MAU34804—Fixed Point Theorems and Economic Equilibria
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Appendix D: Further Results Concerning Barycentric Subdivision

David R. Wilkins

D. Further Results Concerning Barycentric Subdivision

D.1. The Barycentric Subdivision of a Simplex

Proposition D.1

Let σ be a simplex in $\mathbb{R}^N$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$, and let $m_0, m_1, \ldots, m_r$ be integers satisfying

$$0 \leq m_0 < m_1 < \cdots < m_r \leq q.$$

Let ρ be the simplex in $\mathbb{R}^N$ with vertices $\hat{\tau}_0, \hat{\tau}_1, \ldots, \hat{\tau}_r$, where $\hat{\tau}_k$ denotes the barycentre of the simplex τ_k with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{m_k}$ for $k = 1, 2, \ldots, r$. Then the simplex ρ is the set consisting of all points of $\mathbb{R}^N$ that can be represented in the form $\sum_{j=0}^{q} t_j \mathbf{v}_j$, where $t_0, t_1, \ldots, t_q$ are real numbers satisfying the following conditions:

(i) $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$;

(ii) $\sum_{j=0}^{q} t_j = 1$;

(iii) $t_0 \geq t_1 \geq \cdots \geq t_q$;

(iv) $t_j = t_{m_0}$ for all integers j satisfying $j \leq m_0$;

(v) $t_j = t_{m_k}$ for all integers j and k satisfying $0 < k \leq r$ and $m_{k-1} < j \leq m_k$;

(vi) $t_j = 0$ for all integers j satisfying $j > m_r$.

Moreover the interior of the simplex ρ is the set consisting of all points of $\mathbb{R}^N$ that can be represented in the form $\sum_{j=0}^{q} t_j \mathbf{v}_j$, where $t_0, t_1, \ldots, t_q$ are real numbers satisfying conditions (i)–(iv) above together with the following extra condition:

(vii) $t_{m_{k-1}} > t_{m_k} > 0$ for all integers k satisfying $0 < k \leq r$.

Proof

Let $\mathbf{w}_k = \hat{\tau}_k$ for $k = 0, 1, \ldots, r$. Then

$$\mathbf{w}_k = \frac{1}{m_k + 1} \sum_{j=0}^{m_k} \mathbf{v}_j.$$

Let $\mathbf{x} \in \rho$, and let the real numbers $u_0, u_1, \ldots, u_r$ be the barycentric coordinates of the point $\mathbf{x}$ with respect to the vertices $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_r$ of ρ, so that $0 \leq u_k \leq 1$ for $k = 0, 1, \ldots, r$, $\sum_{k=0}^{r} u_k \mathbf{w}_k = \mathbf{x}$, and $\sum_{k=0}^{r} u_k = 1$.

Also let

$$K(j) = \{k \in \mathbb{Z} : 0 \leq k \leq r \text{ and } m_k \geq j\}$$

for $j = 0, 1, \ldots, q$. Then $\mathbf{x} = \sum_{j=0}^{q} t_j \mathbf{v}_j$, where

$$t_j = \sum_{k \in K(j)} \frac{u_k}{m_k + 1}$$

when $0 \leq j \leq m_r$, and $t_j = 0$ when $m_r < j \leq q$.

Moreover

$$\begin{aligned}\sum_{j=0}^{q} t_j &= \sum_{j=0}^{m_r} \sum_{k \in K(j)} \frac{u_k}{m_k + 1} = \sum_{(j,k) \in L} \frac{u_k}{m_k + 1} \\ &= \sum_{k=0}^{r} \sum_{j=0}^{m_k} \frac{u_k}{m_k + 1} = \sum_{k=0}^{r} u_k = 1,\end{aligned}$$

where

$$L = \{(j,k) \in \mathbb{Z}^2 : 0 \leq j \leq q, \;\; 0 \leq k \leq r \text{ and } j \leq m_k\}.$$

Now $t_j \geq 0$ for $j = 0, 1, \ldots, q$, because $u_k \geq 0$ for $k = 0, 1, \ldots, r$, and therefore

$$0 \leq t_j \leq \sum_{j=0}^{q} t_j = 1.$$

Also $t_{j'} \leq t_j$ for all integers j and j' satisfying $0 \leq j < j' \leq m_r$, because $K(j') \subset K(j)$. If $0 \leq j \leq m_0$ then $K(j) = K(m_0)$, and therefore $t_j = t_{m_0}$. Similarly if $0 < k \leq r$, and $m_{k-1} < j \leq m_k$ then $K(j) = K(m_k)$, and therefore $t_j = t_{m_k}$. Thus the real numbers $t_0, t_1, \ldots, t_k$ satisfy conditions (i)–(vi) above.

Now let $t_0, t_1, \ldots, t_q$ be real numbers satisfying conditions (i)-(vi), let

$$u_r = (m_r + 1)t_{m_r}$$

and

$$u_k = (m_k + 1)(t_{m_k} - t_{m_{k+1}})$$

for $k = 0, 1, \ldots, r - 1$. Then

$$t_{m_k} = \sum_{k'=k}^{r} \frac{u_{k'}}{m_{k'} + 1}$$

for $k = 0, 1, \ldots, r$. Also $u_k \geq 0$ for $k = 0, 1, \ldots, r$, and

$$\begin{aligned}
\sum_{k=0}^{r} u_k &= \sum_{k=0}^{r-1} (m_k + 1)(t_{m_k} - t_{m_{k+1}}) + (m_r + 1) t_{m_r} \\
&= (m_0 + 1) t_{m_0} + \sum_{k=1}^{r-1} (m_k + 1) t_{m_k} - \sum_{k=0}^{r-2} (m_k + 1) t_{m_{k+1}} \\
&\qquad - (m_{r-1} + 1) t_{m_r} + (m_r + 1) t_{m_r} \\
&= (m_0 + 1) t_{m_0} + \sum_{k=1}^{r-1} (m_k + 1) t_{m_k} - \sum_{k=1}^{r-1} (m_{k-1} + 1) t_{m_k} \\
&\qquad + (m_r - m_{r-1}) t_{m_r} \\
&= (m_0 + 1) t_{m_0} + \sum_{k=1}^{r} (m_k - m_{k-1}) t_{m_k},
\end{aligned}$$

But

$$\begin{aligned}\sum_{j=0}^{q} t_q &= \sum_{j=0}^{m_0} t_j + \sum_{k=1}^{r} \sum_{j=m_{k-1}+1}^{m_k} t_j \\ &= (m_0+1)t_{m_0} + \sum_{k=1}^{r} (m_k - m_{k-1}) t_{m_k},\end{aligned}$$

because conditions (i)-(vi) satisfied by the real numbers $t_0, t_1, \ldots, t_q$ ensure that $t_j = t_{m_0}$ when $0 \leq j \leq m_0$, $t_j = t_{m_k}$ when $1 \leq k \leq r$, and $m_{k-1} < j \leq m_k$ and $t_j = 0$ when $j > m_r$. Thus

$$\sum_{k=0}^{r} u_k = (m_0+1)t_{m_0} + \sum_{k=1}^{r} (m_k - m_{k-1}) t_{m_k} = \sum_{j=0}^{q} t_j = 1.$$

It follows that $u_0, u_1, \ldots, u_r$ are the barycentric coordinates of a point of the simplex with vertices $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_r$.

Moreover

$$t_j = \sum_{k \in K(j)} \frac{u_k}{m_k + 1}$$

for $j = 0, 1, \ldots, q$, and therefore

$$\begin{aligned}
\sum_{k=0}^{r} u_k \mathbf{w_k} &= \sum_{k=0}^{r} \sum_{j=0}^{m_k} \frac{u_k}{m_k + 1} \mathbf{v}_j \\
&= \sum_{(j,k) \in L} \frac{u_k}{m_k + 1} \mathbf{v}_j \\
&= \sum_{j=0}^{q} \sum_{k \in K(j)} \frac{u_k}{m_k + 1} \mathbf{v}_j \\
&= \sum_{j=0}^{q} t_j \mathbf{v}_j .
\end{aligned}$$

We conclude the the simplex ρ is the set of all points of $\mathbb{R}^N$ that are representable in the form $\sum_{j=0}^{q} t_j \mathbf{v}_j$, where the coefficients $t_0, t_1, \ldots, t_q$ are real numbers satisfying conditions (i)–(vi).

Now the point $\sum_{j=0}^{q} t_j \mathbf{v}_j$ belongs to the interior of the simplex ρ if and only if $u_k > 0$ for $k = 0, 1, \ldots, r$, where $u_r = (m_r + 1)t_{m_r}$ and $u_k = (m_k + 1)(t_{m_k} - t_{m_{k+1}})$ for $k = 0, 1, \ldots, r - 1$.

This point therefore belongs to the interior of the simplex ρ if and only if $t_{m_r} > 0$ and $t_{m_k} > t_{m_{k+1}}$ for $k = 0, 1, \ldots, r-1$. Thus the interior of the simplex ρ consists of those points $\sum_{j=0}^{q} t_j \mathbf{v}_j$ of σ whose barycentric coordinates $t_0, t_1, \ldots, t_q$ with respect to the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of σ satisfy conditions (i)–(vii), as required. ∎

Corollary D.2

Let σ be a simplex in some Euclidean space $\mathbb{R}^N$, and let K_σ be the simplicial complex consisting of the simplex σ together with all of its faces. Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of σ, and let $t_0, t_1, \ldots, t_q$ be the barycentric coordinates of some point $\mathbf{x}$ of σ, so that $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$, $\sum_{j=0}^{q} t_j \mathbf{v}_j = \mathbf{x}$ and $\sum_{j=0}^{q} t_j = 1$. Then there exists a permutation π of the set $\{0, 1, \ldots, q\}$ and integers $m_0, m_1, \ldots, m_r$ satisfying

$$0 \leq m_0 < m_1 < \cdots < m_r \leq q.$$

such the following conditions are satisfied:

(iii) $t_{\pi(0)} \geq t_{\pi(1)} \geq \cdots \geq t_{\pi(q)}$;

(iv) $t_{\pi(j)} = t_{\pi(m_0)}$ for all integers j satisfying $j \leq m_0$;

(v) $t_{\pi(j)} = t_{\pi(m_k)}$ for all integers j and k satisfying $0 < k \leq r$ and $m_{k-1} < j \leq m_k$;

(vi) $t_{\pi(j)} = 0$ for all integers j satisfying $j > m_r$.

(vii) $t_{\pi(m_{k-1})} > t_{\pi(m_k)} > 0$ for all integers k satisfying $0 < k \leq r$.

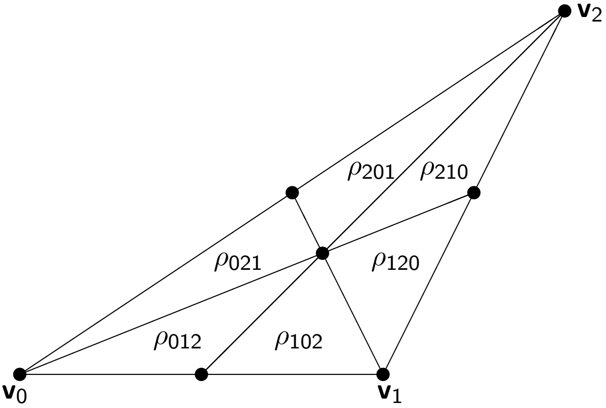
Let ρ be the simplex of the first barycentric subdivision K'_σ of the simplical complex K_σ with vertices $\hat{\tau}_0, \hat{\tau}_1, \ldots, \hat{\tau}_r$, where $\hat{\tau}_k$ is the barycentre of the simplex τ_k with vertices $\mathbf{v}_{\pi(0)}, \mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(m_k)}$ for $k = 0, 1, \ldots, r$. Then ρ is the unique simplex of K'_σ that contains the point $\mathbf{x}$ in its interior.

Proof
The required permutation π can be any permutation that rearranges the barycentric coordinates in descending order, so that $1 \geq t_{\pi(0)} \geq t_{\pi(1)} \geq \ldots \geq t_{\pi(q)} \geq 0$. The required result then follows immediately on applying Proposition D.1. ∎

Corollary D.2 may be applied to determine the simplices of the first barycentric subdivision K'_σ of the simplicial complex K_σ that consists of some simplex σ together with all of its faces.

Example

Let K be the simplicial complex consisting of a triangle with vertices $\mathbf{v}_0$, $\mathbf{v}_1$ and $\mathbf{v}_2$, together with all its edges and vertices, and let K' be the first barycentric subdivision of the simplicial complex K. Then K' consists of six triangles ρ_{012}, ρ_{102}, ρ_{021}, ρ_{120}, ρ_{201} and ρ_{210}, together with all the edges and vertices of those triangles, where


$\mathbf{v}_2$
ρ_{201}
ρ_{210}
ρ_{021}
ρ_{120}
ρ_{012}
ρ_{102}
$\mathbf{v}_0$
$\mathbf{v}_1$

$$\begin{aligned}
\rho_{012} &= \left\{ \sum_{j=0}^{2} t_j \mathbf{v}_j : 1 \geq t_0 \geq t_1 \geq t_2 \geq 0 \text{ and } \sum_{j=0}^{2} t_j = 1 \right\}, \\
\rho_{102} &= \left\{ \sum_{j=0}^{2} t_j \mathbf{v}_j : 1 \geq t_1 \geq t_0 \geq t_2 \geq 0 \text{ and } \sum_{j=0}^{2} t_j = 1 \right\}, \\
\rho_{021} &= \left\{ \sum_{j=0}^{2} t_j \mathbf{v}_j : 1 \geq t_0 \geq t_2 \geq t_1 \geq 0 \text{ and } \sum_{j=0}^{2} t_j = 1 \right\}, \\
\rho_{120} &= \left\{ \sum_{j=0}^{2} t_j \mathbf{v}_j : 1 \geq t_1 \geq t_2 \geq t_0 \geq 0 \text{ and } \sum_{j=0}^{2} t_j = 1 \right\},
\end{aligned}$$

$$\rho_{201} = \left\{ \sum_{j=0}^{2} t_j \mathbf{v}_j : 1 \geq t_2 \geq t_0 \geq t_1 \geq 0 \text{ and } \sum_{j=0}^{2} t_j = 1 \right\},$$

$$\rho_{210} = \left\{ \sum_{j=0}^{2} t_j \mathbf{v}_j : 1 \geq t_2 \geq t_1 \geq t_0 \geq 0 \text{ and } \sum_{j=0}^{2} t_j = 1 \right\}.$$

The intersection of any two of those triangles is a common edge or vertex of those triangles. For example, the intersection of the triangles ρ_{012} and ρ_{102} is the edge $\rho_{012} \cap \rho_{102}$, where

$$\rho_{012} \cap \rho_{102} = \left\{ \sum_{j=0}^{2} t_j \mathbf{v}_j : 1 \geq t_0 = t_1 \geq t_2 \geq 0 \text{ and } \sum_{j=0}^{2} t_j = 1 \right\}.$$

And the intersection of the triangle ρ_{012} and ρ_{120} is the barycentre of the triangle $\mathbf{v}_0\,\mathbf{v}_1\,\mathbf{v}_2$, and is thus the point $\sum_{j=0}^{2} t_j \mathbf{v}_j$ whose barycentric coordinates t_0, t_1, t_2 satisfy $t_0 = t_1 = t_2 = \frac{1}{3}$.

Let σ be a q-simplex with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$, let K_σ be the simplicial complex consisting of the simplex σ, together with all its faces, and let K'_σ be the first barycentric subdivision of the simplicial complex K_σ. Then the q-simplices of K'_σ are the simplices of the form $\rho_{m_0\, m_1\, \ldots\, m_q}$, where the list $m_0, m_1, \ldots, m_q$ is a rearrangement of the list $0, 1, \ldots, q$ (so that each integer between 0 and q occurs exactly one in the list $m_0, m_1, \ldots, m_q$), and where

$$\begin{aligned} &\rho_{m_0\, m_1\, \ldots\, m_q} \\ &= \left\{ \sum_{j=0}^{q} t_j \mathbf{v}_j : 1 \geq t_{m_0} \geq t_{m_1} \geq \cdots \geq t_{m_q} \geq 0 \text{ and } \sum_{j=0}^{q} t_j = 1 \right\}. \end{aligned}$$

A point of σ belongs to the interior of one of the simplices of K'_σ if and only if its barycentric coordinates $t_0, t_1, \ldots, t_q$ are all distinct and strictly positive. Moreover if a point $\sum_{j=0}^{q} t_j \mathbf{v}_j$ of σ with barycentric coordinates $t_0, t_1, \ldots, t_q$ belongs to the interior of some r-simplex of K'_σ then there are exactly $r + 1$ distinct values amongst the real numbers $t_0, t_1, \ldots, t_q$ (i.e., $\{t_0, t_1, \ldots, t_q\}$ is a set with exactly $r + 1$ elements).