MAU34804—Fixed Point Theorems and Economic Equilibria
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Section 8: Walrasian Equilibria

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8. Walrasian Equilibria

8.1. Exchange Economies

We consider an exchange economy consisting of a finite number of commodities and a finite number of households, each provided with an initial endowment of each of the commodities. The commodities are required to be *infinitely divisible*: this means that a household can hold an amount x of that commodity for any non-negative real number x. (Thus salt, for example, could be regarded as an 'infinitely divisible' quantity whereas cars cannot: it makes little sense to talk about a particular household owning 2.637 of a car, for example, though such a household may well own 2.637 kilograms of salt.) Now the households may well wish to exchange commodities with one another so as improve on their initial endowment. They might for example seek to barter commodities with one another: however this method of redistribution would not work very efficiently in a large economy.

Alternatively they might attempt to set up a price mechanism to simplify the task of redistributing the commodities. Thus suppose that each commodity is assigned a given price. Then each household could sell its initial endowment to the market, receiving in return the value of its initial endowment at the given prices. The household could then purchase from the market a quantity of each commodity so as to maximize its own preference, subject to the constraint that the total value of the commodities purchased by any household cannot exceed the value of its initial endowment at the given prices.

The problem of redistribution then becomes one of fixing prices so that there is exactly enough of each commodity to go around: if the price of any commodity is too low then the demand for that commodity is likely to outstrip supply, whereas if the price is too high then supply will exceed demand. A *Walras equilibrium* is achieved if prices can be found so that the supply of each commodity matches its demand. We shall use *Berge's Maximum Theorem* and the *Kakutani fixed point theorem* to prove the existence of a Walras equilibrium in this idealized economy.

Let our exchange economy consist of n commodities and m households. We suppose that household h is provided with an initial endowment $\overline{x}_{hi}$ of commodity i, where $\overline{x}_{hi} \geq 0$. Thus the initial endowment of household h can be represented by a vector $\overline{\mathbf{x}}_h$ in $\mathbb{R}^n$ whose ith component is $\overline{x}_{hi}$. The prices of the commodities are given by a price vector $\mathbf{p}$ whose ith component p_i specifies the price of a unit of the ith commodity: a price vector $\mathbf{p}$ is required to satisfy $p_i \geq 0$ for all i. Then the value of the initial endowment of household h at the given prices is $\mathbf{p} \,.\, \overline{\mathbf{x}}_h$. This quantity represents the *wealth* of household h at prices $\mathbf{p}$.

Definition

For each positive integer n, the *positive orthant* $\mathbb{R}^n_+$ is the subset of $\mathbb{R}^n$ defined so that

$$\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0}\}.$$

In particular $\mathbb{R}_+ = \{t \in \mathbb{R} : t \geq 0\}$.

Definition

A real-valued function $u\colon X \to \mathbb{R}$ defined over a subset X of $\mathbb{R}^n$ is said to be *strictly increasing* on X if $u(\mathbf{x}) < u(\mathbf{x}')$ for all $\mathbf{x}, \mathbf{x}'$ in X satisfying $\mathbf{x} \leq \mathbf{x}'$ and $\mathbf{x} \neq \mathbf{x}'$.

8.2. The Budget Correspondence

We now discuss basic properties of the *budget correspondence*.

The budget correspondence is defined on the set of pairs. A *price-wealth pair* is an ordered pair $(\mathbf{p}, w)$, where $\mathbf{p} \in \mathbb{R}^n$, w is a non-negative real number and $\mathbf{p} \geq 0$. The budget correspondence assigns to each price-wealth pair the bundles of commodities that an economic agent with the specified wealth can afford to purchase at the specified prices.

More formally, the definition of the budget correspondence may be given as follows.

Definition

In a model of an exchange economy with n commodities, The *budget correspondence* $B : \mathbb{R}^n_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^n$ assigns to each price-wealth pair $(\mathbf{p}, w)$ in $\mathbb{R}^n_+ \times \mathbb{R}_+$ the subset $B(\mathbf{p}, w)$ of $\mathbb{R}^n_+$ defined such that

$$B(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0} \text{ and } \mathbf{p} \,.\, \mathbf{x} \leq w\}.$$

Example

Consider the case of two commodities. The budget correspondence $B\colon \mathbb{R}^2_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^2$ is defined so that

$$B(\mathbf{p}, w) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, \;\; x_2 \geq 0 \text{ and } p_1 x_1 + p_2 x_2 \leq w\}$$

for all $\mathbf{p} \in \mathbb{R}^2_+$ and $w \in \mathbb{R}_+$, where $\mathbf{p} = (p_1, p_2)$.

Let $\mathbf{p}_0$ be the vector in $\mathbb{R}^2_+$ with $\mathbf{p}_0 = (1, 0)$, and let V be the open set in $\mathbb{R}^3$ defined so that

$$V = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 1 + \frac{1}{1 + x_2^2}\right\}.$$

Now

$$B(\mathbf{p}_0, w) = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq w \text{ and } x_2 \geq 0\}$$

for all $w > 0$. It follows that $B(\mathbf{p}_0, 1) \subset V$, but $B(\mathbf{p}_0, w) \not\subset V$ for all $w > 1$. Indeed if $w > 1$ then t can be chosen large enough to ensure that

$$w > 1 + \frac{1}{1 + t^2}.$$

But then $(w, t) \in B(\mathbf{p}_0, w)$, but $(w, t) \notin V$. This example demonstrates that the budget correspondence $B \colon \mathbb{R}^2_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^2$ is not upper hemicontinuous at $(\mathbf{p}_0, 1)$, where $\mathbf{p}_0 = (1, 0)$.

Note also that $B(\mathbf{p}, w) = B(w^{-1}\mathbf{p}, 1)$ for all $(\mathbf{p}, w) \in \mathbb{R}^2 \times \mathbb{R}_+$ satisfying $w > 0$. It follows that the budget correspondence $\mathbf{p} \mapsto B(\mathbf{p}, 1)$ is not upper hemicontinuous on $\mathbb{R}^2_+$ at $\mathbf{p}_0$.

Now let $\mathbf{p}_0 = (1, 0)$ as before, and let

$$V = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 1\}.$$

Now

$$B(\mathbf{p}_0, 0) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0 \text{ and } x_2 \geq 0\}.$$

It follows that $B(\mathbf{p}_0, 0) \cap V \neq \emptyset$. But if $\mathbf{p} >> \mathbf{0}$ then $B(\mathbf{p}, 0) = \{(0, 0)\}$. Thus $B(\mathbf{p}, 0) \cap V = \emptyset$ whenever $\mathbf{p} \geq \mathbf{0}$. It follows that the budget correspondence B is not lower hemicontinuous at $(\mathbf{p}_0, 0)$.

Proposition 8.1

Let n be a positive integer, let $\mathbf{c}$ be an element of $\mathbb{R}^n$ satisfying $\mathbf{c} \gg \mathbf{0}$, and let $B_{\mathbf{c}} : \mathbb{R}^n_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^n$ be the correspondence that assigns to each price-wealth pair $(\mathbf{p}, w)$ in $\mathbb{R}^n_+ \times \mathbb{R}_+$ the subset $B_{\mathbf{c}}(\mathbf{p}, w)$ of $\mathbb{R}^n_+$ defined such that

$$B_{\mathbf{c}}(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{0} \leq \mathbf{x} \leq \mathbf{c} \text{ and } \mathbf{p} \,.\, \mathbf{x} \leq w\}.$$

Then the correspondence $B_{\mathbf{c}} : \mathbb{R}^n_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^n$ is upper hemicontinuous on $\mathbb{R}^n_+ \times \mathbb{R}_+$ and lower hemicontinuous on

$$\{(\mathbf{p}, w) \in \mathbb{R}^n_+ \times \mathbb{R} : w > 0\}.$$

Moreover $B_{\mathbf{c}}(\mathbf{p}, w)$ of $\mathbb{R}^n_+$ is non-empty, compact and convex for all $(\mathbf{p}, w) \in \mathbb{R}^n_+ \times \mathbb{R}$.

Proof

The set $B_{\mathbf{c}}(\mathbf{p}, w)$ is a non-empty closed bounded convex subset of $\mathbb{R}^n_+$ for $i = 1, 2, \ldots, n$. Any closed bounded subset of $\mathbb{R}^n$ is compact. It follows that The set $B_{\mathbf{c}}(\mathbf{p}, w)$ is non-empty, compact convex for all $(\mathbf{p}, w) \in \mathbb{R}^n_+ \times \mathbb{R}_+$.

Next we show that the correspondence $B_{\mathbf{c}}$ is upper hemicontinuous on $\mathbb{R}^n_+ \times \mathbb{R}_+$. Let $(\mathbf{p}_0, w_0) \subset \mathbb{R}^n_+ \times \mathbb{R}_+$, and let V be an open set in $\mathbb{R}^n$ for which $B_{\mathbf{c}}(\mathbf{p}_0, w_0) \subset V$. We will show that there exists an open set N in $\mathbb{R}^n_+ \times \mathbb{R}_+$ such that $(\mathbf{p}_0, w_0) \in N$ and $B_{\mathbf{c}}(\mathbf{p}, w) \subset V$ for all $(\mathbf{p}, w) \in N$.

Now $B_{\mathbf{c}}(\mathbf{p}, w) \subset C$ for all $(\mathbf{p}, w) \in \mathbb{R}^n_+ \times \mathbb{R}_+$, where

$$C = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{0} \leq \mathbf{x} \leq \mathbf{c}\}.$$

It follows that if $C \subset V$ then $B_{\mathbf{c}}(\mathbf{p}, w) \subset V$ for all $(\mathbf{p}, w) \in \mathbb{R}^n_+ \times \mathbb{R}_+$. We may therefore take $N = \mathbb{R}^n_+ \times \mathbb{R}_+$ in the case where $C \subset V$.

In the case where C is not contained in V let $F = C \setminus V$. Then F is a non-empty closed subset of C. If $\mathbf{x} \in C$ and $\mathbf{p}_0 \,.\, \mathbf{x} \leq w_0$ then $\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}_0, w_0)$, and therefore $\mathbf{x} \in V$, because $B_{\mathbf{c}}(\mathbf{p}_0, w_0) \subset V$, and thus $\mathbf{x} \notin F$. It follows that $\mathbf{p}_0 \,.\, \mathbf{x} > w_0$ for all $\mathbf{x} \in F$. It then follows from the Extreme Value Theorem that the continuous function sending each point $\mathbf{x}$ of F to $\mathbf{p}_0 \,.\, \mathbf{x}$ attains a minimum value at some point of the set F, and therefore there exists a point $\mathbf{x}_1$ of F with the property that $\mathbf{p}_0 \,.\, \mathbf{x}_1 \leq \mathbf{p}_0 \,.\, \mathbf{x}$ for all $\mathbf{x} \in F$. Let a real number w be chosen so that $w_0 < w_1 < \mathbf{p}_0 \,.\, \mathbf{x}_1$. Then $\mathbf{p}_0 \,.\, \mathbf{x} > w_1$ for all $\mathbf{x} \in F$, and therefore $B_{\mathbf{c}}(\mathbf{p}_0, w_1) \cap F = \emptyset$. But $B_{\mathbf{c}}(\mathbf{p}_0, w_1) \subset C$ and $F = C \setminus V$. It follows that $B_{\mathbf{c}}(\mathbf{p}_0, w_1) \subset V$.

Now let N be the subset of $\mathbb{R}^n_+ \times \mathbb{R}^+$ consisting of those price-wealth pairs $(\mathbf{p}, w)$ with the property that

$$(\mathbf{p})_i > \frac{w}{w_1}(\mathbf{p}_0)_i$$

for those integers i between 1 and n for which $(\mathbf{p}_0)_i > 0$. Then N is open in $\mathbb{R}^n_+ \times \mathbb{R}_+$. Moreover the definition of N and the inequality $w_0 < w_1$ together ensure that $(\mathbf{p}_0, w_0) \in N$.

Care needs to be exercised in cases where $w = 0$. Suppose that $\mathbf{p} \geq 0$ and $(\mathbf{p}, 0) \in N$. Then $(\mathbf{p})_i > 0$ for all integers i between 1 and n for which $(\mathbf{p}_0)_i > 0$. It follows that if $\mathbf{x} \in \mathbb{R}^n$ satisfies $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{p} \,.\, \mathbf{x} = 0$ then $(\mathbf{p})_i = 0$ for those integers i between 1 and n for which $(\mathbf{x})_i > 0$. But then $(\mathbf{p}_0)_i = 0$ for those integers i between 1 and n for which $(\mathbf{x})_i > 0$, and therefore $\mathbf{p}_0 \,.\, \mathbf{x} = 0$. We conclude from this that if $(\mathbf{p}, 0) \in N$ and $\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, 0)$ then $\mathbf{p}_0 \,.\, \mathbf{x} = 0$, and therefore $\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}_0, 0)$. But

$$B_{\mathbf{c}}(\mathbf{p}_0, 0) \subset B_{\mathbf{c}}(\mathbf{p}_0, w_0) \subset V.$$

We conclude therefore that if $(\mathbf{p}, 0) \in N$ then $B_{\mathbf{c}}(\mathbf{p}, 0) \subset V$.

Now let $(\mathbf{p}, w) \in N$, where $w > 0$, and let $\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, w)$. Then $\mathbf{x} \geq 0$ and $\mathbf{p} \,.\, \mathbf{x} \leq w$. Then

$$\mathbf{p}_0 \,.\, \mathbf{x} = \sum_{i=1}^{n} (\mathbf{p}_0)_i (\mathbf{x})_i \leq \frac{w_1}{w} \sum_{i=1}^{n} (\mathbf{p})_i (\mathbf{x})_i = \frac{w_1}{w} \, \mathbf{p} \,.\, \mathbf{x} \leq w_1,$$

and therefore $\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}_0, w_1)$. It follows that if $(\mathbf{p}, w) \in N$ and $w > 0$ then

$$B_{\mathbf{c}}(\mathbf{p}, w) \subset B_{\mathbf{c}}(\mathbf{p}_0, w_1) \subset V.$$

We conclude therefore that $B_{\mathbf{c}}(\mathbf{p}, w) \subset V$ for all $(\mathbf{p}, w) \in N$. The results we have so far obtained combine to show that the correspondence $B_{\mathbf{c}}$ is upper hemicontinuous on $\mathbb{R}^n_+ \times \mathbb{R}_+$.

Now let $(\mathbf{p}_0, w_0) \in \mathbb{R}^n_+ \times \mathbb{R}_+$ satisfy $w_0 > 0$, and let V be an open set in $\mathbb{R}^n$ that satisfies $V \cap B_{\mathbf{c}}(\mathbf{p}_0, w_0) \neq \emptyset$. The constraint $w_0 > 0$ ensures that any open ball of positive radius centred on a point of $B_{\mathbf{c}}(\mathbf{p}_0, w_0)$ intersects the interior of that set. It follows that the open set V must intersect the interior of the set $B_{\mathbf{c}}(\mathbf{p}_0, w_0)$, and therefore there exists $\mathbf{x}_0 \in V$ for which $\mathbf{0} \leq \mathbf{x}_0 \leq \mathbf{c}$ and $\mathbf{p}_0 \,.\, \mathbf{x}_0 < w_0$. Let

$$N = \{(\mathbf{p}, w) \in \mathbb{R}^n_+ \times \mathbb{R}_+ : w - \mathbf{p} \,.\, \mathbf{x}_0 > 0\}.$$

Then N is open in $\mathbb{R}^n$, $(\mathbf{p}_0, \mathbf{w}_0) \in N$, and $\mathbf{x}_0 \in V \cap B_{\mathbf{c}}(\mathbf{p}, w)$ for all $(\mathbf{p}, w) \in N$. We conclude from this that the correspondence $B_{\mathbf{c}}$ is lower hemicontinuous on the set of those price-wealth pairs $(\mathbf{p}, w)$ in $\mathbb{R}^n_+ \times \mathbb{R}_+$ for which $w > 0$. This completes the proof. ∎

Proposition 8.2

Let n be a positive integer, and let $B\colon \mathbb{R}^n_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^n$ be the budget correspondence that assigns to each price-wealth pair $(\mathbf{p}, w)$ in $\mathbb{R}^n_+ \times \mathbb{R}_+$ the subset $B(\mathbf{p}, w)$ of $\mathbb{R}^n_+$ defined such that

$$B(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0} \text{ and } \mathbf{p} \,.\, \mathbf{x} \leq w\}.$$

Then the budget correspondence $B\colon \mathbb{R}^n_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^n$ is both upper hemicontinuous and lower hemicontinuous on the set Γ^n, where

$$\Gamma^n = \{(\mathbf{p}, w) \in \mathbb{R}^n \times \mathbb{R} : \mathbf{p} \gg \mathbf{0} \text{ and } w > 0\}.$$

Moreover $B(\mathbf{p}, w)$ of $\mathbb{R}^n_+$ is non-empty, compact and convex for all $(\mathbf{p}, w) \in \Gamma^n$.

Proof

Let $(\mathbf{p}_0, w_0)$ be a price-wealth pair for which $\mathbf{p}_0 >> \mathbf{0}$ and $w_0 > 0$. Then $(\mathbf{p}_0)_i > 0$ for $i = 1, 2, \ldots, n$. Let a positive vector $\mathbf{c}$ be chosen so that

$$(\mathbf{c})_i > \frac{w_0}{(\mathbf{p}_0)_i}$$

for $i = 1, 2, \ldots, n$. Let

$$N = \{(\mathbf{p}, w) \in \mathbb{R}^n_+ \times \mathbb{R}_+ : w > 0 \text{ and } (\mathbf{p})_i > \frac{w}{(\mathbf{c})_i} \text{ for } i = 1, 2, \ldots, n\}.$$

Then N is an open subset of $\mathbb{R}^n_+ \times \mathbb{R}_+$, $(\mathbf{p}_0, w_0) \in N$. Moreover if $(\mathbf{p}, w) \in N$, and if $\mathbf{x} \in B(\mathbf{p}, w)$, then $\mathbf{x} \geq 0$, $\mathbf{p} \,.\, \mathbf{x} \leq w$ and $w > 0$ But then $(\mathbf{p})_i > 0$ and

$$(\mathbf{p})_i(\mathbf{x})_i \leq w < (\mathbf{p})_i(\mathbf{c})_i$$

for $i = 1, 2, \ldots, n$, and therefore $\mathbf{x} \leq \mathbf{c}$.

It follows that $B(\mathbf{p}, w) = B_{\mathbf{c}}(\mathbf{p}, w)$ for all $(\mathbf{p}, w) \in N$, where

$$B_{\mathbf{c}}(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{0} \leq \mathbf{x} \leq \mathbf{c} \text{ and } \mathbf{p} \,.\, \mathbf{x} \leq w\}.$$

Now the correspondence $B_{\mathbf{c}}$ defined in this fashion is both upper hemicontinuous and lower hemicontinuous on the set of all price-wealth pairs $(\mathbf{p}, w)$ for which $w > 0$. (Proposition 8.1). It follows that, because $w > 0$ and $B(\mathbf{p}, w) = B_{\mathbf{c}}(\mathbf{p}, w)$ for all $(\mathbf{p}, w) \in N$, the budget correspondence B is both upper hemicontinuous and lower hemicontinuous on the open subset N of the set of price-wealth pairs, and is therefore both upper and lower hemicontinuous around the price-wealth pair $(\mathbf{p}_0, w_0)$. The result follows. ∎

8.3. Maximizing Normalized Commodity Prices

Proposition 8.3

Let n be a positive integer, let

$$\Delta = \left\{ \mathbf{p} \in \mathbb{R}^n : \mathbf{p} \geq \mathbf{0} \text{ and } \sum_{i=1}^{n} (\mathbf{p})_i = 1 \right\}.$$

Let $\gamma : \mathbb{R}^n \to \mathbb{R}$ be the function defined so that, for each $\mathbf{x} \in \mathbb{R}^n$, $\gamma(\mathbf{x})$ is the maximum of the components of $\mathbf{x}$, and let $\mu : \mathbb{R}^n \rightrightarrows \Delta$ be the correspondence defined such that

$$\mu(\mathbf{x}) = \{\mathbf{p} \in \Delta : \mathbf{p} \,.\, \mathbf{x} = \gamma(\mathbf{x})\}.$$

Then the correspondence $\mu : \mathbb{R}^n \rightrightarrows \Delta$ is upper hemicontinuous, and $\mu(\mathbf{x})$ is a non-empty compact convex subset of Δ for all $\mathbf{x} \in \mathbb{R}^n$. Also $\mathbf{p} \,.\, \mathbf{x} \leq \mathbf{p}' \,.\, \mathbf{x} = \gamma(\mathbf{x})$ for all $\mathbf{p} \in \Delta$ and $\mathbf{p}' \in \mu(\mathbf{x})$.

Proof

Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{p} \in \Delta$, and let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. Then $p_i \geq 0$ for $i = 1, 2, \ldots, n$, and

$$\gamma(\mathbf{x}) = \max(x_1, x_2, \ldots, x_n).$$

Let $I(\mathbf{x})$ denote those integers i between 1 and n for which $x_i = \gamma(\mathbf{x})$. Now $0 \leq p_i \leq 1$ for $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} p_i = 1$. It follows that

$$\mathbf{p} \,.\, \mathbf{x} = \sum_{i=1}^{n} p_i x_i \leq \gamma(\mathbf{x}) \sum_{i=1}^{n} p_i = \gamma(\mathbf{x}).$$

Moreover if $x_i < \gamma(\mathbf{x})$ and $p_i > 0$ for some integer i between 1 and n then $\mathbf{p} \,.\, \mathbf{x} < \gamma(\mathbf{x})$. It follows that $\mathbf{p} \,.\, \mathbf{x} \leq \gamma(\mathbf{x})$ for all $\mathbf{p} \in \Delta$, and $\mathbf{p} \,.\, \mathbf{x} = \gamma(\mathbf{x})$ if and only if $p_i = 0$ for those integers i between 1 and n for which $x_i < \gamma(\mathbf{x})$. It follows that $\mathbf{p} \,.\, \mathbf{x} = \gamma(\mathbf{x})$ if and only if $p_i = 0$ for those integers i between 1 and n for which $i \notin I(\mathbf{x})$.

Therefore

$$\begin{aligned}\mu(\mathbf{x}) &= \{(p_1, p_2, \ldots, p_n) \in \Delta : p_i = 0 \text{ whenever } (\mathbf{x})_i < \gamma(\mathbf{x})\} \\ &= \{(p_1, p_2, \ldots, p_n) \in \Delta : p_i = 0 \text{ whenever } i \notin I(\mathbf{x})\}.\end{aligned}$$

It follows that, for all $\mathbf{x} \in \mathbb{R}$, the set $\mu(\mathbf{x})$ is a closed subset of the simplex Δ, and is therefore a compact set. It is clearly non-empty and convex. Also

$$\mathbf{p} \,.\, \mathbf{x} \leq \gamma(\mathbf{x}) = \mathbf{p}' \,.\, \mathbf{x}$$

for all $\mathbf{p} \in \Delta$ and $\mathbf{p}' \in \mu(\mathbf{x})$.

Let $\mathbf{x}' \in \mathbb{R}^n$, and let $\mathbf{x}' = (x_1', x_2', \ldots, x_n')$. If $i \in I(\mathbf{x}')$ then $x_i' = \gamma(\mathbf{x}')$, and if $i \notin I(\mathbf{x}')$ then $x_i' < \gamma(\mathbf{x}')$. There then exists a real number θ such that $\theta < \gamma(\mathbf{x}')$ and $x_i' < \theta$ whenever $i \notin I(\mathbf{x}')$. Let N be the subset of $\mathbb{R}^n$ consisting of those elements $(x_1, x_2, \ldots, x_n)$ of $\mathbb{R}^n$ satisfying the following two conditions:

- $x_i > \theta$ if $i \in I(\mathbf{x}')$;
- $x_i < \theta$ if $i \notin I(\mathbf{x}')$.

Then N is open in $\mathbb{R}^n$ and $\mathbf{x}' \in N$. Moreover $I(\mathbf{x}) \subset I(\mathbf{x}')$ for all $\mathbf{x} \in N$, and therefore $\mu(\mathbf{x}) \subset \mu(\mathbf{x}')$ for all $\mathbf{x} \in N$. Thus if V is open in $\mathbb{R}^n$ and if $\mu(\mathbf{x}') \subset V$ then $\mu(\mathbf{x}) \subset V$ for all $\mathbf{x} \in N$. We conclude from this that the correspondence $\mu\colon \mathbb{R}^n \to \Delta$ is upper hemicontinuous on $\mathbb{R}$. This completes the proof. ∎

Remark

Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ is the standard basis of $\mathbb{R}^n$, defined so that, for each integer i between 1 and n, the ith component of $\mathbf{e}_i$ is equal to 1 and the other components are zero. Then the simplex Δ is an $(n-1)$-dimensional simplex with vertices $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, and, for each $\mathbf{x} \in \mathbb{R}^n$, the subset $\mu(\mathbf{x})$ of Δ is the face of the simplex Δ spanned by those vertices $\mathbf{e}_i$ of Δ for which $(\mathbf{x})_i = \gamma(\mathbf{x})$, where $\gamma(\mathbf{x})$ denotes the maximum value of the components of the vector $\mathbf{x}$.

8.4. Consumer Preferences

We next discuss how each household sets out to determine its purchase requirements.

We suppose that the preferences of household h are represented by a *utility function* $u_h : \mathbb{R}^n_+ \to \mathbb{R}$ that is continuous, strictly increasing and quasiconcave. Such a utility function therefore satisfies the following conditions:

- the function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is *continuous*;
- the function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is *strictly increasing*, and thus if $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n_+$ satisfy $\mathbf{x} \leq \mathbf{x}'$ and $\mathbf{x} \neq \mathbf{x}'$ then $u(\mathbf{x}) < u(\mathbf{x}')$;
- the function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is *quasiconcave*, and thus

$$u((1-t)\mathbf{x} + t\mathbf{x}') \geq \min\Big(u(\mathbf{x}), u(\mathbf{x}')\Big)$$

for all $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n_+$ and $t \in [0, 1]$.

Proposition 8.4

Let $u\colon X \to \mathbb{R}$ be a function defined on a closed convex subset X of $\mathbb{R}^n$ that is continuous, strictly increasing and quasiconcave, let $\mathbf{p}$ be a non-zero non-negative price vector in $\mathbb{R}^n$, let w be a positive real number, let

$$B(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0} \text{ and } \mathbf{p} \,.\, \mathbf{x} \leq w\}$$

and let $\mathbf{x}^* \in B(\mathbf{p}, w)$. Suppose that there exists some open neighbourhood N of $\mathbf{x}^*$ in $\mathbb{R}^n_+$ with the property that $u(\mathbf{x}) \leq u(\mathbf{x}^*)$ for all $\mathbf{x} \in B(\mathbf{p}, w) \cap N$. Then $\mathbf{p} \,.\, \mathbf{x}^* = w$ and $u(\mathbf{x}) \leq u(\mathbf{x}^*)$ for all $\mathbf{x} \in B(\mathbf{p}, w)$.

Proof

Suppose that it were the case that $\mathbf{p} \,.\, \mathbf{x}^* < w$. Then it would be possible to find $\mathbf{x} \in N$ satisfying $\mathbf{x} >> \mathbf{x}^*$ and $\mathbf{p} \,.\, \mathbf{x} < w$. Then $\mathbf{x} \in B(\mathbf{p}, w) \cap N$. The strictly increasing property of the utility function u would then ensure that $u(\mathbf{x}) > u(\mathbf{x}^*)$. But this would contradict that assumption that the maximum of the utility function u on $B(\mathbf{p}, w) \cap N$ is attained at $\mathbf{x}^*$.

Next suppose that there were to exist in the set $B(\mathbf{p}, w)$ a commodity bundle $\mathbf{x}'$ for which $u(\mathbf{x}') > u(\mathbf{x}^*)$. It would then follow from the continuity of the utility function u that the value of utility function u would exceed $u(\mathbf{x}^*)$ throughout some open ball of positive radius centred on $\mathbf{x}'$. Now $w > 0$, and therefore $B(\mathbf{p}, w)$ has non-empty interior. Moreover every open ball of positive radius about an element of $B(\mathbf{p}, w)$ would intersect the interior of this set. It follows that there would exist a commodity bundle $\mathbf{x}''$ in the interior of $B(\mathbf{p}, w)$ lying sufficiently close to $\mathbf{x}'$ to ensure that $u(\mathbf{x}'') > u(\mathbf{x}^*)$ and $\mathbf{p} \,.\, \mathbf{x}'' < w$. The quasiconcavity of the utility function would ensure that the utility function u would take values no less than $u(\mathbf{x}^*)$ along the line segment joining the commodity bundles $\mathbf{x}^*$ and $\mathbf{x}''$. Moreover this line segment would be wholly contained within the convex set $B(\mathbf{p}, w)$.

Now $\mathbf{x}^* \in N$. Therefore there would then exist a commodity bundle $\mathbf{x}'''$ on the line segment joining $\mathbf{x}^*$ and $\mathbf{x}''$ that was distinct from $\mathbf{x}^*$ but was close enough to $\mathbf{x}^*$ to ensure that $\mathbf{x}''' \in N$. Then $u(\mathbf{x}''') \geq u(\mathbf{x}^*)$ and $\mathbf{p} \,.\, \mathbf{x}''' < w$. There would then exist a commodity bundle $\mathbf{x}$ satisfying $\mathbf{x} \geq \mathbf{x}'''$ and $\mathbf{x} \neq \mathbf{x}'''$ for which $\mathbf{x} \in N$ and $\mathbf{p} \,.\, \mathbf{x} < w$. Then $\mathbf{x} \in B(\mathbf{p}, w) \cap N$ and

$$u(\mathbf{x}) > u(\mathbf{x}''') \geq u(\mathbf{x}^*),$$

contradicting the fact that the function u achieves is maximum value on $B(\mathbf{p}, w) \cap N$ at $\mathbf{x}^*$. We conclude therefore that the maximum value of the utility function u on $B(\mathbf{p}, w)$ is attained at the point $\mathbf{x}^*$, as required. ∎

Let Γ^n be the set of price-wealth pairs $(\mathbf{p}, w)$ for which $\mathbf{p} \gg \mathbf{0}$ and $w > 0$, so that

$$\Gamma^n = \{(\mathbf{p}, w) \in \mathbb{R}^n \times \mathbb{R} : \mathbf{p} \gg \mathbf{0} \text{ and } w > 0\}.$$

Then the closure $\overline{\Gamma}^n$ of Γ^n in $\mathbb{R}^n \times \mathbb{R}$ satisfies

$$\overline{\Gamma}^n = \mathbb{R}^n_+ \times \mathbb{R}_+ = \{(\mathbf{p}, w) \in \mathbb{R}^n \times \mathbb{R} : \mathbf{p} \geq \mathbf{0} \text{ and } w \geq 0\}.$$

Let $B : \overline{\Gamma}^n \rightrightarrows \mathbb{R}^n$ denote the budget correspondence on $\overline{\Gamma}^n$, where

$$B(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0} \text{ and } \mathbf{p} \,.\, \mathbf{x} \leq w\}$$

for all $(\mathbf{p}, w) \in \overline{\Gamma}^n$.

Let $u\colon \overline{\Gamma}^n \to \mathbb{R}$ be a utility function for a given consumer, defined over $\overline{\Gamma}^n$, that is continuous, strictly increasing and quasiconcave. Then the utility function u and the budget correspondence B together determine a single valued function $V\colon \Gamma^n \to \mathbb{R}$ and a correspondence $\xi\colon \Gamma^n \rightrightarrows \mathbb{R}^n_+$, where

$$V(\mathbf{p}, w) = \sup\{u(\mathbf{x}) : \mathbf{x} \in B(\mathbf{p}, w)\}$$

and

$$\xi(\mathbf{p}, w) = \sup\{\mathbf{x} \in B(\mathbf{p}, w) : u(\mathbf{x}) = V(\mathbf{p}, w)\}.$$

The function $V : \Gamma^n \to \mathbb{R}$ is referred to as the *indirect utility function* for the given consumer, and the correspondence $\xi : \Gamma^n \rightrightarrows \mathbb{R}^n_+$ is referred to as the *demand correspondence* for that consumer. The value of $V(\mathbf{p}, w)$ is the maximum utility that the consumer by purchasing a bundle of commodities that is affordable for that consumer when the commodity prices are given by the price vector $\mathbf{p}$ and the wealth of the consumer is represented by the non-negative real number w. The demand correspondence $\xi : \Gamma^n \rightrightarrows \mathbb{R}^n_+$ associates to a price-wealth pair $(\mathbf{p}, w)$ the set consisting of those bundles of commodities that are most desirable for the consumer with wealth w, subject to being affordable at prices $\mathbf{p}$.

Proposition 8.5

In an exchange economy with n commodities, suppose that the preferences of a given consumer are represented by a utility function $u\colon \overline{\Gamma}^n \to \mathbb{R}$, defined over the line $\overline{\Gamma}^n$ of price-wealth pairs, that is continuous, strictly increasing and quasiconcave. Then the resulting indirect utility function $V\colon \Gamma^n \to \mathbb{R}$ is continuous on the set Γ^n of price-wealth pairs $(\mathbf{p}, w)$ for which $\mathbf{p} \gg \mathbf{0}$ and $w > 0$, and the demand correspondence $\xi\colon \Gamma^n \rightrightarrows \mathbb{R}^n_+$ is upper hemicontinuous and maps each price-wealth pair $(\mathbf{p}, w)$ in Γ^n to a non-empty compact convex subset of $\mathbb{R}^n_+$.

Proof

Proposition 8.2 ensures that the budget correspondence $\xi : \Gamma^n \rightrightarrows \mathbb{R}^n_+$ is both upper hemicontinuous and lower hemicontinuous on Γ^n. Moreover $\xi(\mathbf{p}, w)$ is a non-empty compact subset of $\mathbb{R}^n_+$ for all $(\mathbf{p}, w) \in \Gamma^n$. It follows from a direct application of Berge's Maximum Theorem (Theorem 2.23) that the indirect utility function is continuous and the demand correspondence is upper hemicontinuous and maps each price-wealth pair in Γ^n to a non-empty compact subset of $\mathbb{R}^n_+$. The convexity of $B(\mathbf{p}, w)$ and the quasiconcavity of the utility function u then ensure that $\xi(\mathbf{p}, w)$ is convex for all price-wealth pairs $(\mathbf{p}, w)$ in Γ^n. ∎

Let $\mathbf{c}$ be an element of $\mathbb{R}^n$ satisfying $\mathbf{c} \gg \mathbf{0}$. In what follows we restrict consumer choice to those bundles of commodities that, for a particular price-wealth pair $(\mathbf{p}, w)$, are both affordable and subject to the availability constraint $\mathbf{0} \leq \mathbf{x} \leq \mathbf{c}$. Thus let $B_{\mathbf{c}} : \overline{\Gamma}^n \rightrightarrows \mathbb{R}^n$ denote the budget correspondence on $\overline{\Gamma}^n$ when availability is constrained in this fashion, so that

$$B_{\mathbf{c}}(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{0} \leq \mathbf{x} \leq \mathbf{c} \text{ and } \mathbf{p} \,.\, \mathbf{x} \leq w\}$$

for all $(\mathbf{p}, w) \in \overline{\Gamma}^n$. It follows from Proposition 8.1 that the correspondence $B_{\mathbf{c}} : \overline{\Gamma}^n \rightrightarrows \mathbb{R}^n$ is both upper hemicontinuous and lower hemicontinuous throughout the set $\hat{\Gamma}^n$ defined so that

$$\hat{\Gamma}^n = \{(\mathbf{p}, w) \in \mathbb{R}^n \times \mathbb{R} : \mathbf{p} \geq \mathbf{0} \text{ and } w > 0\}.$$

We still require the utility function $u\colon \overline{\Gamma}^n \to \mathbb{R}$ for the given consumer to be continuous, strictly increasing and quasiconcave. Then the utility function u and the modified budget correspondence $B_{\mathbf{c}}$ together determine a single valued function $\hat{V}_{\mathbf{c}}\colon \Gamma^n \to \mathbb{R}$ and a correspondence $\hat{\xi}_{\mathbf{c}}\colon \Gamma^n \rightrightarrows \mathbb{R}^n_+$, where

$$\hat{V}_{\mathbf{c}}(\mathbf{p}, w) = \sup\{u(\mathbf{x}) : \mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, w)\}$$

and

$$\hat{\xi}_{\mathbf{c}}(\mathbf{p}, w) = \sup\{\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, w) : u(\mathbf{x}) = \hat{V}_{\mathbf{c}}\mathbf{p}, w)\}.$$

Proposition 8.6

In an exchange economy with n commodities, suppose that the preferences of a given consumer are represented by a utility function $u\colon \overline{\Gamma}^n \to \mathbb{R}$, defined over the line $\overline{\Gamma}^n$ of price-wealth pairs, that is continuous, strictly increasing and quasiconcave. Let $\mathbf{c} \in \mathbb{R}^n$ satisfy $\mathbf{c} \gg \mathbf{0}$, and let the consumer be required to select from bundles $\mathbf{x}$ of commodities, represented by non-negative n-dimensional vectors, that, for prices and wealth given by the price-wealth pair $(\mathbf{p}, w)$, satisfy both the budget constraint $\mathbf{p} \,.\, \mathbf{x} \leq w$ and the availability constraint $\mathbf{0} \leq \mathbf{x} \leq \mathbf{c}$. Then the resulting indirect utility function $\hat{V}_{\mathbf{c}}\colon \hat{\Gamma}^n \to \mathbb{R}$ is continuous on the set $\hat{\Gamma}^n$ of price-wealth pairs $(\mathbf{p}, w)$ for which $w > 0$, and the demand correspondence $\hat{\xi}_{\mathbf{c}}\colon \hat{\Gamma}^n \rightrightarrows \mathbb{R}^n_+$ is upper hemicontinuous and maps each price-wealth pair $(\mathbf{p}, w)$ in $\hat{\Gamma}^n$ to a non-empty compact convex subset of $\mathbb{R}^n_+$.

Proof

Proposition 8.1 ensures that the budget correspondence $\hat{\xi}_{\mathbf{c}} : \hat{\Gamma}^n \rightrightarrows \mathbb{R}^n_+$ is both upper hemicontinuous and lower hemicontinuous on $\hat{\Gamma}^n$. Moreover $\hat{\xi}_{\mathbf{c}}(\mathbf{p}, w)$ is a non-empty compact subset of $\mathbb{R}^n_+$ for all $(\mathbf{p}, w) \in \hat{\Gamma}^n$. It follows from a direct application of Berge's Maximum Theorem (Theorem 2.23) that the indirect utility function is continuous and the demand correspondence is upper hemicontinuous and maps each price-wealth pair in $\hat{\Gamma}^n$ to a non-empty compact subset of $\mathbb{R}^n_+$. The convexity of $B_{\mathbf{c}}(\mathbf{p}, w)$ and the quasiconcavity of the utility function u then ensure that $\hat{\xi}_{\mathbf{c}}(\mathbf{p}, w)$ is convex for all price-wealth pairs $(\mathbf{p}, w)$ in $\hat{\Gamma}^n$. ∎

8.6. Addition of Compact-Valued Correspondences

We discuss now the addition of vector-valued correspondences.

Suppose that we have m correspondences $\xi_1, \xi_2, \ldots, \xi_m$ defined over some subset Ω of a Euclidean space, and mapping points of Ω to subsets of a Euclidean space $\mathbb{R}^n$. Let $\sum_{h=1}^{n} \xi_h$ denote the correspondence ξ defined such that

$$\xi(\mathbf{p}) = \left\{ \sum_{h=1}^{m} \mathbf{x}_h : \mathbf{x}_h \in \xi_h(\mathbf{p}) \right\}.$$

Proposition 8.7

Let $\xi_1, \xi_2, \ldots, \xi_m$ be correspondences defined over some subset Ω of a Euclidean space, and mapping points of that space to non-empty compact subsets of the n-dimensional Euclidean space $\mathbb{R}^n$. Suppose that these correspondences are upper hemicontinuous. Then the sum $\sum_{h=1}^{m} \xi_h$ of those correspondences is an upper hemicontinuous correspondence mapping points of Ω to non-empty compact subsets of $\mathbb{R}^n$.

Proof

Let $\xi\colon \Omega \rightrightarrows \mathbb{R}^n$ be the correspondence that is the sum $\sum_{h=1}^{m} \xi_h$ of the correspondences $\xi_1, \xi_2, \ldots, \xi_m$. Now, for each $\mathbf{p} \in \Omega$, the set $\xi(\mathbf{p})$ is the image of the Cartesian product

$$\xi_1(\mathbf{p}) \times \xi_2(\mathbf{p}) \times \cdots \times \xi_m(\mathbf{p})$$

under the continuous function that maps each m-tuple of vectors in $\mathbb{R}^n$ to the sum of its components. Moreover $\xi_h(\mathbf{p})$ is, by assumption, a non-empty compact subset of $\mathbb{R}^n$, and any Cartesian product of non-empty compact sets is non-empty and compact, and the image of a non-empty compact set under a continuous map is non-empty and compact. We conclude therefore that $\xi(\mathbf{p})$ is a non-empty compact subset of $\mathbb{R}^n$ for all $\mathbf{p} \in \Omega$.

We can therefore apply the "ϵ–δ" criterion for upper hemicontinuity of compact-valued correspondences established by Proposition 2.16. Given any subset K of $\mathbb{R}^n$, and given any positive real number r, we denote by $B(K, r)$ the subset of $\mathbb{R}^n$ that lie within a distance less than r of a point of K.

Let $\mathbf{p} \in \Omega$, and let some strictly positive real number ε be given. It follows from Proposition 2.16 that, for each integer h between 1 and m, there exists some open neighbourhood N_h of $\mathbf{p}$ in Ω such that $\xi_h(\mathbf{p}') \subset B(\xi_h(\mathbf{p}), \varepsilon/m)$ for all $\mathbf{p}' \in N_h$. Let N be the open neighbourhood of $\mathbf{p}$ in Ω that is the intersection of $N_1, N_2, \ldots, N_h$. Then a straightforward application of the triangle inequality ensures that $\xi(\mathbf{p}') \subset B(\xi(\mathbf{p}), \varepsilon)$ for all $\mathbf{p}' \in N$. It then follows from Proposition 2.16 that the correspondence $\xi : \Omega \rightrightarrows \mathbb{R}^n$ is upper hemicontinuous at $\mathbf{p}$. Its values are non-empty compact subsets of $\mathbb{R}^n$. The result follows. ∎

8.7. Aggregate Supply and Demand in an Exchange Economy

We now consider the properties of aggregate supply and demand in a pure exchange economy, or market, in which n commodities are traded between m households. Each household is provided with an initial endowment of commodities. The initial endowment of household h is then represented by an n-dimensional vector $\overline{\mathbf{x}}_h$ whose ith component specifies the initial endowment (relative to some appropriate unit) of the ith commodity traded in the market. The *aggregate supply* is then represented by a vector $\mathbf{s}$ that is the sum of the initial endowment vectors of all households. Thus

$$\mathbf{s} = \sum_{h=1}^{m} \overline{\mathbf{x}}_h.$$

We restrict our attention to the situation in which $\bar{\mathbf{x}}_h \gg \mathbf{0}$ for $h = 1, 2, \ldots, m$. This restriction requires that each household be given an initial endowment of every commodity traded in the market. This ensures that, provided all commodity prices are non-negative, and at least one commodity price is strictly positive, then initial endowment of each household has strictly positive value, and thus each household has wealth to enable it to trade in the market. Within the mathematical model, this ensures that the demand correspondences of each household are lower hemicontinuous (see Proposition 8.6). The requirement that $\bar{\mathbf{x}}_h \gg \mathbf{0}$ for all households h also ensures that $\mathbf{s} \gg \mathbf{0}$.

The prices of the commodities are encoded in a price vector $\mathbf{p}$ whose components are non-negative real numbers. The ith component of this price vector $\mathbf{p}$ specifies the price of a unit of the ith commodity. We suppose that the price of at least one commodity is non-zero.

Each household seeks to trade its initial endowment for the bundle of commodities that provides it with maximum utility within the budget constraint that requires the value of purchased commodities to be less than or equal to the value of the initial endowment traded in. A consequence of this is that the demand of the ith consumers at prices $\lambda\mathbf{p}$ is identical to the demand at prices $\mathbf{p}$ for all positive real numbers λ. Indeed the bundles of commodities available to household h at prices $\mathbf{p}$ are those represented by vectors $\mathbf{x}_h$ satisfying the budget constraint

$$\mathbf{p} \,.\, \mathbf{x}_h \leq \mathbf{p} \,.\, \overline{\mathbf{x}_h}.$$

It follows that the price vector $\mathbf{p}$ may be replaced by the scalar multiple $\lambda\mathbf{p}$ for any positive real number λ without altering the set of bundles of commodities that the households individually can afford.

It is appropriate therefore to normalize prices in some fashion so that all non-zero non-negative price vectors can be expressed uniquely as a scalar multiple of a normalized price vector. We adopt the normalization scheme in which the sum of the prices of the commodities is required to be equal to one.

Definition

A price vector $\mathbf{p}$ (with non-negative components) is said to be *normalized* if $\sum_{i=1}^{n}(\mathbf{p})_i = 1$.

Normalized price vectors are therefore represented by the points of the *price simplex* Δ, where

$$\Delta = \left\{ \mathbf{p} \in \mathbb{R}^n : \mathbf{p} \geq \mathbf{0} \text{ and } \sum_{i=1}^{n}(\mathbf{p})_i = 1 \right\}.$$

We suppose that the demand for each household is determined by the appropriate budget constraint and by a utility function that is continuous, strictly increasing and quasiconcave. This being the case, if the price of the ith commodity is zero, with the result that the commodity is free, then every household can afford to acquire unlimited quantities of it, and because the utility functions are required to be strictly increasing, demand for that commodity cannot be satisfied: the households have an *insatiable* appetite for free commodities.

This might suggest constraining price variation to price vectors whose components are strictly positive. However the fixed point theorems that are used to prove the existence of equilibria in which supply balances demand apply to functions or correspondences defined on compact sets. Therefore the correspondences that specify the demands of the consumers as prices vary should assign a non-empty compact set not only to the normalized price vectors in the interior of the price simplex Δ but also to the price vectors on the boundary of the price simplex.

Accordingly we impose an additional constraint on the purchases of the households. In addition to the budget constraint, we place limits on the amount of each commodity in the bundles available to the households. These limits may be specified by a fixed positive vector $\mathbf{c}$. Accordingly we require that $\mathbf{c} >> \mathbf{0}$ and that, for each integer h between 1 and m, household h selects a bundle at prices $\mathbf{p}$ to maximize utility amongst bundles $\mathbf{x}$ that satisfy both the budget constraint

$$\mathbf{p} \,.\, \mathbf{x} \leq \mathbf{p} \,.\, \overline{\mathbf{x}_h}$$

and the additional constraint $\mathbf{0} \leq \mathbf{x} \leq \mathbf{c}$.

We denote by $B_{\mathbf{c},h}(\mathbf{p})$ the set of bundles of commodities from which household $\mathbf{x}$ makes its selection. Accordingly, with this additional constraint, for each price vector $\mathbf{p}$ belonging to the price simplex Δ, household h selects the bundle of commodities that maximizes its utility function u_h over the non-empty compact set $B_{\mathbf{c},h}(\mathbf{p})$, where

$$B_{\mathbf{c},h}(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{0} \leq \mathbf{x} \leq \mathbf{c} \text{ and } \mathbf{p} \,.\, \mathbf{x} \leq \mathbf{p} \,.\, \bar{\mathbf{x}}_h\}.$$

We denote the set of bundles of commodities that maximizes utility for household h under these constraints by $\hat{\xi}_{\mathbf{c},h}(\mathbf{p})$. We obtain in this fashion a correspondence $\hat{\xi}_{\mathbf{c},h} \colon \Delta \rightrightarrows \mathbb{R}^n_+$ that determines the set of bundles maximizing utility for household h at prices $\mathbf{p}$, subject to the budget constraint and the additional constraint that available bundles of commodities by bounded above by the positive vector $\mathbf{c}$.

Proposition 8.8

Suppose that, in a model of an exchange economy with n goods and m households, every household receives a strictly positive initial endowment of every commodity, so that the initial endowment vector $\overline{\mathbf{x}}_h$ of household h satisfies $\overline{\mathbf{x}}_h >> \mathbf{0}$ for $h = 1, 2, \ldots, m$. Suppose also that the preferences of household h are determined by a utility function u_h that is continuous, strictly increasing and quasiconcave. Then, for each household, and for each $\mathbf{c} \in \mathbb{R}^n$ satisfying $\mathbf{c} >> \mathbf{0}$ the demand correspondences $\hat{\xi}_{\mathbf{c},h} \colon \Delta \rightrightarrows \mathbb{R}^n_+$ is upper hemicontinuous on the set Δ of normalized price vectors, and maps each normalized price vector $\mathbf{p}$ to a non-empty compact convex subset $\hat{\xi}_{\mathbf{c},h}(\mathbf{p})$ of $\mathbb{R}^n_+$ that consists of those bundles $\mathbf{x}$ of commodities that maximize utility for household h at prices $\mathbf{p}$ subject to both the affordability constraint $\mathbf{p} \,.\, \mathbf{x} \leq \mathbf{p} \,.\, \overline{\mathbf{x}}$ and the constraint $\mathbf{0} \leq \mathbf{x} \leq \mathbf{c}$.

Proof

Let

$$\hat{\Gamma}^n = \{(\mathbf{p}, w) \in \mathbb{R}^n \times \mathbb{R} : \mathbf{p} \geq \mathbf{0} \text{ and } w > 0\},$$

and, for all $(\mathbf{p}, w) \in \hat{\Gamma}^n$, let us denote by $\hat{\xi}'_{\mathbf{c},h}(\mathbf{p}, w)$ the demand of household h at prices $\mathbf{p}$, when the household has wealth h, where $\hat{\xi}'_{\mathbf{c},h}(\mathbf{p}, w)$ is the set of bundles $\mathbf{x}$ of commodities maximizing utility for household h at prices $\mathbf{p}$ subject to the constraints $\mathbf{p} \,.\, \mathbf{x} \leq w$ and $\mathbf{0} \leq \mathbf{x} \leq \mathbf{c}$. It follows from Proposition 8.6 that this correspondence $\hat{\xi}'_{\mathbf{c},h}$ is upper hemicontinuous on $\hat{\Gamma}^n$, and moreover it maps each price-wealth pair in $\hat{\Gamma}^n$ to a non-empty compact convex subset of $\mathbb{R}^n$. Let $\psi_h \colon \Delta \to \hat{\Gamma}^n$ be the continuous mapping that sends $\mathbf{p} \in \Delta$ to $(\mathbf{p}, \mathbf{p} \,.\, \overline{\mathbf{x}}_h)$. Then $\hat{\xi}_{\mathbf{c},h} = \hat{\xi}'_{\mathbf{c},h} \circ \psi_h$: in other words,

$$\hat{\xi}_{\mathbf{c},h}(\mathbf{p}) = \hat{\xi}'_{\mathbf{c},h}(\mathbf{p}, \mathbf{p} \,.\, \overline{\mathbf{x}}_h) = \hat{\xi}'_{\mathbf{c},h}(\psi_h(\mathbf{p})).$$

It follows that the correspondence $\hat{\xi}_{\mathbf{c},h}\colon \Delta \to \mathbb{R}^n_+$ is the composition of a continuous mapping followed by an upper hemicontinuous correspondence. Any correspondence of this type must itself be an upper hemicontinuous correspondence. Moreover the images of normalized price vectors in Δ are subsets of $\mathbb{R}^n_+$ that have the required properties. ∎

Now, because the demand correspondences $\hat{\xi}_{\mathbf{c},h} \colon \Delta \rightrightarrows \mathbb{R}^n_+$ for the individual households assign to each normalized price vector $\mathfrak{l}$ in the price simplex a non-empty compact subset of $\mathbb{R}^n_+$, these demand correspondences may be added together to obtain an correspondence $\hat{\xi}_{\mathbf{c}} \colon \Delta \rightrightarrows \mathbb{R}^n_+$ that represents aggregate demand from the entire economy for each normalized price vector belonging to the price simplex Δ.

An immediate application of Proposition 8.7 yields the following result.

Corollary 8.9

Suppose that, in a model of an exchange economy with n goods and m households, every household receives a strictly positive initial endowment of every commodity, so that the initial endowment vector $\overline{\mathbf{x}}_h$ of household h satisfies $\overline{\mathbf{x}}_h \gg \mathbf{0}$ for $h = 1, 2, \ldots, m$. Suppose also that the preferences of household h are determined by a utility function u_h that is continuous, strictly increasing and quasiconcave. Let Δ denote the simplex whose elements are the normalized price vectors, and, for each $\mathbf{c} \in \mathbb{R}^n$ satisfying $\mathbf{c} \gg \mathbf{0}$, let the demand correspondence $\hat{\xi}_{\mathbf{c},h} \colon \Delta \rightrightarrows \mathbb{R}^n_+$ be defined as specified in the statement of Proposition 8.8, let $\mathbf{s} = \sum_{h=1}^{m} \overline{\mathbf{x}}_h$, and let $\hat{\xi}_{\mathbf{c}} = \sum_{h=1}^{m} \hat{\xi}_{\mathbf{c},h}$. Then the the aggregate demand correspondence $\hat{\xi}_{\mathbf{c}} \colon \Delta \rightrightarrows \mathbb{R}^n_+$ is upper hemicontinuous on Δ, and maps each element of Δ to a non-empty compact convex subset of $\mathbb{R}^n_+$. Moreover $\mathbf{p} \,.\, \mathbf{x} \leq \mathbf{p} \,.\, \mathbf{s}$ for all $\mathbf{p} \in \Delta$ and $\mathbf{x} \in \hat{\xi}_{\mathbf{c}}(\mathbf{p})$.

8.8. Walrasian Equilibria in Exchange Economies

Theorem 8.10

Let n be a positive integer, let

$$\Delta = \left\{ \mathbf{p} \in \mathbb{R}^n : \mathbf{p} \geq \mathbf{0} \text{ and } \sum_{i=1}^{n} (\mathbf{p})_i = 1 \right\},$$

let K be a compact subset of $\mathbb{R}^n$, and let $\zeta : \Delta \rightrightarrows K$ be an upper hemicontinuous correspondence mapping points of the simplex Δ to non-empty closed convex subsets of K. Suppose that $\mathbf{p} \cdot \mathbf{z} \leq 0$ for all $\mathbf{p} \in \Delta$ and $\mathbf{z} \in \zeta(\mathbf{p})$. Then there exist $\mathbf{p}^* \in \Delta$ and $\mathbf{z}^* \in \zeta(\mathbf{p}^*)$ for which $\mathbf{z}^* \leq \mathbf{0}$.

Proof

The set K is clearly non-empty. We may assume, without loss of generality, that the set K is both compact and convex, because if K were not convex, then it could be replaced by a compact convex set containing it.

Let $\gamma\colon \mathbb{R}^n \to \mathbb{R}$ be the function defined so that, for each $\mathbf{x} \in \mathbb{R}^n$, $\gamma(\mathbf{x})$ is the maximum of the components of $\mathbf{x}$, and let $\mu\colon \mathbb{R}^n \rightrightarrows \Delta$ be the correspondence defined such that

$$\mu(\mathbf{x}) = \{\mathbf{p} \in \Delta : \mathbf{p} \,.\, \mathbf{x} = \gamma(\mathbf{x})\}.$$

It was shown in Proposition 8.3 that the correspondence $\mu\colon \mathbb{R}^n \rightrightarrows \Delta$ is upper hemicontinuous, and $\mu(\mathbf{x})$ is a non-empty compact convex subset of Δ for all $\mathbf{x} \in \mathbb{R}^n$. Moreover $\mathbf{p} \,.\, \mathbf{x} \leq \mathbf{p}' \,.\, \mathbf{x} = \gamma(\mathbf{x})$ for all $\mathbf{p} \in \Delta$ and $\mathbf{p}' \in \mu(\mathbf{x})$. (The upper hemicontinuity of μ also follows directly on applying Berge's Maximum Theorem, which is Theorem 2.23 above.)

Let $\Phi\colon \Delta \times K \rightrightarrows \Delta \times K$ be the correspondence defined such that

$$\Phi(\mathbf{p}, \mathbf{z}) = \mu(\mathbf{z}) \times \zeta(\mathbf{p}) = \{(\mathbf{p}', \mathbf{z}') : \mathbf{p}' \in \mu(\mathbf{z}) \text{ and } \mathbf{z}' \in \zeta(\mathbf{p}))\}$$

for all $\mathbf{p} \in \Delta$ and $\mathbf{z} \in K$. The correspondences μ and ζ are upper hemicontinuous and closed-valued, and every upper hemicontinuous closed-valued correspondence has a closed graph (Proposition 2.11). It follows that the correspondence Φ has closed graph. Moreover $\Phi(\mathbf{p}, \mathbf{z})$ is a non-empty closed convex subset of the compact convex set $\Delta \times K$ for all $\mathbf{p} \in \Delta$ and $\mathbf{z} \in K$. It follows from the Kakutani Fixed Point Theorem (Theorem 5.4) that there exists $(\mathbf{p}^*, \mathbf{z}^*) \in \Delta \times K$ for which $(\mathbf{p}^*, \mathbf{z}^*) \in \Phi(\mathbf{p}^*, \mathbf{z}^*)$. Then $\mathbf{p}^* \in \mu(\mathbf{z}^*)$ and $\mathbf{z}^* \in \zeta(\mathbf{p}^*)$.

Now the conditions of the theorem require that $\mathbf{p}^* \,.\, \mathbf{z} \leq 0$ for all $\mathbf{z} \in \zeta(\mathbf{p}^*)$. Combining this inequality with the definition of the correspondence μ, and noting that $\mathbf{p}^* \in \mu(\mathbf{z}^*)$ and $\mathbf{z}^* \in \zeta(\mathbf{p}^*)$, we find that

$$\mathbf{p} \,.\, \mathbf{z}^* \leq \mathbf{p}^* \,.\, \mathbf{z}^* \leq 0$$

for all $\mathbf{p} \in \Delta$. Applying this result when $\mathbf{p}$ is the vertex of Δ whose ith component is equal to 1 and whose other components are zero, we find that $(\mathbf{z}^*)_i \leq 0$ for $i = 1, 2, \ldots, n$, and thus $\mathbf{z}^* \leq \mathbf{0}$, as required. ∎

Remark

For Theorem 8.10, and its proof, see Gérard Debreu, *Theory of Value* (Cowles Foundation Monograph 17, 1959), Section 5.6. In his notes on Chapter 5 of that monograph, Debreu notes that the result was obtained and published independently by D. Gale (published 1955) and H. Nikaido (published 1956). Debreu also thanks A. Borel, P. Samuel and A. Weil for conversations that he had with them on an early formulation of the result.

Theorem 8.11

Suppose that, in a model of an exchange economy with n goods and m households, every household receives a strictly positive initial endowment of every commodity, so that the initial endowment vector $\overline{\mathbf{x}}_h$ of household h satisfies $\overline{\mathbf{x}}_h \gg \mathbf{0}$ for $h = 1, 2, \ldots, m$. Suppose also that the preferences of household h are determined by a utility function u_h that is continuous, strictly increasing and quasiconcave. Then there exists a normalized price vector $\mathbf{p}^*$ satisfying $\mathbf{p}^* \gg \mathbf{0}$ and, for each household h, a corresponding bundle $\mathbf{x}_h^*$ of commodities that maximizes utility for that household subject to the affordability constraint $\mathbf{p} \,.\, \mathbf{x}_h^* \leq \mathbf{p} \,.\, \overline{\mathbf{x}}_h$, so that the total supply is redistributed amongst the households, and thus

$$\sum_{h=1}^{m} \mathbf{x}_h^* = \sum_{h=1}^{m} \overline{\mathbf{x}}_h.$$

Proof

Let $\mathbf{s} = \sum_{h=1}^{m} \mathbf{x}_h$, and let $\mathbf{c} \in \mathbb{R}^n$ be chosen so that $\mathbf{c} >> \mathbf{s}$. Let

$$\Delta = \left\{ \mathbf{p} \in \mathbb{R}^n : \mathbf{p} \geq \mathbf{0} \text{ and } \sum_{i=1}^{n} (\mathbf{p})_i = 1 \right\},$$

and, for each household, let $\hat{\xi}_{\mathbf{c},h} \colon \Delta \rightrightarrows \mathbb{R}^n_+$ be the demand correspondence that sends each normalized price vector $\mathbf{p}$ in Δ to the set $\hat{\xi}_{\mathbf{c},h}(\mathbf{p})$ of bundles of commodities that maximize utility for household h subject to the affordability constraint $\mathbf{p}^* \,.\, \mathbf{x}_h \leq \mathbf{p}^* \,.\, \overline{\mathbf{x}}_h$, and the additional constraint $\mathbf{0} \leq \mathbf{x} \leq \mathbf{c}$.

Let the correspondence $\hat{\xi}_{\mathbf{c}} \colon \Delta \rightrightarrows \mathbb{R}^n_+$ be defined so that $\hat{\xi}_{\mathbf{c}} = \sum_{h=1}^{m} \hat{\xi}_{\mathbf{c},h}$. Then the correspondence $\hat{\xi}_{\mathbf{c}}$ is upper hemicontinuous and maps each normalized price vector in Δ to a non-empty compact convex subset of $\mathbb{R}^n_+$ whose elements $\mathbf{x}$ satisfy $\mathbf{p} \,.\, \mathbf{x} \leq \mathbf{p} \,.\, \mathbf{s}$ (see Corollary 8.9).

Let the correspondence $\zeta_{\mathbf{c}} \colon \Delta \to \mathbb{R}^n$ be defined so that

$$\zeta_{\mathbf{c}} = \{\mathbf{x} - \mathbf{s} : \mathbf{x} \in \hat{\xi}_{\mathbf{c}}(\mathbf{p})\}$$

for all $\mathbf{p} \in \Delta$. Then $\mathbf{p} \,.\, \mathbf{z} \leq 0$ for all $\mathbf{p} \in \Delta$ and $\mathbf{z} \in \zeta(\mathbf{p})$. Moreover $\zeta_{\mathbf{c}}$ maps Δ into the compact set

$$\{\mathbf{z} \in \mathbb{R}^n : -\mathbf{s} \leq \mathbf{z} \leq \mathbf{c} - \mathbf{s}\}.$$

It then follows from Theorem 8.10 that there exist $\mathbf{p}^* \in \Delta$ and $\mathbf{z}^* \in \zeta_{\mathbf{c}}(\mathbf{p}^*)$ for which $\mathbf{z}^* \leq \mathbf{0}$.

Now $\mathbf{z}^* + \mathbf{s} \in \hat{\xi}_{\mathbf{c}}(\mathbf{p}^*)$. It follows from the definition of $\hat{\xi}_{\mathbf{c}}(\mathbf{p}^*)$. that there exist $\mathbf{x}_h^* \in \hat{\xi}_{\mathbf{c},h}(\mathbf{p}^*)$ for $h = 1, 2, \ldots, n$ for which $\sum_{h=1}^{m} \mathbf{x}_h^* = \mathbf{z}^* + \mathbf{s}$. Then $\sum_{h=1}^{m} \mathbf{x}_h^* \leq \mathbf{s}$, because $\mathbf{z}^* \leq \mathbf{0}$. Now $\mathbf{x}_h^* \geq \mathbf{0}$ for $h = 1, 2, \ldots, m$. It follows that $\mathbf{0} \leq \mathbf{x}_h^* \leq \mathbf{s}$ and therefore $\mathbf{x}_h^* << \mathbf{c}$ for $h = 1, 2, \ldots, m$.

Now $\mathbf{x}_h^*$ maximizes the utility function u_h on the set $B_{\mathbf{c},h}(\mathbf{p}^*)$, where

$$B_{\mathbf{c},h}(\mathbf{p}^*) = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{0} \leq \mathbf{x} \leq \mathbf{c} \text{ and } \mathbf{p}^* \,.\, \mathbf{x} \leq \mathbf{p}^* \,.\, \overline{\mathbf{x}}_h\}.$$

Let

$$B_h(\mathbf{p}^*) = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{x} \geq \mathbf{0} \text{ and } \mathbf{p}^* \,.\, \mathbf{x} \leq \mathbf{p}^* \,.\, \overline{\mathbf{x}}_h\}.$$

and let

$$N = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} << \mathbf{c}\}.$$

Then the set N is open in $\mathbb{R}^n$, $\mathbf{x}_h^* \in N$ and the maximum value of the utility function u_h for household h on $B_h(\mathbf{p}^*) \cap N$ is achieved at $\mathbf{x}_h^*$. It follows directly from Proposition 8.4 that

$$\mathbf{p}^* \,.\, \mathbf{x}_h^* = \mathbf{p}^* \,.\, \overline{\mathbf{x}}_h,$$

and moreover the maximum value of the utility function u_h for household h on $B_h(\mathbf{p}^*)$ is achieved at $\mathbf{x}_h^*$.

Next we note that were it the case that $(\mathbf{p}^*)_i = 0$ for some index i between 1 and n then the amount of the ith commodity in the bundle $\mathbf{x}_h^*$ could be increased to obtain a bundle $\mathbf{x}$ for which $\mathbf{x} \neq \mathbf{x}_h^*$, $\mathbf{x} \geq \mathbf{x}_h^*$ and $\mathbf{p}^* \,.\, \mathbf{x} = \mathbf{p}^* \,.\, \mathbf{x}_h^*$. But then $u_h(\mathbf{x}) > u_h(\mathbf{x}_h^*)$, because the utility function u_h is strictly increasing, and thus $\mathbf{x}_h^*$ would not maximize utility for for household h subject to the affordability constraint. We conclude therefore that $\mathbf{p}^* >> 0$.

Finally we note that

$$\mathbf{s} - \sum_{h=1}^{m} \mathbf{x}_h^* \geq \mathbf{0}$$

and

$$\mathbf{p}^* \,.\, \left(\mathbf{s} - \sum_{h=1}^{m} \mathbf{x}_h^* \right) = \sum_{h=1}^{m} \mathbf{p}^* \,.\, (\overline{\mathbf{x}}_h - \mathbf{x}_h^*) = 0.$$

It follows that

$$\mathbf{s} = \sum_{h=1}^{m} \mathbf{x}_h^*.$$

This completes the proof. ∎

8.9. Walras's Law

In the exchange economy model under discussion, let $\mathbf{p}$ be a price vector satisfying $\mathbf{p} \gg 0$, and let $\xi_h(\mathbf{p})$ be the set of bundles of commodities maximizing utility for household h, subject only to the budget constraint requiring that $\mathbf{p} \,.\, \mathbf{x} \leq \mathbf{p} \,.\, \overline{\mathbf{x}}_h$ for all bundles $\mathbf{x}$ available to household h. Then $\mathbf{p} \,.\, \mathbf{x} = \mathbf{p} \,.\, \overline{\mathbf{x}}_h$ for all $\mathbf{x} \in \xi_h(\mathbf{p})$. Summing over all households, we find that $\mathbf{p} \,.\, \mathbf{x} = \mathbf{p} \,.\, \mathbf{s}$, for all $\mathbf{x} \in \xi(\mathbf{p})$, where $\mathbf{s}$ denotes the aggregate supply, defined so that $\mathbf{s} = \sum_{h=1}^{m} \overline{\mathbf{x}}_h$, and $\xi(\mathbf{p})$ denotes the value of the aggregate demand correspondence at prices p, defined so that $\xi = \sum_{h=1}^{m} \xi_h$.

It follows that $\mathbf{p} \,.\, \mathbf{z} = 0$ for all $\mathbf{z} \in \zeta(\mathbf{p})$, where ζ denotes the *excess demand correspondence*, defined such that

$$\zeta(\mathbf{p}) = \{\mathbf{x} - \mathbf{s} : \mathbf{x} \in \xi(\mathbf{p})\}$$

for all $\mathbf{p} \in \Delta$ satisfying $\mathbf{p} >> \mathbf{0}$. This property of the excess demand correspondence is often referred to as *Walras's Law*.

8.10. Walrasian Equilibria with Strictly Quasiconcave Utility

We consider an exchange economy with n commodities and m households, retaining the notation of the previous discussion. We now consider the situation in which the utility function of each household is strictly quasiconcave.

Definition

A real-valued function $u\colon X \to \mathbb{R}$ defined on a convex subset X of $\mathbb{R}^n$ is said to be *strictly quasiconcave* on X if

$$u((1-t)\mathbf{x} + t\mathbf{x}') > \min\Big(u(\mathbf{x}), u(\mathbf{x}')\Big)$$

for all distinct points $\mathbf{x}$ and $\mathbf{x}'$ of X and for all real numbers t satisfying $0 < t < 1$.

Suppose that, in the exchange economy, the utility function u_h of household h is continuous, strictly increasing and strictly quasiconcave for $h = 1, 2, \ldots, m$. The utility function of household h cannot then be maximized at two distinct points of any non-empty compact convex set. Let $\mathbf{c}$ be an n-dimensional vector satisfying $\mathbf{c} \gg 0$. Then, given any normalized price vector $\mathbf{p}$, and given an initial endowment $\overline{x}_h$ for the ith household, there is a unique bundle of commodities $\hat{\mathbf{x}}_{\mathbf{c},h}(\mathbf{p})$ satisfying the budget constraint $\mathbf{p} \,.\, \hat{\mathbf{x}}_{\mathbf{c},h}(\mathbf{p}) \leq \mathbf{p} \,.\, \overline{\mathbf{x}}_h$ and the total availability constraint $\hat{\mathbf{x}}_{\mathbf{c},h}(\mathbf{p}) \leq \mathbf{c}$ which maximizes the utility function for household h for all bundles of commodities that satisfy the budget constraint and the total availability constraint. Moreover if $\hat{\mathbf{x}}_{\mathbf{c},h}(\mathbf{p}) \ll \mathbf{c}$ then $\mathbf{p} \,.\, \hat{\mathbf{x}}_{\mathbf{c},h}(\mathbf{p}) = \mathbf{p} \,.\, \overline{\mathbf{x}}_h$.

The preferences of household h, given normalized prices, given its initial endowment, and given the upper bounds on the availability of each commodity specified by the components of the vector $\mathbf{c}$, therefore determine a *demand function* $\hat{\mathbf{x}}_{\mathbf{c},h} \colon \Delta \to \mathbb{R}^{+}_{-}$ on the price simplex Δ, where

$$\begin{aligned} \Delta \;=\; \{(p_1, p_2, \ldots, p_n) \in \mathbb{R}^n :& \\ p_i \geq 0 \text{ for } i = 1, 2, \ldots, n, \text{ and } & \sum_{i=1}^{n} p_i = 1\}. \end{aligned}$$

The results obtained in more generality for demand correspondences, using Berge's Maximum Theorem, ensure that this demand function $\hat{\mathbf{x}}_{\mathbf{c},h}$ is continuous on Δ.

Summing the demand functions for the households, and subtracting the initial endowments, we obtain an *excess demand function* $\hat{\mathbf{z}}_{\mathbf{c}} : \Delta \to \mathbb{R}^n$ on the price simplex Δ whose value at normalized prices $\mathbf{p}$ specifies the excess demand for the commodities traded, when each household seeks to purchase commodities to maximize its utility, subject to the budget constraint determined by the prices and its initial endowment, and subject to the availability constraint that no household can purchase an amount of the ith commodity exceeding in amount the ith component of the vector $\mathbf{c}$. This excess demand function on the price simplex Δ is continuous, and satisfies $\mathbf{p} \,.\, \hat{z}_{\mathbf{c}}(\mathbf{p}) \leq 0$ for all $\mathbf{p} \in \Delta$.

The existence of Walrasian equilibria at which supply at least matches demand can then be established on the basis of the following proposition, whose proof makes use of the Brouwer Fixed Point Theorem.

Proposition 8.12

Let

$$\Delta = \{(p_1, p_2, \ldots, p_n) \in \mathbb{R}^n : p_i \geq 0 \text{ for } i = 1, 2, \ldots, n, \text{ and } \sum_{i=1}^{n} p_i = 1\},$$

let $\mathbf{z}\colon \Delta \to \mathbb{R}^n$ *be a continuous function mapping* Δ *into* $\mathbb{R}^n$, *and let*

$$\mathbf{z}(\mathbf{p}) = (z_1(\mathbf{p}), z_2(\mathbf{p}), \ldots, z_n(\mathbf{p}))$$

for all $\mathbf{p} \in \Delta$. *Suppose that* $\mathbf{p} \,.\, \mathbf{z}(\mathbf{p}) \leq 0$ *for all* $\mathbf{p} \in \Delta$. *Then there exists* $\mathbf{p}^* \in \Delta$ *such that* $z_i(\mathbf{p}^*) \leq 0$ *for* $i = 1, 2, \ldots, n$.

Proof

Let $\mathbf{v}\colon \Delta \to \mathbb{R}^n$ be the function with ith component v_i given by

$$v_i(\mathbf{p}) = \begin{cases} p_i + z_i(\mathbf{p}) & \text{if } z_i(\mathbf{p}) > 0; \\ p_i & \text{if } z_i(\mathbf{p}) \leq 0. \end{cases}$$

Note that $\mathbf{v}(\mathbf{p}) \neq \mathbf{0}$ and the components of $\mathbf{v}(\mathbf{p})$ are non-negative for all $\mathbf{p} \in \Delta$. It follows that there is a well-defined map $\varphi\colon \Delta \to \Delta$ given by

$$\varphi(\mathbf{p}) = \frac{1}{\sum_{i=1}^{n} v_i(\mathbf{p})} \mathbf{v}(\mathbf{p}),$$

The Brouwer Fixed Point Theorem (Theorem 5.3) ensures that there exists $\mathbf{p}^* \in \Delta$ satisfying $\varphi(\mathbf{p}^*) = \mathbf{p}^*$. Then $\mathbf{v}(\mathbf{p}^*) = \lambda \mathbf{p}^*$ for some $\lambda \geq 1$. We claim that $\lambda = 1$.

Suppose that it were the case that $\lambda > 1$. Then $v_i(\mathbf{p}^*) > p_i^*$, and thus $z_i(\mathbf{p}^*) > 0$ whenever $p_i^* > 0$. But $p_i^* \geq 0$ for all i, and $p_i^* > 0$ for at least one value of i, since $\mathbf{p}^* \in \Delta$. It would follow that $\mathbf{p}^* \,.\, \mathbf{z}(\mathbf{p}^*) > 0$, contradicting the requirement that $\mathbf{p} \,.\, \mathbf{z}(\mathbf{p}) \leq 0$ for all $\mathbf{p} \in \Delta$. We conclude that $\lambda = 1$, and thus $v_i = p_i^*$ and $z_i(\mathbf{p}^*) \leq 0$ for all i, as required. ∎

8.11. Historical Note

The proof of the existence of Walrasian equilibria in exchange economies can be generalized to *Arrow-Debreu* models where economic activity is carried out by both households and firms. The problem of existence of equilibria was studied by L. Walras in the 1870s, though a rigorous proof of the existence of equilibria was not found till the 1930s, when A. Wald proved existence for a limited range of economic models. Proofs of existence using topological fixed point theorems such as the Brouwer Fixed Point Theorem or the Kakutani Fixed Point Theorem were first published in 1954 by K. J. Arrow and G. Debreu and by L. McKenzie. Subsequent research has centred on problems of uniqueness and stability, and the existence theorems have been generalized to economies with an infinite number of commodities and economic agents (households and firms). An alternative approach to the existence theorems using techniques of differential topology was pioneered by G. Debreu and by S. Smale.

More detailed accounts of the theory of 'general equilibrium' can be found in, for example, *The theory of value*, by G. Debreu, *General competitive analysis*, by K. J. Arrow and F. H. Hahn, or *Economics for mathematicians* by J. W. S. Cassels.