MAU34804—Fixed Point Theorems and Economic Equilibria
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Section 4: Simplicial Complexes

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4. Simplicial Complexes

4.1. Simplical Complexes in Euclidean Spaces

Definition

A finite collection K of simplices in $\mathbb{R}^k$ is said to be a *simplicial complex* if the following two conditions are satisfied:—

- if σ is a simplex belonging to K then every face of σ also belongs to K,
- if σ_1 and σ_2 are simplices belonging to K then either $\sigma_1 \cap \sigma_2 = \emptyset$ or else $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2.

Definition

The *dimension* of a simplicial complex K is the greatest non-negative integer n with the property that K contains an n-simplex.

Definition

The *polyhedron* of a simplicial complex K is the union of all the simplices of K.

The polyhedron $|K|$ of a simplicial complex K is a subset of a Euclidean space that is both closed and bounded. It is therefore a compact subset of that Euclidean space.

Example

Let K_σ consist of some n-simplex σ together with all of its faces. Then K_σ is a simplicial complex of dimension n, and $|K_\sigma| = \sigma$.

Lemma 4.1

Let K be a simplicial complex, and let X be a subset of some Euclidean space. A function $f: |K| \to X$ is continuous on the polyhedron $|K|$ of K if and only if the restriction of f to each simplex of K is continuous on that simplex.

Proof

Each simplex of the simplicial complex K is a closed subset of the polyhedron $|K|$ of the simplicial complex K. The numbers of simplices belonging to the simplicial complex is finite. The result therefore follows from a straightforward application of Lemma 1.18. ∎

We shall denote by $\mathrm{Vert}\, K$ the set of vertices of a simplicial complex K (i.e., the set consisting of all vertices of all simplices belonging to K). A collection of vertices of K is said to *span* a simplex of K if these vertices are the vertices of some simplex belonging to K.

Definition

Let K be a simplicial complex in $\mathbb{R}^k$. A *subcomplex* of K is a collection L of simplices belonging to K with the following property:—

- if σ is a simplex belonging to L then every face of σ also belongs to L.

Note that every subcomplex of a simplicial complex K is itself a simplicial complex.

Proposition 4.2

Let K be a finite collection of simplices in some Euclidean space $\mathbb{R}^k$, and let $|K|$ be the union of all the simplices in K. Then K is a simplicial complex (with polyhedron $|K|$) if and only if the following two conditions are satisfied:—

- *K contains the faces of its simplices,*
- *every point of $|K|$ belongs to the interior of a unique simplex of K.*

Proof

Suppose that K is a simplicial complex. Then K contains the faces of its simplices. We must show that every point of $|K|$ belongs to the interior of a unique simplex of K. Let $\mathbf{x} \in |K|$. Then $\mathbf{x} \in \rho$ for some simplex ρ of K. It follows from Lemma 3.3 that there exists a unique face σ of ρ such that the point $\mathbf{x}$ belongs to the interior of σ. But then $\sigma \in K$, because $\rho \in K$ and K contains the faces of all its simplices. Thus $\mathbf{x}$ belongs to the interior of at least one simplex of K.

Suppose that **x** were to belong to the interior of two distinct simplices σ and τ of K. Then **x** would belong to some common face $\sigma \cap \tau$ of σ and τ (since K is a simplicial complex). But this common face would be a proper face of one or other of the simplices σ and τ (since $\sigma \neq \tau$), contradicting the fact that **x** belongs to the interior of both σ and τ. We conclude that the simplex σ of K containing **x** in its interior is uniquely determined.

Conversely, we must show that if K is some finite collection of simplices in some Euclidean space, if K contains the faces of all its simplices, and if every point of the union $|K|$ of those simplices belongs the the interior of a unique simplex in the collection, then that collection is a simplicial complex. To achieve this, we must prove that if σ and τ are simplices belonging to the collection K, and if $\sigma \cap \tau \neq \emptyset$, then $\sigma \cap \tau$ is a common face of σ and τ.

Let $\mathbf{x} \in \sigma \cap \tau$. Then $\mathbf{x}$ belongs to the interior of a unique simplex ω belonging to the collection K. However any point of σ or τ belongs to the interior of a unique face of that simplex, and all faces of σ and τ belong to K. It follows that ω is a common face of σ and τ, and thus the vertices of ω are vertices of both σ and τ. It follows that the simplices σ and τ have vertices in common.

Let ρ be the simplex whose vertex set is the intersection of the vertex sets of σ and τ. Then ρ is a common face of both σ and τ, and therefore $\rho \in K$. Moreover if $\mathbf{x} \in \sigma \cap \tau$ and if ω is the unique simplex of K whose interior contains the point $\mathbf{x}$, then (as we have already shown), all vertices of ω are vertices of both σ and τ. But then the vertex set of ω is a subset of the vertex set of ρ, and thus ω is a face of ρ. Thus each point $\mathbf{x}$ of $\sigma \cap \tau$ belongs to ρ, and therefore $\sigma \cap \tau \subset \rho$. But ρ is a common face of σ and τ and therefore $\rho \subset \sigma \cap \tau$. It follows that $\sigma \cap \tau = \rho$, and thus $\sigma \cap \tau$ is a common face of σ and τ. This completes the proof that the collection K of simplices satisfying the given conditions is a simplicial complex. ∎

4.2. Barycentric Subdivision of a Simplicial Complex

Let σ be a q-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. The *barycentre* of σ is defined to be the point

$$\hat{\sigma} = \frac{1}{q+1}(\mathbf{v}_0 + \mathbf{v}_1 + \cdots + \mathbf{v}_q).$$

Let σ and τ be simplices in some Euclidean space. If σ is a proper face of τ then we denote this fact by writing $\sigma < \tau$.
A simplicial complex K_1 is said to be a *subdivision* of a simplicial complex K if $|K_1| = |K|$ and each simplex of K_1 is contained in a simplex of K.

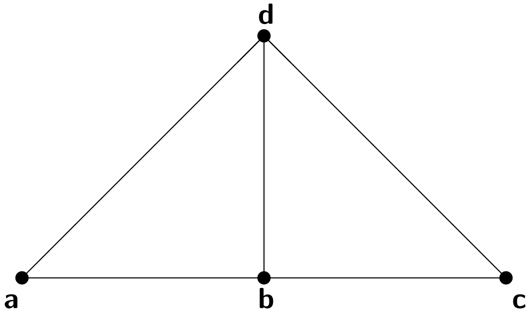
Definition

Let K be a simplicial complex in some Euclidean space $\mathbb{R}^k$. The *first barycentric subdivision* K' of K is defined to be the collection of simplices in $\mathbb{R}^k$ whose vertices are $\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_r$ for some sequence $\sigma_0, \sigma_1, \ldots, \sigma_r$ of simplices of K with $\sigma_0 < \sigma_1 < \cdots < \sigma_r$. Thus the set of vertices of K' is the set of all the barycentres of all the simplices of K.

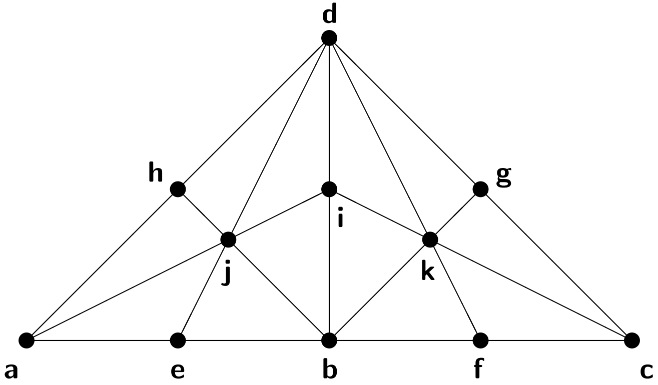
Note that every simplex of K' is contained in a simplex of K. Indeed if $\sigma_0, \sigma_1, \ldots, \sigma_r \in K$ satisfy $\sigma_0 < \sigma_1 < \cdots < \sigma_r$ then the simplex of K' spanned by $\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_r$, is contained in the simplex σ_r of K.

Example

Let K be the simplicial complex consisting of two triangles $\mathbf{a\,b\,d}$ and $\mathbf{b\,c\,d}$ that intersect along a common edge $\mathbf{b\,d}$, together with all the edges and vertices of the two triangles, as depicted in the following diagram:



The barycentric subdivision K' of this simplicial complex is then as depicted in the following diagram:



We see that K' consists of 12 triangles, together with all the edges and vertices of those triangles. Of the 11 vertices of K', the vertices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are the vertices of the original complex K, the vertices $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$ and $\mathbf{i}$ are the barycentres of the edges $\mathbf{a}\,\mathbf{b}$, $\mathbf{b}\,\mathbf{c}$, $\mathbf{c}\,\mathbf{d}$, $\mathbf{a}\,\mathbf{d}$ and $\mathbf{b}\,\mathbf{d}$ respectively, and are located at the midpoints of those edges, and the vertices $\mathbf{j}$ and $\mathbf{k}$ are the barycentres of the triangles $\mathbf{a}\,\mathbf{b}\,\mathbf{d}$ and $\mathbf{b}\,\mathbf{c}\,\mathbf{d}$ of K. Thus $\mathbf{e} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$, $\mathbf{f} = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c}$, etc., and $\mathbf{j} = \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b} + \frac{1}{3}\mathbf{d}$ and $\mathbf{k} = \frac{1}{3}\mathbf{b} + \frac{1}{3}\mathbf{c} + \frac{1}{3}\mathbf{d}$.

Proposition 4.3

Let K be a simplicial complex in some Euclidean space, and let K' be the first barycentric subdivision of K. Then K' is itself a simplicial complex, and $|K'| = |K|$.

Proof

We prove the result by induction on the number of simplices in K. The result is clear when K consists of a single simplex, since that simplex must then be a point and therefore $K' = K$. We prove the result for a simplicial complex K, assuming that it holds for all complexes with fewer simplices.

It is clear from the definition of the barycentric subdivision K' that any face of a simplex of K' must itself belong to K'. We must verify that any two simplices of K' are disjoint or else intersect in a common face.

Choose a simplex σ of K for which $\dim \sigma = \dim K$, and let $L = K \setminus \{\sigma\}$. Then L is a subcomplex of K, since σ is not a proper face of any simplex of K. Now L has fewer simplices than K. It follows from the induction hypothesis that L' is a simplicial complex and $|L'| = |L|$. Also it follows from the definition of K' that K' consists of the following simplices:—

- the simplices of L',
- the barycentre $\hat{\sigma}$ of σ,
- simplices $\hat{\sigma}\rho$ whose vertex set is obtained by adjoining $\hat{\sigma}$ to the vertex set of some simplex ρ of L', where the vertices of ρ are barycentres of proper faces of σ.

By checking all possible intersections of simplices of the above types, it is easy to verify that any two simplices of K' intersect in a common face. Indeed any two simplices of L' intersect in a common face, since L' is a simplicial complex. If ρ_1 and ρ_2 are simplices of L' whose vertices are barycentres of proper faces of σ, then $\rho_1 \cap \rho_2$ is a common face of ρ_1 and ρ_2 which is of this type, and $\hat{\sigma}\rho_1 \cap \hat{\sigma}\rho_2 = \hat{\sigma}(\rho_1 \cap \rho_2)$. Thus $\hat{\sigma}\rho_1 \cap \hat{\sigma}\rho_2$ is a common face of $\hat{\sigma}\rho_1$ and $\hat{\sigma}\rho_2$. Also any simplex τ of L' is disjoint from the barycentre $\hat{\sigma}$ of σ, and $\hat{\sigma}\rho \cap \tau = \rho \cap \tau$. We conclude that K' is indeed a simplicial complex.

It remains to verify that $|K'| = |K|$. Now $|K'| \subset |K|$, since every simplex of K' is contained in a simplex of K. Let $\mathbf{x}$ be a point of the chosen simplex σ. Then there exists a point $\mathbf{y}$ belonging to a proper face of σ and some $t \in [0, 1]$ such that $\mathbf{x} = (1 - t)\hat{\sigma} + t\,\mathbf{y}$. But then $\mathbf{y} \in |L|$, and $|L| = |L'|$ by the induction hypothesis. It follows that $\mathbf{y} \in \rho$ for some simplex ρ of L' whose vertices are barycentres of proper faces of σ. But then $\mathbf{x} \in \hat{\sigma}\rho$, and therefore $\mathbf{x} \in |K'|$. Thus $|K| \subset |K'|$, and hence $|K'| = |K|$, as required. ∎

We define (by induction on j) the jth barycentric subdivision $K^{(j)}$ of K to be the first barycentric subdivision of $K^{(j-1)}$ for each $j > 1$.

Lemma 4.4

Let σ be a q-simplex and let τ be a face of σ. Let $\hat{\sigma}$ and $\hat{\tau}$ be the barycentres of σ and τ respectively. If all the 1-simplices (edges) of σ have length not exceeding d for some $d > 0$ then

$$|\hat{\sigma} - \hat{\tau}| \leq \frac{qd}{q+1}.$$

Proof

Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of σ. Let $\mathbf{x}$ and $\mathbf{y}$ be points of σ. We can write $\mathbf{y} = \sum_{j=0}^{q} t_j \mathbf{v}_j$, where $0 \leq t_i \leq 1$ for $i = 0, 1, \ldots, q$ and $\sum_{j=0}^{q} t_j = 1$. Now

$$\begin{aligned} |\mathbf{x} - \mathbf{y}| &= \left| \sum_{i=0}^{q} t_i (\mathbf{x} - \mathbf{v}_i) \right| \leq \sum_{i=0}^{q} t_i |\mathbf{x} - \mathbf{v}_i| \\ &\leq \operatorname{maximum} \left(|\mathbf{x} - \mathbf{v}_0|, |\mathbf{x} - \mathbf{v}_1|, \ldots, |\mathbf{x} - \mathbf{v}_q| \right). \end{aligned}$$

Applying this result with $\mathbf{x} = \hat{\sigma}$ and $\mathbf{y} = \hat{\tau}$, we find that

$$|\hat{\sigma} - \hat{\tau}| \leq \operatorname{maximum} \left(|\hat{\sigma} - \mathbf{v}_0|, |\hat{\sigma} - \mathbf{v}_1|, \ldots, |\hat{\sigma} - \mathbf{v}_q| \right).$$

4. Simplicial Complexes (continued)

But

$$\hat{\sigma} = \frac{1}{q+1}\mathbf{v}_i + \frac{q}{q+1}\mathbf{z}_i$$

for $i = 0, 1, \ldots, q$, where $\mathbf{z}_i$ is the barycentre of the $(q-1)$-face of σ opposite to $\mathbf{v}_i$, given by

$$\mathbf{z}_i = \frac{1}{q}\sum_{j \neq i} \mathbf{v}_j.$$

Moreover $\mathbf{z}_i \in \sigma$. It follows that

$$|\hat{\sigma} - \mathbf{v}_i| = \frac{q}{q+1}|\mathbf{z}_i - \mathbf{v}_i| \leq \frac{qd}{q+1}$$

for $i = 1, 2, \ldots, q$, and thus

$$|\hat{\sigma} - \hat{\tau}| \leq \text{maximum}\left(|\hat{\sigma} - \mathbf{v}_0|, |\hat{\sigma} - \mathbf{v}_1|, \ldots, |\hat{\sigma} - \mathbf{v}_q|\right) \leq \frac{qd}{q+1},$$

as required. $\blacksquare$

Definition

The *mesh* $\mu(K)$ of a simplicial complex K is the length of the longest edge of K.

Lemma 4.5

Let K be a simplicial complex, and let n be the dimension of K. Let K' be the first barycentric subdivision of K. Then

$$\mu(K') \leq \frac{n}{n+1}\mu(K).$$

Proof
A 1-simplex of K' is of the form $(\hat{\tau}, \hat{\sigma})$, where σ is a q-simplex of K for some $q \leq n$ and τ is a proper face of σ. Then

$$|\hat{\tau} - \hat{\sigma}| \leq \frac{q}{q+1}\mu(K) \leq \frac{n}{n+1}\mu(K)$$

by Lemma 4.4, as required. ∎

Lemma 4.6

Let K be a simplicial complex, let $K^{(j)}$ be the jth barycentric subdivision of K for all positive integers j, and let $\mu(K^{(j)})$ be the mesh of $K^{(j)}$. Then $\lim_{j \to +\infty} \mu(K^{(j)}) = 0$.

Proof

The dimension of all barycentric subdivisions of a simplicial complex is equal to the dimension of the simplicial complex itself. It therefore follows from Lemma 4.5 that

$$\mu(K^{(j)}) \leq \left(\frac{n}{n+1} \right)^j \mu(K).$$

The result follows. ∎

4.3. Piecewise Linear Maps on Simplicial Complexes

Definition

Let K be a simplicial complex in n-dimensional Euclidean space. A function $f\colon |K| \to \mathbb{R}^m$ mapping the polyhedron $|K|$ of K into m-dimensional Euclidean space $\mathbb{R}^m$ is said to be *piecewise linear* on each simplex of K if

$$f\left(\sum_{i=0}^{q} t_i \mathbf{v}_i\right) = \sum_{i=0}^{q} t_i f(\mathbf{v}_i)$$

for all vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of K that span a simplex of K, and for all non-negative real numbers $t_0, t_1, \ldots, t_q$ satisfying $\sum\limits_{i=0}^{q} t_i = 1$.

Lemma 4.7

Let K be a simplicial complex in n-dimensional Euclidean space, and let $f\colon |K| \to \mathbb{R}^m$ be a function mapping the polyhedron $|K|$ of K into m-dimensional Euclidean space $\mathbb{R}^m$ that is piecewise linear on each simplex of K. Then $f\colon |K| \to \mathbb{R}^m$ is continuous.

Proof

The definition of piecewise linear functions ensures that the restriction of $f\colon |K| \to \mathbb{R}^m$ to each simplex of K is continuous on that simplex. The result therefore follows from Lemma 4.1. ∎

Proposition 4.8

Let K be a simplicial complex in n-dimensional Euclidean space and let $\alpha\colon \mathrm{Vert}(K) \to \mathbb{R}^m$ be a function mapping the set $\mathrm{Vert}(K)$ of vertices of K into m-dimensional Euclidean space $\mathbb{R}^m$. Then there exists a unique function $f\colon |K| \to \mathbb{R}^m$ defined on the polyhedron $|K|$ of K that is piecewise linear on each simplex of K and satisfies $f(\mathbf{v}) = \alpha(\mathbf{v})$ for all vertices $\mathbf{v}$ of K.

Proof

Given any point $\mathbf{x}$ of K, there exists a unique simplex of K whose interior contains the point $\mathbf{x}$ (Proposition 4.2). Let the vertices of this simplex be $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_p$, where $p \leq n$. Then there exist uniquely-determined strictly positive real numbers $t_0, t_1, \ldots, t_p$ satisfying $\sum_{i=0}^{p} t_i = 1$ for which $\mathbf{x} = \sum_{i=0}^{p} t_i \mathbf{v}_i$. We then define $f(\mathbf{x})$ so that

$$f(\mathbf{x}) = \sum_{i=0}^{p} t_i \alpha(\mathbf{v}_i).$$

Defining $f(\mathbf{x})$ in this fashion at each point $\mathbf{x}$ of $|K|$, we obtain a function $f\colon |K| \to \mathbb{R}^m$ mapping Δ into $\mathbb{R}^m$.

Now let $\mathbf{x} \in \sigma$ for some q-simplex of K. We can order the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of σ so that the point $\mathbf{x}$ belongs to the interior of the face of σ spanned by $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_p$ where $p \leq q$. Let $t_1, t_2, \ldots, t_q$ be the barycentric coordinates of the point $\mathbf{x}$ with respect to the simplex σ. Then $\mathbf{x} = \sum_{i=0}^{q} t_i \mathbf{v}_i$, where $t_i > 0$ for those integers i satisfying $0 \leq i \leq p$, $t_i = 0$ for those integers i (if any) satisfying $p < i \leq q$, and $\sum_{i=0}^{p} t_i = \sum_{i=0}^{q} t_i = 1$. Then

$$f\left(\sum_{i=0}^{q} t_i \mathbf{v}_i\right) = f(\mathbf{x}) = \sum_{i=0}^{p} t_i \alpha(\mathbf{v}_i) = \sum_{i=0}^{q} t_i f(\mathbf{v}_i).$$

The result follows. ∎

Corollary 4.9

Let K be a simplicial complex in $\mathbb{R}^n$ and let L be simplicial complexes in $\mathbb{R}^m$, where m and n are positive integers, and let $\varphi\colon \mathrm{Vert}(K) \to \mathrm{Vert}(L)$ be a function mapping vertices of K to vertices of L. Suppose that

$$\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$$

span a simplex of L for all vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of K that span a simplex of K. Then there exists a unique continuous map $\overline{\varphi}\colon |K| \to |L|$ mapping the polyhedron $|K|$ of K into the polyhedron $|L|$ of L that is piecewise linear on each simplex of K and satisfies $\overline{\varphi}(\mathbf{v}) = \varphi(\mathbf{v})$ for all vertices $\mathbf{v}$ of K. Moreover this function maps the interior of a simplex of K spanned by vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ into the interior of the simplex of L spanned by $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$.

Proof
It follows from Proposition 4.8 that there is a unique piecewise linear function $f \colon |K| \to \mathbb{R}^m$ that satisfies $f(\mathbf{v}) = \varphi(\mathbf{v})$ for all $\mathbf{v} \in \mathrm{Vert}(K)$. We show that $f(|K|) \subset |L|$.
Let

$$\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$$

be vertices of a simplex σ of K, and let $t_0, t_1, \ldots, t_q$ be non-negative real numbers satisfying $\sum_{j=0}^{q} t_j = 1$. Then

$$\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$$

span a simplex of L. Let τ be the simplex of L spanned by these vertices of L, and let $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_r$ be the vertices of τ. Then, for each integer j between 1 and r, let u_j be the sum of those t_i for which $\varphi(\mathbf{v}_i) = \mathbf{w}_j$.

Then

$$f\left(\sum_{i=0}^{q} t_i \mathbf{v}_i\right) = \sum_{i=0}^{q} t_i \varphi(\mathbf{v}_i) = \sum_{j=0}^{r} u_j \mathbf{w}_j$$

and $\sum_{j=0}^{r} u_j = 1$. It follows that $f(\sigma) \subset \tau$. Moreover, given any integer j between 1 and r, there exists at least one integer i between 1 and q for which $\varphi(\mathbf{v}_i) = \mathbf{w}_j$. It follows that if $t_0, t_1, t_2, \ldots, t_q$ are all strictly positive then $u_0, u_1, \ldots, u_r$ are also all strictly positive. Therefore the piecewise linear function f maps the interior of σ into the interior of τ.

We have already shown that $f\colon |K| \to \mathbb{R}^m$ maps each simplex of K into a simplex of L. Therefore there exists a uniquely-determined linear function $\overline{\varphi}\colon |K| \to |L|$ satisfying $\overline{\varphi}(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in |K|$. The result follows. ∎

4.4. Simplicial Maps

Definition

A *simplicial map* $\varphi\colon K \to L$ between simplicial complexes K and L is a function $\varphi\colon \operatorname{Vert} K \to \operatorname{Vert} L$ from the vertex set of K to that of L such that $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$ span a simplex belonging to L whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K.

Note that a simplicial map $\varphi\colon K \to L$ between simplicial complexes K and L can be regarded as a function from K to L: this function sends a simplex σ of K with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ to the simplex $\varphi(\sigma)$ of L spanned by the vertices $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$.

It follows from Corollary 4.9 that simplicial map $\varphi\colon K \to L$ also induces in a natural fashion a continuous map $\varphi\colon |K| \to |L|$ between the polyhedra of K and L, where

$$\varphi\left(\sum_{j=0}^{q} t_j \mathbf{v}_j\right) = \sum_{j=0}^{q} t_j \varphi(\mathbf{v}_j)$$

whenever $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$, $\sum_{j=0}^{q} t_j = 1$, and $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. Moreover it also follows from Corollary 4.9 that the interior of a simplex σ of K is mapped into the interior of the simplex $\varphi(\sigma)$ of L.

There are thus three equivalent ways of describing a simplicial map: as a function between the vertex sets of two simplicial complexes, as a function from one simplicial complex to another, and as a continuous map between the polyhedra of two simplicial complexes. In what follows, we shall describe a simplicial map using the representation that is most appropriate in the given context.

4.5. Simplicial Approximations

Definition

Let $f\colon |K| \to |L|$ be a continuous map between the polyhedra of simplicial complexes K and L. A simplicial map $s\colon K \to L$ is said to be a *simplicial approximation* to f if, for each $\mathbf{x} \in |K|$, $s(\mathbf{x})$ is an element of the unique simplex of L which contains $f(\mathbf{x})$ in its interior.

Definition

Let X and Y be subsets of Euclidean spaces. Continuous maps $f\colon X \to Y$ and $g\colon X \to Y$ from X to Y are said to be *homotopic* if there exists a continuous map $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.

Lemma 4.10

Let K and L be simplicial complexes, let $f\colon |K| \to |L|$ be a continuous map between the polyhedra of K and L, and let $s\colon K \to L$ be a simplicial approximation to the map f. Then there is a well-defined homotopy $H\colon |K| \times [0,1] \to |L|$, between the maps f and s, where

$$H(\mathbf{x}, t) = (1 - t)f(\mathbf{x}) + ts(\mathbf{x})$$

for all $\mathbf{x} \in |K|$ and $t \in [0,1]$.

Proof

Let $\mathbf{x} \in |K|$. Then there is a unique simplex σ of L such that the point $f(\mathbf{x})$ belongs to the interior of σ. Then $s(\mathbf{x}) \in \sigma$. But, given any two points of a simplex embedded in some Euclidean space, the line segment joining those two points is contained within the simplex. It follows that $(1-t)f(\mathbf{x}) + ts(\mathbf{x}) \in |L|$ for all $\mathbf{x} \in K$ and $t \in [0,1]$. Thus the homotopy $H\colon |K| \times [0,1] \to |L|$ is a well-defined map from $|K| \times [0,1]$ to $|L|$. Moreover it follows directly from the definition of this map that $H(\mathbf{x}, 0) = f(\mathbf{x})$ and $H(\mathbf{x}, 1) = s(\mathbf{x})$ for all $\mathbf{x} \in |K|$ and $t \in [0,1]$. The map H is thus a homotopy between the maps f and s, as required. ∎

Definition

Let K be a simplicial complex, and let $\mathbf{x} \in |K|$. The *star neighbourhood* $\mathrm{st}_K(\mathbf{x})$ of $\mathbf{x}$ in K is the union of the interiors of all simplices of K that contain the point $\mathbf{x}$.

Lemma 4.11

Let K be a simplicial complex and let $\mathbf{x} \in |K|$. Then the star neighbourhood $\mathrm{st}_K(\mathbf{x})$ of $\mathbf{x}$ is open in $|K|$, and $\mathbf{x} \in \mathrm{st}_K(\mathbf{x})$.

Proof

Every point of $|K|$ belongs to the interior of a unique simplex of K (Proposition 4.2). It follows that the complement $|K| \setminus \mathrm{st}_K(\mathbf{x})$ of $\mathrm{st}_K(\mathbf{x})$ in $|K|$ is the union of the interiors of those simplices of K that do not contain the point $\mathbf{x}$. But if a simplex of K does not contain the point $\mathbf{x}$, then the same is true of its faces. Moreover the union of the interiors of all the faces of some simplex is the simplex itself. It follows that $|K| \setminus \mathrm{st}_K(\mathbf{x})$ is the union of all simplices of K that do not contain the point $\mathbf{x}$. But each simplex of K is closed in $|K|$. It follows that $|K| \setminus \mathrm{st}_K(\mathbf{x})$ is a finite union of closed sets, and is thus itself closed in $|K|$. We deduce that $\mathrm{st}_K(\mathbf{x})$ is open in $|K|$. Also $\mathbf{x} \in \mathrm{st}_K(\mathbf{x})$, since $\mathbf{x}$ belongs to the interior of at least one simplex of K. ∎

Proposition 4.12

A function $s\colon \operatorname{Vert} K \to \operatorname{Vert} L$ *between the vertex sets of simplicial complexes* K *and* L *is a simplicial map, and a simplicial approximation to some continuous map* $f\colon |K| \to |L|$*, if and only if* $f(\operatorname{st}_K(\mathbf{v})) \subset \operatorname{st}_L(s(\mathbf{v}))$ *for all vertices* $\mathbf{v}$ *of* K*.*

Proof

Let $s\colon K \to L$ be a simplicial approximation to $f\colon |K| \to |L|$, let $\mathbf{v}$ be a vertex of K, and let $\mathbf{x} \in \operatorname{st}_K(\mathbf{v})$. Then $\mathbf{x}$ and $f(\mathbf{x})$ belong to the interiors of unique simplices $\sigma \in K$ and $\tau \in L$. Moreover $\mathbf{v}$ must be a vertex of σ, by definition of $\operatorname{st}_K(\mathbf{v})$. Now $s(\mathbf{x})$ must belong to τ (since s is a simplicial approximation to the map f), and therefore $s(\mathbf{x})$ must belong to the interior of some face of τ.

But $s(\mathbf{x})$ must belong to the interior of $s(\sigma)$, because $\mathbf{x}$ is in the interior of σ (see Corollary 4.9). It follows that $s(\sigma)$ must be a face of τ, and therefore $s(\mathbf{v})$ must be a vertex of τ. Thus $f(\mathbf{x}) \in \mathrm{st}_L(s(\mathbf{v}))$. We conclude that if $s\colon K \to L$ is a simplicial approximation to $f\colon |K| \to |L|$, then $f\left(\mathrm{st}_K(\mathbf{v})\right) \subset \mathrm{st}_L\left(s(\mathbf{v})\right)$.

Conversely let $s\colon \mathrm{Vert}\, K \to \mathrm{Vert}\, L$ be a function with the property that $f\left(\mathrm{st}_K(\mathbf{v})\right) \subset \mathrm{st}_L\left(s(\mathbf{v})\right)$ for all vertices $\mathbf{v}$ of K. Let $\mathbf{x}$ be a point in the interior of some simplex of K with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. Then $\mathbf{x} \in \mathrm{st}_K(\mathbf{v}_j)$ and hence $f(\mathbf{x}) \in \mathrm{st}_L\left(s(\mathbf{v}_j)\right)$ for $j = 0, 1, \ldots, q$. It follows that each vertex $s(\mathbf{v}_j)$ must be a vertex of the unique simplex $\tau \in L$ that contains $f(\mathbf{x})$ in its interior. In particular, $s(\mathbf{v}_0), s(\mathbf{v}_1), \ldots, s(\mathbf{v}_q)$ span a face of τ, and $s(\mathbf{x}) \in \tau$. We conclude that the function $s\colon \mathrm{Vert}\, K \to \mathrm{Vert}\, L$ represents a simplicial map which is a simplicial approximation to $f\colon |K| \to |L|$, as required. ∎

Corollary 4.13

If $s: K \to L$ and $t: L \to M$ are simplicial approximations to continuous maps $f: |K| \to |L|$ and $g: |L| \to |M|$, where K, L and M are simplicial complexes, then $t \circ s: K \to M$ is a simplicial approximation to $g \circ f: |K| \to |M|$.

4.6. The Simplicial Approximation Theorem

Theorem 4.14

(Simplicial Approximation Theorem) Let K and L be simplicial complexes, and let $f\colon |K| \to |L|$ be a continuous map. Then, for some sufficiently large integer j, there exists a simplicial approximation $s\colon K^{(j)} \to L$ to f defined on the jth barycentric subdivision $K^{(j)}$ of K.

Proof

The collection consisting of the stars $\mathrm{st}_L(\mathbf{w})$ of all vertices $\mathbf{w}$ of L is an open cover of $|L|$, since each star $\mathrm{st}_L(\mathbf{w})$ is open in $|L|$ (Lemma 4.11) and the interior of any simplex of L is contained in $\mathrm{st}_L(\mathbf{w})$ whenever $\mathbf{w}$ is a vertex of that simplex. It follows from the continuity of the map $f\colon |K| \to |L|$ that the collection consisting of the preimages $f^{-1}(\mathrm{st}_L(\mathbf{w}))$ of the stars of all vertices $\mathbf{w}$ of L is an open cover of $|K|$.

Now the set $|K|$ is a closed bounded subset of a Euclidean space. It follows that there exists a Lebesgue number δ_L for the open cover consisting of the preimages of the stars of all the vertices of L (see Proposition 1.19). This Lebesgue number δ_L is a positive real number with the following property: every subset of $|K|$ whose diameter is less than δ_L is contained in the preimage of the star of some vertex $\mathbf{w}$ of L. It follows that every subset of $|K|$ whose diameter is less than δ_L is mapped by f into $\mathrm{st}_L(\mathbf{w})$ for some vertex $\mathbf{w}$ of L.

Now the mesh $\mu(K^{(j)})$ of the jth barycentric subdivision of K tends to zero as $j \to +\infty$ (see Lemma 4.6). Thus we can choose j such that $\mu(K^{(j)}) < \frac{1}{2}\delta_L$. If $\mathbf{v}$ is a vertex of $K^{(j)}$ then each point of $\mathrm{st}_{K^{(j)}}(\mathbf{v})$ is within a distance $\frac{1}{2}\delta_L$ of $\mathbf{v}$, and hence the diameter of $\mathrm{st}_{K^{(j)}}(\mathbf{v})$ is at most δ_L. We can therefore choose, for each vertex $\mathbf{v}$ of $K^{(j)}$ a vertex $s(\mathbf{v})$ of L such that $f(\mathrm{st}_{K^{(j)}}(\mathbf{v})) \subset \mathrm{st}_L(s(\mathbf{v}))$. In this way we obtain a function $s\colon \mathrm{Vert}\, K^{(j)} \to \mathrm{Vert}\, L$ from the vertices of $K^{(j)}$ to the vertices of L. It follows directly from Proposition 4.12 that this is the desired simplicial approximation to f. $\blacksquare$