MAU34804—Fixed Point Theorems and Economic Equilibria
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Section 5: Fixed Point Theorems

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5. Fixed Point Theorems

5.1. Sperner's Lemma

Definition

Let K be a simplicial complex which is a subdivision of some n-dimensional simplex Δ. We define a *Sperner labelling* of the vertices of K to be a function, labelling each vertex of K with an integer between 0 and n, with the following properties:—

- for each $j \in \{0, 1, \ldots, n\}$, there is exactly one vertex of Δ labelled by j,
- if a vertex $\mathbf{v}$ of K belongs to some face of Δ, then some vertex of that face has the same label as $\mathbf{v}$.

Lemma 5.1 (Sperner's Lemma)

Let K be a simplicial complex which is a subdivision of an n-simplex Δ. Then, for any Sperner labelling of the vertices of K, the number of n-simplices of K whose vertices are labelled by $0, 1, \ldots, n$ is odd.

Proof

Given integers $i_0, i_1, \ldots, i_q$ between 0 and n, let $N(i_0, i_1, \ldots, i_q)$ denote the number of q-simplices of K whose vertices are labelled by $i_0, i_1, \ldots, i_q$ (where an integer occurring k times in the list labels exactly k vertices of the simplex). We must show that $N(0, 1, \ldots, n)$ is odd.

We prove the result by induction on the dimension n of the simplex Δ; it is clearly true when $n = 0$. Suppose that the result holds in dimensions less than n. For each simplex σ of K of dimension n, let $p(\sigma)$ denote the number of $(n-1)$-faces of σ labelled by $0, 1, \ldots, n-1$. If σ is labelled by $0, 1, \ldots, n$ then $p(\sigma) = 1$; if σ is labelled by $0, 1, \ldots, n-1, j$, where $j < n$, then $p(\sigma) = 2$; in all other cases $p(\sigma) = 0$. Therefore

$$\sum_{\substack{\sigma \in K \\ \dim \sigma = n}} p(\sigma) = N(0, 1, \ldots, n) + 2 \sum_{j=0}^{n-1} N(0, 1, \ldots, n-1, j).$$

Now the definition of Sperner labellings ensures that the only $(n-1)$-face of Δ containing simplices of K labelled by $0, 1, \ldots, n-1$ is that with vertices labelled by $0, 1, \ldots, n-1$.

Thus if M is the number of $(n-1)$-simplices of K labelled by $0, 1, \ldots, n-1$ that are contained in this face, then $N(0, 1, \ldots, n-1) - M$ is the number of $(n-1)$-simplices labelled by $0, 1, \ldots, n-1$ that intersect the interior of Δ. It follows that

$$\sum_{\substack{\sigma \in K \\ \dim \sigma = n}} p(\sigma) = M + 2\bigl(N(0, 1, \ldots, n-1) - M\bigr),$$

since any $(n-1)$-simplex of K that is contained in a proper face of Δ must be a face of exactly one n-simplex of K, and any $(n-1)$-simplex that intersects the interior of Δ must be a face of exactly two n-simplices of K. On combining these equalities, we see that $N(0, 1, \ldots, n) - M$ is an even integer. But the induction hypothesis ensures that Sperner's Lemma holds in dimension $n-1$, and thus M is odd. It follows that $N(0, 1, \ldots, n)$ is odd, as required. $\blacksquare$

5.2. Proof of Brouwer's Fixed Point Theorem

Proposition 5.2

Let Δ be an n-simplex with boundary $\partial\Delta$. Then there does not exist any continuous map $r\colon \Delta \to \partial\Delta$ with the property that $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial\Delta$.

Proof

Suppose that such a map $r\colon \Delta \to \partial\Delta$ were to exist. It would then follow from the Simplicial Approximation Theorem (Theorem 4.14) that there would exist a simplicial approximation $s\colon K \to L$ to the map r, where L is the simplicial complex consisting of all of the proper faces of Δ, and K is the jth barycentric subdivision, for some sufficiently large j, of the simplicial complex consisting of the simplex Δ together with all of its faces.

If $\mathbf{v}$ is a vertex of K belonging to some proper face Σ of Δ then $r(\mathbf{v}) = \mathbf{v}$, and hence $s(\mathbf{v})$ must be a vertex of Σ, since $s\colon K \to L$ is a simplicial approximation to $r\colon \Delta \to \partial\Delta$. In particular $s(\mathbf{v}) = \mathbf{v}$ for all vertices $\mathbf{v}$ of Δ. Thus if $\mathbf{v} \mapsto m(\mathbf{v})$ is a labelling of the vertices of Δ by the integers $0, 1, \ldots, n$, then $\mathbf{v} \mapsto m(s(\mathbf{v}))$ is a Sperner labelling of the vertices of K. Thus Sperner's Lemma (Lemma 5.1) guarantees the existence of at least one n-simplex σ of K labelled by $0, 1, \ldots, n$. But then $s(\sigma) = \Delta$, which is impossible, since Δ is not a simplex of L. We conclude therefore that there cannot exist any continuous map $r\colon \Delta \to \partial\Delta$ satisfying $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial\Delta$. $\blacksquare$

Theorem 5.3 (Brouwer Fixed Point Theorem)

(Brouwer Fixed Point Theorem) Let X be a subset of a Euclidean space that is homeomorphic to the closed n-dimensional ball E^n, where

$$E^n = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \leq 1\}.$$

Then any continuous function $f : X \to X$ mapping the set X into itself has at least one fixed point $\mathbf{x}^*$ for which $f(\mathbf{x}^*) = \mathbf{x}^*$.

Proof

The closed n-dimensional ball E^n is itself homeomorphic to an n-dimensional simplex Δ. Therefore there exists a homeomorphism $h\colon X \to \Delta$ mapping the set X onto the simplex Δ. Then the continuous map $f\colon X \to X$ determines a continuous map $g\colon \Delta \to \Delta$, where $g(h(\mathbf{x})) = h(f(\mathbf{x}))$ for all $\mathbf{x} \in X$. Suppose that it were the case that $f(\mathbf{x}) \neq \mathbf{x}$ for all $\mathbf{x} \in X$. Then $g(\mathbf{z}) \neq \mathbf{z}$ for all $\mathbf{z} \in \Delta$. There would then exist a well-defined continuous map $r\colon \Delta \to \partial\Delta$ mapping each point $\mathbf{z}$ of Δ to the unique point $r(\mathbf{z})$ of the boundary $\partial\Delta$ of Δ at which the half line starting at $g(\mathbf{z})$ and passing through $\mathbf{z}$ intersects $\partial\Delta$. Then $r\colon \Delta \to \partial\Delta$ would be continuous, and $r(\mathbf{z}) = \mathbf{z}$ for all $\mathbf{z} \in \partial\Delta$. However Proposition 5.2 guarantees that there does not exist any continuous map $r\colon \Delta \to \partial\Delta$ with these properties. Therefore the map f must have at least one fixed point, as required. ∎

5.3. The Kakutani Fixed Point Theorem

Theorem 5.4 (Kakutani's Fixed Point Theorem)

Let X be a non-empty, compact and convex subset of n-dimensional Euclidean space $\mathbb{R}^n$, and let $\Phi\colon X \rightrightarrows X$ be a correspondence mapping X into itself. Suppose that the graph of the correspondence Φ is closed and that $\Phi(\mathbf{x})$ is non-empty and convex for all $\mathbf{x} \in X$. Then there exists a point $\mathbf{x}^*$ of X that satisfies $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$.

Proof

There exists a continuous map $r\colon \mathbb{R}^n \to X$ from $\mathbb{R}^n$ to X with the property that $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$. (see Proposition 3.8). Let Δ be an n-dimensional simplex chosen such that $X \subset \Delta$, and let $\Psi(\mathbf{x}) = \Phi(r(\mathbf{x}))$ for all $\mathbf{x} \in \Delta$. If $\mathbf{x}^* \in \Delta$ satisfies $\mathbf{x}^* \in \Psi(\mathbf{x}^*)$ then $\mathbf{x}^* \in X$ and $r(\mathbf{x}^*) = \mathbf{x}^*$, and therefore $\mathbf{x} \in \Phi(\mathbf{x}^*)$. It follows that the result in the general case follows from that in the special case in which the closed bounded convex subset X of $\mathbb{R}^n$ is an n-dimensional simplex.

Thus let Δ be an n-dimensional simplex contained in $\mathbb{R}^n$, and let $\Phi\colon \Delta \rightrightarrows \Delta$ be a correspondence with closed graph, where $\Phi(\mathbf{x})$ is a non-empty closed convex subset of Δ for all $\mathbf{x} \in X$. We must prove that there exists some point $\mathbf{x}^*$ of Δ with the property that $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$.

Let K be the simplicial complex consisting of the n-simplex Δ together with all its faces, and let $K^{(j)}$ be the jth barycentric subdivision of K for all positive integers j. Then $|K^{(j)}| = \Delta$ for all positive integers j. Now $\Phi(\mathbf{v})$ is non-empty for all vertices $\mathbf{v}$ of $K^{(j)}$. Now any function mapping the vertices of a simplicial complex into a Euclidean space extends uniquely to a piecewise linear map defined over the polyhedron of that simplicial complex (Proposition 4.8). Therefore there exists a sequence $f_1, f_2, f_3, \ldots$ of continuous functions mapping the simplex Δ into itself such that, for each positive integer j, the continuous map $f_j \colon \Delta \to \Delta$ is piecewise linear on the simplices of $K^{(j)}$ and satisfies $f_j(\mathbf{v}) \in \Phi(\mathbf{v})$ for all vertices $\mathbf{v}$ of $K^{(j)}$.

Now it follows from the Brouwer Fixed Point Theorem (Theorem 5.3) that, for each positive integer j, there exists $\mathbf{z}_j \in \Delta$ for which $f_j(\mathbf{z}_j) = \mathbf{z}_j$. For each positive integer j, there exist vertices

$$\mathbf{v}_{0,j}, \mathbf{v}_{1,j}, \ldots, \mathbf{v}_{n,j}$$

of $K^{(j)}$ spanning a simplex of K and non-negative real numbers $t_{0,j}, t_{1,j}, \ldots, t_{n,j}$ satisfying $\sum_{i=0}^{n} t_{i,j} = 1$ such that

$$\mathbf{z}_j = \sum_{i=0}^{n} t_{i,j} \mathbf{v}_{i,j}$$

for all positive integers j. Let $\mathbf{y}_{i,j} = f_j(\mathbf{v}_{i,j})$ for $i = 0, 1, \ldots, n$ and for all positive integers j. Then $\mathbf{y}_{i,j} \in \Phi(\mathbf{v}_{i,j})$ for $i = 0, 1, \ldots, n$ and for all positive integers j.

The function f_j is piecewise linear on the simplices of $K^{(j)}$. It follows that

$$\begin{aligned}\sum_{i=0}^{n} t_{i,j}\mathbf{v}_{i,j} &= \mathbf{z}_j = f_j(\mathbf{z}_j) = f_j\left(\sum_{i=0}^{n} t_{i,j}\mathbf{v}_{i,j}\right) \\ &= \sum_{i=0}^{n} t_{i,j} f_j(\mathbf{v}_{i,j}) = \sum_{i=0}^{n} t_{i,j}\mathbf{y}_{i,j}\end{aligned}$$

for all positive integers j. Also $|\mathbf{v}_{i,j} - \mathbf{v}_{0,j}| \leq \mu(K^{(j)})$ for $i = 0, 1, \ldots, n$ and for all positive integers j, where $\mu(K^{(j)})$ denotes the mesh of the simplicial complex $K^{(j)}$ (i.e., the length of the longest side of that simplicial complex). Moreover $\mu(K^{(j)}) \to 0$ as $j \to +\infty$ (see Lemma 4.6). It follows that

$$\lim_{j\to+\infty} |\mathbf{v}_{i,j} - \mathbf{v}_{0,j}| = 0.$$

Now the multidimensional Bolzano-Weierstrass Theorem (Theorem 1.2) ensures the existence of points

$$\mathbf{x}^*, \mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_n$$

of the simplex Δ, non-negative real numbers $t_0, t_1, \ldots, t_n$ and an infinite sequence $m_1, m_2, m_3, \ldots$ of positive integers, where

$$m_1 < m_2 < m_3 < \cdots ,$$

such that

$$\begin{aligned} \mathbf{x}^* &= \lim_{j \to +\infty} \mathbf{v}_{0,m_j}, \\ \mathbf{y}_i &= \lim_{j \to +\infty} \mathbf{y}_{i,m_j} \quad (0 \leq i \leq n), \\ t_i &= \lim_{j \to +\infty} t_{i,m_j} \quad (0 \leq i \leq n). \end{aligned}$$

Now

$$|\mathbf{v}_{i,m_j} - \mathbf{x}^*| \leq |\mathbf{v}_{i,m_j} - \mathbf{v}_{0,m_j}| + |\mathbf{v}_{0,m_j} - \mathbf{x}^*|$$

for $i = 0, 1, \ldots, n$ and for all positive integers j. Moreover $\lim_{j \to +\infty} |\mathbf{v}_{i,m_j} - \mathbf{v}_{0,m_j}| = 0$ and $\lim_{j \to +\infty} |\mathbf{v}_{0,m_j} - \mathbf{x}^*| = 0$. It follows that $\lim_{j \to +\infty} \mathbf{v}_{i,m_j} = \mathbf{x}^*$ for $i = 0, 1, \ldots, n$. Also

$$\sum_{i=0}^{n} t_i = \lim_{j \to +\infty} \left(\sum_{i=0}^{n} t_{i,m_j} \right) = 1.$$

It follows that

$$\begin{aligned} \lim_{j \to +\infty} \left(\sum_{i=0}^{n} t_{i,m_j} \mathbf{v}_{i,m_j} \right) &= \sum_{i=0}^{n} \left(\lim_{j \to +\infty} t_{i,m_j} \right) \left(\lim_{j \to +\infty} \mathbf{v}_{i,m_j} \right) \\ &= \sum_{i=0}^{n} t_i \mathbf{x}^* = \mathbf{x}^*. \end{aligned}$$

But we have also shown that $\sum_{i=0}^{n} t_{i,j}\mathbf{y}_{i,j} = \sum_{i=0}^{n} t_{i,j}\mathbf{v}_{i,j}$ for all positive integers j. It follows that

$$\sum_{i=0}^{n} t_i \mathbf{y}_i = \lim_{j \to +\infty} \left(\sum_{i=0}^{n} t_{i,m_j} \mathbf{y}_{i,m_j} \right) = \lim_{j \to +\infty} \left(\sum_{i=0}^{n} t_{i,m_j} \mathbf{v}_{i,m_j} \right) = \mathbf{x}^*.$$

Next we show that $\mathbf{y}_i \in \Phi(\mathbf{x}^*)$ for $i = 0, 1, \ldots, n$. Now

$$(\mathbf{v}_{i,m_j}, \mathbf{y}_{i,m_j}) \in \mathrm{Graph}(\Phi)$$

for all positive integers j, and the graph $\mathrm{Graph}(\Phi)$ of the correspondence Φ is closed. It follows that

$$(\mathbf{x}^*, \mathbf{y}_i) = \lim_{j \to +\infty} (\mathbf{v}_{i,m_j}, \mathbf{y}_{i,m_j}) \in \mathrm{Graph}(\Phi)$$

and thus $\mathbf{y}_i \in \Phi(\mathbf{x}^*)$ for $i = 0, 1, \ldots, m$ (see Proposition 2.6).

It follows from the convexity of $\Phi(\mathbf{x}^*)$ that

$$\sum_{i=0}^{n} t_i \mathbf{y}_i \in \Phi(\mathbf{x}^*).$$

(see Lemma 3.5). But $\sum_{i=0}^{n} t_i \mathbf{y}_i = \mathbf{x}^*$. It follows that $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$, as required. ∎