3}x + 1.$$

In order to verify these formulae it suffices to note that $(t_0, t_1, t_2) = (1, 0, 0)$ when $(x, y) = (1, 2)$, $(t_0, t_1, t_2) = (0, 1, 0)$ when $(x, y) = (3, 3)$ and $(t_0, t_1, t_2) = (0, 0, 1)$ when $(x, y) = (4, 5)$.

3.5. The Interior of a Simplex

Definition

The *interior* of a simplex σ is defined to be the set consisting of all points of σ that do not belong to any proper face of σ.

Lemma 3.2

Let σ be a q-simplex in some Euclidean space with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. Let $\mathbf{x}$ be a point of σ, and let $t_0, t_1, \ldots, t_q$ be the barycentric coordinates of the point $\mathbf{x}$ with respect to $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$, so that $t_j \geq 0$ for $j = 0, 1, \ldots, q$, $\mathbf{x} = \sum_{j=0}^{q} t_j \mathbf{v}_j$, and $\sum_{j=0}^{q} t_j = 1$. Then the point $\mathbf{x}$ belongs to the interior of σ if and only if $t_j > 0$ for $j = 0, 1, \ldots, q$.

Proof

The point $\mathbf{x}$ belongs to the face of σ spanned by vertices

$$\mathbf{v}_{j_0}, \mathbf{v}_{j_1}, \ldots, \mathbf{v}_{j_r},$$

where $0 \leq j_0 < j_1 < \cdots < j_r \leq q$, if and only if $t_j = 0$ for all integers j between 0 and q that do not belong to the set $\{j_0, j_1, \ldots, j_r\}$. Thus the point $\mathbf{x}$ belongs to a proper face of the simplex σ if and only if at least one of the barycentric coordinates t_j of that point is equal to zero. The result follows. ∎

Example

A 0-simplex consists of a single vertex $\mathbf{v}$. The interior of that 0-simplex is the vertex $\mathbf{v}$ itself.

Example

A 1-simplex is a line segment. The interior of a line segment in a Euclidean space $\mathbb{R}^k$ with endpoints $\mathbf{v}$ and $\mathbf{w}$ is

$$\{(1-t)\,\mathbf{v} + t\,\mathbf{w} : 0 < t < 1\}.$$

Thus the interior of the line segment consists of all points of the line segement that are not endpoints of the line segment.

Example

A 2-simplex is a triangle. The interior of a triangle with vertices **u**, **v** and **w** is the set

$$\{r\,\mathbf{u} + s\,\mathbf{v} + t\,\mathbf{w} : 0 < r, s, t < 1 \text{ and } r + s + t = 1\}.$$

The interior of this triangle consists of all points of the triangle that do not lie on any edge of the triangle.

Remark

Let σ be a q-dimensional simplex in some Euclidean space $\mathbb{R}^k$, where $k \geq q$. If $k > q$ then the interior of the simplex (defined according to the definition given above) will not coincide with the topological interior determined by the usual topology on $\mathbb{R}^k$. Consider for example a triangle embedded in three-dimensional Euclidean space $\mathbb{R}^3$. The interior of the triangle (defined according to the definition given above) consists of all points of the triangle that do not lie on any edge of the triangle. But of course no three-dimensional ball of positive radius centred on any point of that triangle is wholly contained within the triangle. It follows that the topological interior of the triangle is the empty set when that triangle is considered as a subset of three-dimensional space $\mathbb{R}^3$.

Lemma 3.3

Any point of a simplex belongs to the interior of a unique face of that simplex.

Proof

let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of a simplex σ, and let $\mathbf{x} \in \sigma$. Then $\mathbf{x} = \sum_{j=0}^{q} t_j \mathbf{v}_j$, where $t_0, t_1, \ldots, t_q$ are the barycentric coordinates of the point $\mathbf{x}$. Moreover $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$ and $\sum_{j=0}^{q} t_j = 1$. The unique face of σ containing $\mathbf{x}$ in its interior is then the face spanned by those vertices $\mathbf{v}_j$ for which $t_j > 0$. ∎

3.6. Convex Subsets of Euclidean Spaces

Definition

A subset X of n-dimensional Euclidean space $\mathbb{R}^n$ is said to be *convex* if $(1-t)\mathbf{u} + t\mathbf{v} \in X$ for all points $\mathbf{u}$ and $\mathbf{v}$ of X and for all real numbers t satisfying $0 \leq t \leq 1$.

It follows from the above definition that a subset X of $\mathbb{R}^{\gtrdot}$ is a convex subset of $\mathbb{R}^m$ if and only if, given any two points of X, the line segment joining those two points is wholly contained in X.

Lemma 3.4

An simplex in a Euclidean space is a convex subset of that Euclidean space.

Proof

Let σ be a q-simplex in n-dimensional Euclidean space with vertices $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q$, and let $\mathbf{u}$ and $\mathbf{v}$ be points of σ. Then there exist non-negative real numbers $y_0, y_1, \ldots, y_q$ and $z_0, z_1, \ldots, z_q$, where $\sum_{i=0}^{q} y_i = 1$ and $\sum_{i=0}^{q} z_i = 1$, such that

$$\mathbf{u} = \sum_{i=0}^{q} y_i \mathbf{w}_i, \quad \mathbf{v} = \sum_{i=0}^{q} z_i \mathbf{w}_i.$$

Then

$$(1-t)\mathbf{u} + t\mathbf{v} = \sum_{i=0}^{q} ((1-t)y_i + tz_i)\mathbf{w}_i.$$

3. Simplices and Convexity (continued)

Moreover $(1-t)y_i + tz_i \geq 0$ for $i = 0, 1, \ldots, q$ and for all real numbers t satisfying $0 \leq t \leq 1$. Also

$$\sum_{i=0}^{q} ((1-t)y_i + tz_i) = (1-t)\sum_{i=0}^{q} y_i + t\sum_{i=0}^{q} z_i = 1.$$

It follows that $(1-t)\mathbf{u} + t\mathbf{v} \in \sigma$. Thus σ is a convex subset of $\mathbb{R}^n$. $\blacksquare$

Lemma 3.5

Let X be a convex subset of n-dimensional Euclidean space $\mathbb{R}^n$, and let σ be a simplex contained in $\mathbb{R}^n$. Suppose that the vertices of σ belong to X. Then $\sigma \subset X$.

Proof

We prove the result by induction on the dimension q of the simplex σ. The result is clearly true when $q = 0$, because in that case the simplex σ consists of a single point which is the unique vertex of the simplex.

Thus let σ be a q-dimensional simplex, and suppose that the result is true for all $(q-1)$-dimensional simplices whose vertices belong to the convex set X. Let $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q$ be the vertices of σ. Let $\mathbf{x}$ be a point of σ. Then there exist non-negative real numbers $t_0, t_1, \ldots, t_q$ satisfying $\sum_{i=0}^{q} t_i = 1$ such that $\mathbf{x} = \sum_{i=0}^{q} t_i \mathbf{w}_i$. If $t_0 = 1$ then $\mathbf{x} = \mathbf{w}_0$, and therefore $\mathbf{x} \in X$.

It remains to consider the case when $t_0 < 1$. In that case let $s_i = t_i/(1 - t_0)$ for $i = 1, 2, \ldots, q$, and let

$$\mathbf{v} = \sum_{i=1}^{q} s_i \mathbf{w}_i.$$

Now $s_i \geq 0$ for $i = 1, 2, \ldots, q$, and

$$\sum_{i=1}^{q} s_i = \frac{1}{1 - t_0} \sum_{i=1}^{q} t_i = \frac{1}{1 - t_0} \left(\sum_{i=0}^{q} t_i - t_0 \right) = 1,$$

It follows that $\mathbf{v}$ belongs to the proper face of σ that is spanned by the vertices $\mathbf{w}_1, \ldots, \mathbf{w}_q$. The induction hypothesis then ensures that $\mathbf{v} \in X$. But then

$$\mathbf{x} = t_0 \mathbf{w}_0 + (1 - t_0)\mathbf{v},$$

where $\mathbf{w}_0 \in X$ and $\mathbf{v} \in X$ and $0 \leq t_0 \leq 1$. It follows from the convexity of X that $\mathbf{x} \in X$, as required. $\blacksquare$

3. Simplices and Convexity (continued)

Let X be a convex set in n-dimensional Euclidean space $\mathbb{R}^{\ltimes}$. A point $\mathbf{x}$ of X is said to belong to the *topological interior* of X if there exists some $\delta > 0$ such that $B(\mathbf{x}, \delta) \subset X$, where

$$B(\mathbf{x}, \delta) = \{\mathbf{x}' \in \mathbb{R}^n : |\mathbf{x}' - \mathbf{x}| < \delta\}.$$

Lemma 3.6

Let X be a convex set in n-dimensional Euclidean space $\mathbb{R}^n$, and let $\mathbf{x} = (1-t)\mathbf{u} + t\mathbf{v}$ where $\mathbf{u}, \mathbf{v} \in X$ and $0 < t < 1$. Suppose that either $\mathbf{u}$ or $\mathbf{v}$ belongs to the topological interior of X. Then $\mathbf{x}$ belongs to the topological interior of X.

Proof

Suppose that $\mathbf{v}$ belongs to the topological interior of X. Then there exists $\delta > 0$ such that $B(\mathbf{v}, \delta) \subset X$, where

$$B(\mathbf{v}, \delta) = \{\mathbf{x}' \in \mathbb{R}^n : |\mathbf{x}' - \mathbf{v}| < \delta\}.$$

We claim that $B(\mathbf{x}, t\delta) \subset X$.

Let $\mathbf{x}' \in B(\mathbf{x}, t\delta)$, and let

$$\mathbf{z} = \frac{1}{t}(\mathbf{x}' - \mathbf{x}).$$

Then $\mathbf{v} + \mathbf{z} \in B(\mathbf{v}, \delta)$ and

$$\mathbf{x}' = (1 - t)\mathbf{u} + t(\mathbf{v} + \mathbf{z}),$$

and therefore $\mathbf{x}' \in X$. This proves the result when $\mathbf{v}$ belongs to the topological interior of X. The result when $\mathbf{u}$ belongs to the topological interior of X then follows on interchanging $\mathbf{u}$ and $\mathbf{v}$ and replacing t by $1 - t$. The result follows. ∎

Proposition 3.7

Let X be a closed bounded convex subset of n-dimensional Euclidean space $\mathbb{R}^n$ whose topological interior contains the origin, let S^{n-1} be the unit sphere in $\mathbb{R}^n$, defined such that

$$S^{n-1} = \{\mathbf{u} \in \mathbb{R}^n : |\mathbf{u}| = 1\},$$

and let $\lambda\colon S^{n-1} \to \mathbb{R}$ be the real-valued function on S^{n-1} defined such that

$$\lambda(\mathbf{u}) = \sup\{t \in \mathbb{R} : t\mathbf{u} \in X\}$$

for all $\mathbf{u} \in S^{n-1}$. Then the function $\lambda\colon S^{n-1} \to \mathbb{R}$ is continuous.

Proof

Let $\mathbf{u}_0 \in S^{n-1}$, let $t_0 = \lambda(\mathbf{u}_0)$, and let some positive real number ε be given, where $0 < \varepsilon < t_0$. It follows from Lemma 3.6 that $(t_0 - \varepsilon)\mathbf{u}_0$ belongs to the topological interior of X. It then follows from the continuity of the function sending $\mathbf{u} \in S^{n-1}$ to $(t_0 - \varepsilon)\mathbf{u}$ that there exists some positive real number δ_1 such that $(t_0 - \varepsilon)\mathbf{u} \in X$ for all $\mathbf{u} \in S^{n-1}$ satisfying $|\mathbf{u} - \mathbf{u}_0| < \delta_1$. Therefore $\lambda(\mathbf{u}) \geq t_0 - \varepsilon$ whenever $|\mathbf{u} - \mathbf{u}_0| < \delta_1$.

Next we note that $(t_0 + \varepsilon)\mathbf{u}_0 \notin X$. Now X is closed in $\mathbb{R}^n$, and therefore the complement $\mathbb{R}^n \setminus X$ of X in $\mathbb{R}^n$ is open. It follows that there exists an open ball of positive radius about the point $(t_0 + \varepsilon)\mathbf{u}_0$ that is wholly contained in the complement of X. It then follows from the continuity of the function sending $\mathbf{u} \in S^{n-1}$ to $(t_0 + \varepsilon)\mathbf{u}$ that there exists some positive real number δ_2 such that $(t_0 + \varepsilon)\mathbf{u} \notin X$ for all $\mathbf{u} \in S^{n-1}$ satisfying $|\mathbf{u} - \mathbf{u}_0| < \delta_2$. It then follows from the convexity of X that $t\mathbf{u} \notin X$ for all positive real numbers t satisfying $t \geq t_0 + \varepsilon$. Therefore $\lambda(\mathbf{u}) \leq t_0 + \varepsilon$ whenever $|\mathbf{u} - \mathbf{u}_0| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and

$$\lambda(\mathbf{u}_0) - \varepsilon \leq \lambda(\mathbf{u}) \leq \lambda(\mathbf{u}_0) + \varepsilon$$

for all $\mathbf{u} \in S^{n-1}$ satisfying $|\mathbf{u} - \mathbf{u}_0| < \delta$. The result follows. ∎

Proposition 3.8

Let X be a closed bounded convex subset of n-dimensional Euclidean space $\mathbb{R}^n$. Then there exists a continuous map $r\colon \mathbb{R}^n \to X$ such that $r(\mathbb{R}^n) = X$ and $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$.

Proof

We first prove the result in the special case in which the convex set X has non-empty topological interior. Without loss of generality, we may assume that the origin of $\mathbb{R}^n$ belongs to the topological interior of X. Let

$$S^{n-1} = \{\mathbf{u} \in \mathbb{R}^n : |\mathbf{u}| = 1\},$$

and let $\lambda\colon S^{n-1} \to \mathbb{R}$ be the real-valued function on S^{n-1} defined such that

$$\lambda(\mathbf{u}) = \sup\{t \in \mathbb{R} : t\mathbf{u} \in X\}$$

for all $\mathbf{u} \in S^{n-1}$. Then the function $\lambda\colon S^{n-1} \to \mathbb{R}$ is continuous (Proposition 3.7).

We may therefore define a function $r\colon \mathbb{R}^n \to X$ such that

$$r(\mathbf{x}) = \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \in X; \\ |\mathbf{x}|^{-1}\lambda(|\mathbf{x}|^{-1}\mathbf{x})\mathbf{x} & \text{if } \mathbf{x} \notin X. \end{cases}$$

Let $\mathbf{x} \in X$ and let $\mathbf{u} = |\mathbf{x}|^{-1}\mathbf{x}$. Then $\mathbf{x} = |\mathbf{x}|\,\mathbf{u}$, $|\mathbf{x}| \leq \lambda(\mathbf{u})$ and $\lambda(\mathbf{u})\mathbf{u} \in X$. It follows from Lemma 3.6 that if $|\mathbf{x}| < \lambda(\mathbf{u})$ then the point $\mathbf{x}$ belongs to the topological interior of $\mathbf{u}$. Thus if the point $\mathbf{x}$ of X belongs to the closure of the complement $\mathbb{R}^n \setminus X$ of X then it does not belong to the topological interior of X, and therefore $|\mathbf{x}| = \lambda(|\mathbf{x}|^{-1}\mathbf{x})$, and therefore

$$\mathbf{x} = |\mathbf{x}|^{-1}\lambda(|\mathbf{x}|^{-1}\mathbf{x})\mathbf{x}.$$

The function r defined above is therefore continuous on the closure of $\mathbb{R}^n \setminus X$. It is obviously continuous on X itself. It follows that $r\colon \mathbb{R}^n \to X$ is continuous. This proves the result in the case when the topological interior of the set X is non-empty.

We now extend the result to the case where the topological interior of X is empty. Now the number of points in an affinely independent list of points of $\mathbb{R}^n$ cannot exceed $n+1$. It follows that there exists an integer q not exceeding n such that the convex set X contains a $q+1$ affinely independent points but does not contain $q+2$ affinely independent points. Let $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q$ be affinely independent points of X. Let V be the q-dimensional subspace of $\mathbb{R}^n$ spanned by the vectors

$$\mathbf{w}_1 - \mathbf{w}_0, \mathbf{w}_2 - \mathbf{w}_0, \ldots, \mathbf{w}_q - \mathbf{w}_0.$$

Now if there were to exist a point $\mathbf{x}$ of X for which $\mathbf{x} - \mathbf{w}_0 \notin V$ then the points $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q, \mathbf{x}$ would be affinely independent. The definition of q ensures that this is not the case. Thus if

$$X_V = \{\mathbf{x} - \mathbf{w}_0 : \mathbf{x} \in X\}.$$

then $X_V \subset V$. Moreover X_V is a closed convex subset of V.

Now it follows from Lemma 3.5 that the convex set X_V contains the q-simplex with vertices

$$\mathbf{0},\ \mathbf{w}_1 - \mathbf{w}_0,\ \mathbf{w}_2 - \mathbf{w}_0, \ldots\ \mathbf{w}_q - \mathbf{w}_0.$$

This q-simplex has non-empty topological interior with respect to the vector space V. It follows that X_V has non-empty topological interior with respect to V. It therefore follows from the result already proved that there exists a continuous function $r_V \colon V \to X_V$ that satisfies $r_V(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X_V$. Basic linear algebra ensures the existence of a linear transformation $T \colon \mathbb{R}^n \to V$ satisfying $T(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in V$. Let

$$r(\mathbf{x}) = r_V(T(\mathbf{x} - \mathbf{w}_0)) + \mathbf{w}_0$$

for all $\mathbf{x} \in \mathbb{R}^n$. Then the function $r \colon \mathbb{R}^n \to X$ is continuous, and $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$, as required. ∎

3.7. Convex Sets and Supporting Hyperplanes

Lemma 3.9

Let m be a positive integer, let F be a non-empty closed set in $\mathbb{R}^m$, and let $\mathbf{b}$ be a vector in $\mathbb{R}^m$. Then there exists an element $\mathbf{g}$ of F such that $|\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$ for all $\mathbf{x} \in F$.

Proof

Let R be a positive real number chosen large enough to ensure that the set F_0 is non-empty, where

$$F_0 = F \cap \{\mathbf{x} \in \mathbb{R}^m : |\mathbf{x} - \mathbf{b}| \leq R\}.$$

Then F_0 is a closed bounded subset of $\mathbb{R}^m$. Let $f\colon F_0 \to \mathbb{R}$ be defined such that $f(\mathbf{x}) = |\mathbf{x} - \mathbf{b}|$ for all $\mathbf{x} \in F$. Then $f\colon F_0 \to \mathbb{R}$ is a continuous function on F_0.

Now it is a standard result of real analysis that any continuous real-valued function on a closed bounded subset of a finite-dimensional Euclidean space attains a minimum value at some point of that set. It follows that there exists an element $\mathbf{g}$ of F_0 such that

$$|\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$$

for all $\mathbf{x} \in F_0$. If $\mathbf{x} \in F \setminus F_0$ then

$$|\mathbf{x} - \mathbf{b}| \geq R \geq |\mathbf{g} - \mathbf{b}|.$$

It follows that

$$|\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$$

for all $\mathbf{x} \in F$, as required. ∎

3.8. A Separating Hyperplane Theorem

Theorem 3.10

Let m be a positive integer, let X be a non-empty closed convex set in $\mathbb{R}^m$, and let $\mathbf{b}$ be point of $\mathbb{R}^m$, where $\mathbf{b} \notin X$. Then there exists a linear functional $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ and a real number c such that $\varphi(\mathbf{x}) > c$ for all $\mathbf{x} \in X$ and $\varphi(\mathbf{b}) < c$.

Proof

It follows from Lemma 3.9 that there exists a point $\mathbf{g}$ of X such that $|\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$ for all $\mathbf{x} \in X$. Let $\mathbf{x} \in X$. Then $(1-t)\mathbf{g} + t\mathbf{x} \in X$ for all real numbers t satisfying $0 \leq t \leq 1$, because the set X is convex, and therefore

$$|(1-t)\mathbf{g} + t\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$$

for all real numbers t satisfying $0 \leq t \leq 1$.

Now

$$(1-t)\mathbf{g} + t\mathbf{x} - \mathbf{b} = \mathbf{g} - \mathbf{b} + t(\mathbf{x} - \mathbf{g}).$$

It follows by a straightforward calculation from the definition of the Euclidean norm that

$$\begin{aligned} |\mathbf{g} - \mathbf{b}|^2 &\leq |(1-t)\mathbf{g} + t\mathbf{x} - \mathbf{b}|^2 \\ &= |\mathbf{g} - \mathbf{b}|^2 + 2t(\mathbf{g} - \mathbf{b}) \,.\, (\mathbf{x} - \mathbf{g}) \\ &\qquad + t^2|\mathbf{x} - \mathbf{g}|^2 \end{aligned}$$

for all real numbers t satisfying $0 \leq t \leq 1$. In particular, this inequality holds for all sufficiently small positive values of t, and therefore

$$(\mathbf{g} - \mathbf{b}) \,.\, (\mathbf{x} - \mathbf{g}) \geq 0$$

for all $\mathbf{x} \in X$.

3. Simplices and Convexity (continued)

Let

$$\varphi(\mathbf{x}) = (\mathbf{g} - \mathbf{b}) \,.\, \mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^m$. Then $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ is a linear functional on $\mathbb{R}^m$, and $\varphi(\mathbf{x}) \geq \varphi(\mathbf{g})$ for all $\mathbf{x} \in X$. Moreover

$$\varphi(\mathbf{g}) - \varphi(\mathbf{b}) = |\mathbf{g} - \mathbf{b}|^2 > 0,$$

and therefore $\varphi(\mathbf{g}) > \varphi(\mathbf{b})$. It follows that $\varphi(\mathbf{x}) > c$ for all $\mathbf{x} \in X$, where $c = \frac{1}{2}\varphi(\mathbf{b}) + \frac{1}{2}\varphi(\mathbf{g})$, and that $\varphi(\mathbf{b}) < c$. The result follows. $\blacksquare$

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. A point $\mathbf{b}$ lies on the *boundary* of X if every open ball of positive radius centred on the point $\mathbf{b}$ intersects both the set X itself and the complement $\mathbb{R}^n \setminus X$ of X in $\mathbb{R}^n$.
If a subset X of $\mathbb{R}^n$ is open in $\mathbb{R}^n$ then every point belonging to the boundary of the set X belongs to the complement of X. If the subset X of $\mathbb{R}^m$ is closed in $\mathbb{R}^m$ then every point belonging to the boundary of the set X belongs to the set X itself.

Theorem 3.11 (Supporting Hyperplane Theorem)

Let m be a positive integer, let X be a non-empty closed convex set in $\mathbb{R}^m$, and let $\mathbf{b}$ be point of $\mathbb{R}^m$ that belongs to the boundary of the closed convex set X. Then there exists a linear functional $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ and a real number c such that $\varphi(\mathbf{x}) \geq c$ for all $\mathbf{x} \in X$ and $\varphi(\mathbf{b}) = c$.

Proof

We may assume without loss of generality, that $\mathbf{b} = (0, 0, \ldots, 0)$. We must then prove the existence of a linear functional $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ with the property that $\varphi(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in X$.

Now, because the $\mathbf{b}$ is located on the boundary of the set X, there exists an infinite sequence $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \ldots$ of points of the complement $\mathbb{R}^m \setminus X$ of the set X that converges to $\mathbf{b}$. It follows from basic linear algebra that, given any linear functional $\psi\colon \mathbb{R}^m \to \mathbb{R}$ on $\mathbb{R}^m$, there exists a vector $\mathbf{w}$ in $\mathbb{R}^m$ such that $\psi(\mathbf{x}) = \mathbf{w} \,.\, \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^m$. It therefore follows from Theorem 3.10, that there exists an infinite sequence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ of non-zero vectors in $\mathbb{R}^m$ such that $\mathbf{v}_j \,.\, \mathbf{b}_j < 0$ and $\mathbf{v}_j \,.\, \mathbf{x} \geq 0$ for all $\mathbf{x} \in X$. We may assume, without loss of generality, that $|\mathbf{v}_j| = 1$ for all positive integers j.

It follows from the Bolzano-Weierstrass Theorem (Theorem 1.2) that the infinite sequence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ has a convergent subsequence $\mathbf{v}_{k_1}, \mathbf{v}_{k_2}, \mathbf{v}_{k_3}, \ldots$, where

$$k_1 < k_2 < k_3 < \cdots .$$

Let $\mathbf{v} = \lim_{j \to +\infty} \mathbf{v}_{k_j}$. Then $|\mathbf{v}| = 1$. Let $\varphi(\mathbf{x}) = \mathbf{v} \,.\, \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^m$. Then

$$\varphi(\mathbf{x}) = \lim_{j \to +\infty} \mathbf{v}_{k_j} \,.\, \mathbf{x} \geq 0$$

for all $\mathbf{x} \in X$. The result follows. ∎