MAU34804—Fixed Point Theorems and Economic Equilibria
School of Mathematics, Trinity College
Hilary Term 2024
Section 1: Review of Basic Results of Analysis in Euclidean Spaces

David R. Wilkins

1. Review of Basic Results of Analysis in Euclidean Spaces

1.1. Basic Properties of Vectors and Norms

We denote by $\mathbb{R}^n$ the set consisting of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers. The set $\mathbb{R}^n$ represents n-dimensional *Euclidean space* (with respect to the standard Cartesian coordinate system). Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$, where

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \quad \mathbf{y} = (y_1, y_2, \ldots, y_n),$$

and let λ be a real number. We define

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n), \\
\mathbf{x} - \mathbf{y} &= (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n), \\
\lambda \mathbf{x} &= (\lambda x_1, \lambda x_2, \ldots, \lambda x_n), \\
\mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n, \\
|\mathbf{x}| &= \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.
\end{aligned}$$

The quantity $\mathbf{x} \cdot \mathbf{y}$ is the *scalar product* (or *inner product*) of $\mathbf{x}$ and $\mathbf{y}$, and the quantity $|\mathbf{x}|$ is the *Euclidean norm* of $\mathbf{x}$. Note that $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$. The *Euclidean distance* between two points $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^n$ is defined to be the Euclidean norm $|\mathbf{y} - \mathbf{x}|$ of the vector $\mathbf{y} - \mathbf{x}$.

Let $\mathbf{x}$ and $\mathbf{y}$ be elements in $\mathbb{R}^n$, Let $p(t) = |t\mathbf{x} + \mathbf{y}|^2$ for all real numbers t. Then

$$\begin{aligned} p(t) &= (t\mathbf{x} + \mathbf{y}) \,.\, (t\mathbf{x} + \mathbf{y}) \\ &= t^2|\mathbf{x}|^2 + 2t\mathbf{x} \,.\, \mathbf{y} + |\mathbf{y}|^2 \end{aligned}$$

for all real numbers t. But $p(t) \geq 0$ for all real numbers t. It follows that $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$. This inquality is known as *Schwarz's Inequality*.

Moreover, given any elements **x** and **y** of $\mathbf{R}^n$,

$$\begin{aligned} |\mathbf{x}+\mathbf{y}|^2 &= (\mathbf{x}+\mathbf{y}).(\mathbf{x}+\mathbf{y}) = |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\mathbf{x}\cdot\mathbf{y} \\ &\leq |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2|\mathbf{x}||\mathbf{y}| = (|\mathbf{x}| + |\mathbf{y}|)^2. \end{aligned}$$

It follows that $|\mathbf{x}+\mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$. It follows from this inequality that

$$|\mathbf{x}-\mathbf{z}| \leq |\mathbf{x}-\mathbf{y}| + |\mathbf{y}-\mathbf{z}|$$

for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$. This identity is known as the *Triangle Inequality*. It expresses the geometric result that the length of any side of a triangle in a Euclidean space of any dimension is the sum of the lengths of the other two sides of that triangle.

Definition

A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to *converge* to a point $\mathbf{p}$ if and only if the following criterion is satisfied:—

> *given any real number ε satisfying $\varepsilon > 0$ there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$.*

We refer to $\mathbf{p}$ as the *limit* $\lim\limits_{j \to +\infty} \mathbf{x}_j$ of the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$.

Lemma 1.1

Let $\mathbf{p}$ *be a point of* $\mathbb{R}^n$, *where* $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. *Then a sequence* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ *of points in* $\mathbb{R}^n$ *converges to* $\mathbf{p}$ *if and only if the* i*th components of the elements of this sequence converge to* p_i *for* $i = 1, 2, \ldots, n$.

A proof of Lemma 1.1 is to be found in Appendix A.

Definition

Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be an infinite sequence of points in n-dimensional Euclidean space $\mathbb{R}^n$. A *subsequence* of this infinite sequence is a sequence of the form $\mathbf{x}_{j_1}, \mathbf{x}_{j_2}, \mathbf{x}_{j_3}, \ldots$ where $j_1, j_2, j_3, \ldots$ is an infinite sequence of positive integers with

$$j_1 < j_2 < j_3 < \cdots .$$

Theorem 1.2 (Multidimensional Bolzano-Weierstrass Theorem)

Every bounded sequence of points in a Euclidean space has a convergent subsequence.

A proof of Theorem 1.2 is to be found in Appendix A.

Definition

Let X be a subset of $\mathbb{R}^n$. Given a point $\mathbf{p}$ of X and a non-negative real number r, the *open ball* $B_X(\mathbf{p}, r)$ in X of *radius* r about $\mathbf{p}$ is defined to be the subset of X defined so that

$$B_X(\mathbf{p}, r) = \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < r\}.$$

(Thus $B_X(\mathbf{p}, r)$ is the set consisting of all points of X that lie within a sphere of radius r centred on the point $\mathbf{p}$.)

Definition

Let X be a subset of $\mathbb{R}^n$. A subset V of X is said to be *open* in X if, given any point $\mathbf{p}$ of V, there exists some strictly positive real number δ such that $B_X(\mathbf{p}, \delta) \subset V$, where $B_X(\mathbf{p}, \delta)$ is the open ball in X of radius δ about on the point $\mathbf{p}$. The empty set $\emptyset$ is also defined to be an open set in X.

Lemma 1.3

Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any positive real number r, the open ball $B_X(\mathbf{p}, r)$ in X of radius r about $\mathbf{p}$ is open in X.

A proof of Lemma 1.3 is to be found in Appendix A.

Proposition 1.4

Let X be a subset of $\mathbb{R}^n$. The collection of open sets in X has the following properties:—

(i) *the empty set $\emptyset$ and the whole set X are both open in X;*

(ii) *the union of any collection of open sets in X is itself open in X;*

(iii) *the intersection of any finite collection of open sets in X is itself open in X.*

A proof of Proposition 1.4 is to be found in Appendix A.

Proposition 1.5

Let X be a subset of $\mathbb{R}^n$, and let U be a subset of X. Then U is open in X if and only if there exists some open set V in $\mathbb{R}^n$ for which $U = V \cap X$.

A proof of Proposition 1.5 is to be found in Appendix A.

Lemma 1.6

A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to a point $\mathbf{p}$ if and only if, given any open set U which contains $\mathbf{p}$, there exists some positive integer N such that $\mathbf{x}_j \in U$ for all j satisfying $j \geq N$.

A proof of Lemma 1.6 is to be found in Appendix A.

Definition

Let X be a subset of $\mathbb{R}^n$. A subset F of X is said to be *closed* in X if and only if its complement $X \setminus F$ in X is open in X. (Recall that $X \setminus F = \{\mathbf{x} \in X : \mathbf{x} \notin F\}$.)

Proposition 1.7

Let X be a subset of $\mathbb{R}^n$. The collection of closed sets in X has the following properties:—

(i) *the empty set $\emptyset$ and the whole set X are both closed in X;*

(ii) *the intersection of any collection of closed sets in X is itself closed in X;*

(iii) *the union of any finite collection of closed sets in X is itself closed in X.*

A proof of Proposition 1.7 is to be found in Appendix A.

Lemma 1.8

Let X be a subset of $\mathbb{R}^n$, and let F be a subset of X which is closed in X. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of F which converges to a point $\mathbf{p}$ of X. Then $\mathbf{p} \in F$.

A proof of Lemma 1.8 is to be found in Appendix A.

Definition

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. A function $f: X \to Y$ from X to Y is said to be *continuous* at a point $\mathbf{p}$ of X if and only if the following criterion is satisfied:—

> *given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$.*

The function $f: X \to Y$ is said to be continuous on X if and only if it is continuous at every point $\mathbf{p}$ of X.

Lemma 1.9

Let X, Y and Z be subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively, and let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions satisfying $f(X) \subset Y$. Suppose that f is continuous at some point $\mathbf{p}$ of X and that g is continuous at $f(\mathbf{p})$. Then the composition function $g \circ f\colon X \to Z$ is continuous at $\mathbf{p}$.

A proof of Lemma 1.9 is to be found in Appendix A.

Lemma 1.10

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f\colon X \to Y$ be a continuous function from X to Y. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of X which converges to some point $\mathbf{p}$ of X. Then the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$.

A proof of Lemma 1.10 is to be found in Appendix A.

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f\colon X \to Y$ be a function from X to Y. Then

$$f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$$

for all $\mathbf{x} \in X$, where $f_1, f_2, \ldots, f_n$ are functions from X to $\mathbb{R}$, referred to as the *components* of the function f.

Proposition 1.11

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $\mathbf{p} \in X$. A function $f\colon X \to Y$ is continuous at the point $\mathbf{p}$ if and only if its components are all continuous at $\mathbf{p}$.

A proof of Proposition 1.11 is to be found in Appendix A.

Proposition 1.12

Let X be a subset of $\mathbb{R}^n$, and let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be continuous functions from X to $\mathbb{R}$. Then the functions $f + g$, $f - g$ and $f \cdot g$ are continuous. If in addition $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ then the quotient function f/g is continuous.

A proof of Proposition 1.12 is to be found in Appendix A.

Lemma 1.13

Let X be a subset of $\mathbb{R}^m$, let $f: X \to \mathbb{R}^n$ be a continuous function mapping X into $\mathbb{R}^n$, and let $|f|: X \to \mathbb{R}$ be defined such that $|f|(\mathbf{x}) = |f(\mathbf{x})|$ for all $\mathbf{x} \in X$. Then the real-valued function $|f|$ is continuous on X.

A proof of Proposition 1.13 is to be found in Appendix A.

Given any function $f \colon X \to Y$, we denote by $f^{-1}(V)$ the *preimage* of a subset V of Y under the map f, defined by
$f^{-1}(V) = \{\mathbf{x} \in X : f(\mathbf{x}) \in V\}$.

Proposition 1.14

Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f \colon X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(V)$ is open in X for every open subset V of Y.

A proof of Proposition 1.14 is to be found in Appendix A.

Let X be a subset of $\mathbb{R}^n$, let $f \colon X \to \mathbb{R}$ be continuous, and let c be some real number. Proposition 1.14 ensures that the sets $\{\mathbf{x} \in X : f(\mathbf{x}) > c\}$ and $\{\mathbf{x} \in X : f(\mathbf{x}) < c\}$ are open in X. Moreover given real numbers a and b satisfying $a < b$, the set $\{\mathbf{x} \in X : a < f(\mathbf{x}) < b\}$ is open in X.

Corollary 1.15

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi\colon X \to Y$ be a continuous function from X to Y. Then $\varphi^{-1}(F)$ is closed in X for every subset F of Y that is closed in Y.

A proof of Corollary 1.15 is to be found in Appendix A.

Lemma 1.16

Let X be a closed subset of n-dimensional Euclidean space $\mathbb{R}^n$. Then a subset of X is closed in X if and only if it is closed in $\mathbb{R}^n$.

A proof of Lemma 1.16 is to be found in Appendix A.

1.3. The Multidimensional Extreme Value Theorem

Theorem 1.17 (The Multidimensional Extreme Value Theorem)

Let X be a non-empty closed bounded set in $\mathbb{R}^m$, and let $f\colon X \to \mathbb{R}$ be a continuous real-valued function defined on X. Then there exist points $\mathbf{u}$ and $\mathbf{v}$ of X such that $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$.

A proof of Theorem 1.17 is to be found in Appendix A.

1.4. The Glueing Lemma

The following result, together with its generalizations, is sometimes referred to as the *Glueing Lemma*.

Lemma 1.18 (Glueing Lemma)

Let $\varphi\colon X \to \mathbb{R}^n$ *be a function mapping a subset* X *of* $\mathbb{R}^m$ *into* $\mathbb{R}^n$*. Let* $F_1, F_2, \ldots, F_k$ *be a finite collection of subsets of* X *such that* F_i *is closed in* X *for* $i = 1, 2, \ldots, k$ *and*

$$F_1 \cup F_2 \cup \cdots \cup F_k = X.$$

Then the function φ *is continuous on* X *if and only if the restriction of* φ *to* F_i *is continuous on* F_i *for* $i = 1, 2, \ldots, k$*.*

Proof

Suppose that $\varphi\colon X \to \mathbb{R}^n$ is continuous. Then it follows directly from the definition of continuity that the restriction of φ to each subset of X is continuous on that subset. Therefore the restriction of φ to F_i is continuous on F_i for $i = 1, 2, \ldots, k$.

Conversely we must prove that if the restriction of the function φ to F_i is continuous on F_i for $i = 1, 2, \ldots, k$ then the function $\varphi \colon X \to \mathbb{R}^m$ is continuous. Let $\mathbf{p}$ be a point of X, and let some positive real number ε be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots \delta_k$ satisfying the following conditions:—

(i) if $\mathbf{p} \in F_i$, where $1 \leq i \leq k$, and if $\mathbf{x} \in F_i$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_i$ then $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$;

(ii) if $\mathbf{p} \notin F_i$, where $1 \leq i \leq k$, and if $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_i$ then $\mathbf{x} \notin F_i$.

Indeed the continuity of the function φ on each set F_i ensures that δ_i may be chosen to satisfy (i) for each integer i between 1 and k for which $\mathbf{p} \in F_i$. Also the requirement that F_i be closed in X ensures that $X \setminus F_i$ is open in X and therefore δ_i may be chosen to to satisfy (ii) for each integer i between 1 and k for which $\mathbf{p} \notin F_i$.

Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. Let $\mathbf{x} \in X$ satisfy $|\mathbf{x} - \mathbf{p}| < \delta$. If $\mathbf{p} \notin F_i$ then the choice of δ_i ensures that if $\mathbf{x} \notin F_i$. But X is the union of the sets $F_1, F_2, \ldots, F_k$, and therefore there must exist some integer i between 1 and k for which $\mathbf{x} \in F_i$. Then $\mathbf{p} \in F_i$, and the choice of δ_i ensures that $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$. We have thus shown that $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$ for all points $\mathbf{x}$ of X that satisfy $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $\varphi\colon X \to \mathbb{R}^n$ is continuous, as required. ∎

1.5. Lebesgue Numbers

Definition

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. A collection of subsets of $\mathbb{R}^n$ is said to *cover* X if and only if every point of X belongs to at least one of these subsets.

Definition

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. An *open cover* of X is a collection of subsets of X that are open in X and cover the set X.

Proposition 1.19

Let X be a closed bounded set in n-dimensional Euclidean space, and let $\mathcal{V}$ be an open cover of X. Then there exists a positive real number δ_L with the property that, given any point $\mathbf{u}$ of X, there exists a member V of the open cover $\mathcal{V}$ for which

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta_L\} \subset V.$$

Proof

Let

$$B_X(\mathbf{u}, \delta) = \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta\}$$

for all $\mathbf{u} \in X$ and for all positive real numbers δ. Suppose that there did not exist any positive real number δ_L with the stated property.

Then, given any positive number δ, there would exist a point $\mathbf{u}$ of X for which the set $B_X(\mathbf{u}, \delta)$ would not be wholly contained within any open set V belonging to the open cover $\mathcal{V}$. Consequently there would exist an infinite sequence

$$\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$$

of points of X with the property that, for each positive integer j, the set $B_X(\mathbf{u}_j, 1/j)$ would not be wholly contained within any open set V belonging to the open cover $\mathcal{V}$. The multidimensional Bolzano-Weierstrass Theorem (Theorem 1.2) would then ensure the existence of a convergent subsequence

$$\mathbf{u}_{j_1}, \mathbf{u}_{j_2}, \mathbf{u}_{j_3}, \ldots$$

of this infinite sequence.

Let $\mathbf{p}$ be the limit of this convergent subsequence. Then the point $\mathbf{p}$ would then belong to X, because X is closed (see Lemma 1.8). But then the point $\mathbf{p}$ would belong to an open set V belonging to the open cover $\mathcal{V}$. It would then follow from the definition of open sets that there would exist a positive real number δ for which $B_X(\mathbf{p}, 2\delta) \subset V$. Let $j = j_k$ for a positive integer k large enough to ensure that both $1/j < \delta$ and $\mathbf{u}_j \in B_X(\mathbf{p}, \delta)$. The Triangle Inequality would then ensure that every point of X within a distance $1/j$ of the point $\mathbf{u}_j$ would lie within a distance 2δ of the point $\mathbf{p}$, and therefore

$$B_X(\mathbf{u}_j, 1/j) \subset B_X(\mathbf{p}, 2\delta) \subset V.$$

But we supposed that the point $\mathbf{u}_j$ was chosen so as to ensure that the set $B_X(\mathbf{u}_j, 1/j)$ was not wholly contained within any open set V belonging to the open cover $\mathcal{V}$. Thus a logical contradiction as resulted from the assumption that there is no positive real number δ_L with the property that, given any point $\mathbf{u}$ of X, the set $B_X(\mathbf{u}, \delta_L)$ is not wholly contained within any open set belonging to the open cover $\mathcal{V}$. Consequently some positive real number δ_L satisfying this property must exist, and thus the required result has been proved. ∎

Definition

Let X be a subset of n-dimensional Euclidean space, and let $\mathcal{V}$ be an open cover of X. A positive real number δ_L is said to be a *Lebesgue number* for the open cover $\mathcal{V}$ if, given any point $\mathbf{p}$ of X, there exists some member V of the open cover $\mathcal{V}$ for which

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta_L\} \subset V.$$

Proposition 1.19 ensures that, given any open cover of a closed bounded subset of n-dimensional Euclidean space, there exists a positive real number that is a Lebesgue number for that open cover.

Definition

The *diameter* $\mathrm{diam}(A)$ of a bounded subset A of n-dimensional Euclidean space is defined so that

$$\mathrm{diam}(A) = \sup\{|\mathbf{x} - \mathbf{y}| : \mathbf{x}, \mathbf{y} \in A\}.$$

It follows from this definition that $\mathrm{diam}(A)$ is the smallest real number K with the property that $|\mathbf{x} - \mathbf{y}| \leq K$ for all $\mathbf{x}, \mathbf{y} \in A$.

Lemma 1.20

Let X be a bounded subset of n-dimensional Euclidean space, and let δ be a positive real number. Then there exists a finite collection $A_1, A_2, \ldots, A_s$ of subsets of X such that the $\operatorname{diam}(A_i) < \delta$ for $i = 1, 2, \ldots, s$ and

$$X = A_1 \cup A_2 \cup \cdots \cup A_s.$$

Proof

Let b be a real number satisfying $0 < \sqrt{n}\, b < \delta$ and, for each n-tuple $(j_1, j_2, \ldots, j_n)$ of integers, let $H_{(j_1, j_2, \ldots, j_n)}$ denote the hypercube in $\mathbb{R}^n$ defined such that

$$\begin{aligned} H_{(j_1, j_2, \ldots, j_n)} &= \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \\ &\qquad j_i b \leq x_i \leq (j_i + 1) b \text{ for } i = 1, 2, \ldots, n\}. \end{aligned}$$

Note that if $\mathbf{u}$ and $\mathbf{v}$ are points of $H_{(j_1,j_2,\ldots,j_n)}$ for some n-tuple $(j_1, j_2, \ldots, j_n)$ of integers then $|u_i - v_i| < b$ for $i = 1, 2, \ldots, n$, and therefore $|\mathbf{u} - \mathbf{v}| \leq \sqrt{n}\, b < \delta$. Therefore the diameter of each hypercube $H_{(j_1,j_2,\ldots,j_n)}$ is less than δ.

The boundedness of the set X ensures that there are only finitely many n-tuples $(j_1, j_2, \ldots, j_n)$ of integers for which $X \cap H_{(j_1,j_2,\ldots,j_n)}$ is non-empty. It follows that X is covered by a finite collection $A_1, A_2, \ldots, A_k$ of subsets of X, where each of these subsets is of the form $X \cap H_{(j_1,j_2,\ldots,j_n)}$ for some n-tuple $(j_1, j_2, \ldots, j_n)$ of integers. These subsets all have diameter less than δ. The result follows. ∎

Definition

Let $\mathcal{V}$ and $\mathcal{W}$ be open covers of some subset X of a Euclidean space. Then $\mathcal{W}$ is said to be a *subcover* of $\mathcal{V}$ if and only if every open set belonging to $\mathcal{W}$ also belongs to $\mathcal{V}$.

Definition

A subset X of a Euclidean space is said to be *compact* if and only if every open cover of X possesses a finite subcover.

Theorem 1.21 (The Multidimensional Heine-Borel Theorem)

A subset of n-dimensional Euclidean space $\mathbb{R}^n$ is compact if and only if it is both closed and bounded.

Proof

Let X be a compact subset of $\mathbb{R}^n$ and let

$$V_j = \{\mathbf{x} \in X : |\mathbf{x}| < j\}$$

for all positive integers j. Then the sets $V_1, V_2, V_3, \ldots$ constitute an open cover of X. This open cover has a finite subcover, and therefore there exist positive integers $j_1, j_2, \ldots, j_k$ such that

$$X \subset V_{j_1} \cup V_{j_2} \cup \cdots \cup V_{j_k}.$$

Let M be the largest of the positive integers $j_1, j_2, \ldots, j_k$. Then $|\mathbf{x}| \leq M$ for all $\mathbf{x} \in X$. Thus the set X is bounded.

Let $\mathbf{q}$ be a point of $\mathbb{R}^n$ that does not belong to X, and let

$$W_j = \left\{ \mathbf{x} \in X : |\mathbf{x} - \mathbf{q}| > \frac{1}{j} \right\}$$

for all positive integers j. Then the sets $W_1, W_2, W_3, \ldots$ constitute an open cover of X. This open cover has a finite subcover, and therefore there exist positive integers $j_1, j_2, \ldots, j_k$ such that

$$X \subset W_{j_1} \cup W_{j_2} \cup \cdots \cup W_{j_k}.$$

Let $\delta = 1/M$, where M is the largest of the positive integers $j_1, j_2, \ldots, j_k$. Then $|\mathbf{x} - \mathbf{q}| \geq \delta$ for all $\mathbf{x} \in X$ and thus the open ball of radius δ about the point $\mathbf{q}$ does not intersect the set X. We conclude that the set X is closed. We have now shown that every compact subset of $\mathbb{R}^n$ is both closed and bounded.

We now prove the converse. Let X be a closed bounded subset of $\mathbb{R}^n$, and let $\mathcal{V}$ be an open cover of X. It follows from Proposition 1.19 that there exists a Lebesgue number δ_L for the open cover $\mathcal{V}$. It then follows from Lemma 1.20 that there exist subsets $A_1, A_2, \ldots, A_s$ of X such that $\mathrm{diam}(A_i) < \delta_L$ for $i = 1, 2, \ldots, s$ and

$$X = A_1 \cup A_2 \cup \cdots \cup A_s.$$

We may suppose that A_i is non-empty for $i = 1, 2, \ldots, s$ (because if $A_i = \emptyset$ then A_i could be deleted from the list). Choose $\mathbf{p}_i \in A_i$ for $i = 1, 2, \ldots, s$. Then $A_i \subset B_X(\mathbf{p}_i, \delta_L)$ for $i = 1, 2, \ldots, s$. The definition of the Lebesgue number δ_L then ensures that there exist members $V_1, V_2, \ldots, V_s$ of the open cover $\mathcal{V}$ such that $B_X(\mathbf{p}_i, \delta_L) \subset V_i$ for $i = 1, 2, \ldots, s$. Then $A_i \subset V_i$ for $i = 1, 2, \ldots, s$, and therefore

$$X \subset V_1 \cup V_2 \cup \cdots \cup V_s.$$

Thus $V_1, V_2, \ldots, V_s$ constitute a finite subcover of the open cover $\mathcal{V}$. We have therefore proved that every closed bounded subset of n-dimensional Euclidean space is compact, as required. ∎