MAU34804—Fixed Point Theorems and Economic Equilibria
School of Mathematics, Trinity College
Hilary Term 2022
Section 6: Perron-Frobenius Theory

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6. Perron-Frobenius Theory

6.1. Eigenvectors of Non-Negative Matrices

We establish some notation that will be used throughout this section.

Let m and n be positive integers. Given any $m \times n$ matrix T, we denote by $(T)_{i,j}$ the coefficient in the ith row and jth column of the matrix T for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Also given any n-dimensional vector $\mathbf{v}$, we denote by $(\mathbf{v})_j$ the jth coefficient of the vector j for $j = 1, 2, \ldots, n$.

Definition

A matrix T is said to be *non-negative* if all its coefficients are non-negative real numbers.

Definition

A matrix T is said to be *positive* if all its coefficients are strictly positive real numbers.

Let S and T be $m \times n$ matrices. If $(S)_{i,j} \leq (T)_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, then we denote this fact by writing $S \leq T$, or by writing $T \geq S$. If $(S)_{i,j} < (T)_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, then we denote this fact by writing $S << T$, or by writing $T >> S$.

Let $\mathbf{u}$ and $\mathbf{u}$ be n-dimensional vectors. If $(\mathbf{u})_j \leq (\mathbf{v})_j$ for $j = 1, 2, \ldots, n$, then we denote this fact by writing $\mathbf{u} \leq \mathbf{v}$, or by writing $\mathbf{v} \geq \mathbf{u}$. If $(\mathbf{u})_j < (\mathbf{v})_j$ for $j = 1, 2, \ldots, n$, then we denote this fact by writing $\mathbf{u} << \mathbf{v}$, or by writing $\mathbf{v} >> \mathbf{u}$.

A matrix T with real coefficients is thus *non-negative* if and only if $T \geq 0$. A matrix T with real coefficients is *positive* if and only if $T >> 0$.

Lemma 6.1

Let T be an $m \times n$ matrix with real coefficients. Then T is a non-negative matrix if and only if $T\mathbf{v} \geq 0$ for all $\mathbf{v} \in \mathbb{R}^n$ satisfying $\mathbf{v} \geq \mathbf{0}$.

Proof

Suppose that the matrix T is non-negative. Let $\mathbf{v} \in \mathbb{R}^n$ satisfy $\mathbf{v} \geq \mathbf{0}$. Then

$$(T\mathbf{v})_j = \sum_{k=1}^{n} (T)_{j,k}(\mathbf{v})_k \geq 0$$

for each integer j between 1 and m, because $(T)_{j,k}(\mathbf{v})_k \geq 0$ for $k = 1, 2, \ldots, n$. Therefore $T\mathbf{v} \geq \mathbf{0}$.

Conversely suppose that T is an $m \times n$ matrix with with real coefficients which has the property that if and only if $T\mathbf{v} \geq 0$ for all non-zero n-dimensional vectors $\mathbf{v}$ with non-negative real coefficients. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{R}^n$ with

$$\mathbf{e}_1 = (1, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, \ldots, 0), \ldots, \quad \mathbf{e}_n = (0, 0, \ldots, 1).$$

Then $T\mathbf{e}_k \geq \mathbf{0}$ for $k = 1, 2, \ldots, n$, and therefore $(T)_{j,k} = (T\mathbf{e}_k)_j \geq 0$ for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, n$. The result follows. ∎

Lemma 6.2

Let T be an $m \times n$ matrix with real coefficients. Then T is a positive matrix if and only if $T\mathbf{v} >> \mathbf{0}$ for all $\mathbf{v} \in \mathbb{R}^n$ satisfying both $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{v} \geq \mathbf{0}$.

Proof

Suppose that the matrix T is positive. Then $T_{j,k} > 0$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Let $\mathbf{v} \in \mathbb{R}^n$ satisfy both $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{v} \geq \mathbf{0}$. Then

$$(T\mathbf{v})_j = \sum_{k=1}^{n} (T)_{j,k} (\mathbf{v})_k > 0$$

for each integer j between 1 and m, because $(T)_{j,k}(\mathbf{v})_k \geq 0$ for $k = 1, 2, \ldots, n$ and $(T)_{j,k}(\mathbf{v})_k > 0$ for at least one integer k between 1 and n. Therefore $T\mathbf{v} >> \mathbf{0}$.

Conversely suppose that T is an $m \times n$ matrix with with real coefficients which has the property that if and only if $T\mathbf{v} >> \mathbf{0}$ for all non-zero n-dimensional vectors $\mathbf{v}$ with non-negative real coefficients. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{R}^n$ with

$$\mathbf{e}_1 = (1, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, \ldots, 0), \ldots, \quad \mathbf{e}_n = (0, 0, \ldots, 1).$$

Then $T\mathbf{e}_k >> \mathbf{0}$ for $k = 1, 2, \ldots, n$, and therefore $(T)_{j,k} = (T\mathbf{e}_k)_j > 0$ for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, n$. The result follows. ∎

Proposition 6.3

Let T be a non-negative $n \times n$ (square) matrix. Then there exists a well-defined non-negative real number μ (referred to as the Perron root of T) that may be characterized as the greatest real number ρ for which there exists a non-zero vector $\mathbf{v}$ with real coefficients satisfying the conditions $\mathbf{v} \geq \mathbf{0}$ and $T\mathbf{v} \geq \rho\mathbf{v}$.

Proof

Let

$$\Delta = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \geq \mathbf{0},\ \sum_{j=1}^{n} (\mathbf{v})_j = 1\},$$

and, for each non-negative real number ρ, let E_ρ be the subset of Δ defined so that

$$E_\rho = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \geq \mathbf{0},\ \sum_{j=1}^{n} (\mathbf{v})_j = 1 \text{ and } T\mathbf{v} \geq \rho\mathbf{v}\}.$$

Clearly $E_0 = \Delta$. Also if ρ exceeds the largest coefficient of the matrix T then clearly E_ρ is the empty set. Let

$$I = \{\rho \in \mathbb{R} : \rho \geq 0 \quad \text{and} \quad E_\rho \neq \emptyset\}.$$

Then I is a non-empty set of real numbers which is bounded above. It follows from the Least Upper Bound Principle that the set I has a least upper bound $\sup I$. Let $\mu = \sup I$.

Let ρ be a real number satisfying $0 \leq \rho < \mu$. Then there exists $\rho' \in I$ satisfying $\rho < \rho' \leq \mu$. The set $E_{\rho'}$ must then be non-empty, and moreover $E_{\rho'} \subset E_\rho$. It follows that $E_\rho \neq \emptyset$, and thus $\rho \in I$. It follows that

$$\{\rho \in \mathbb{R} : 0 \leq \rho < \mu\} \subset I,$$

and thus the subset I of $\mathbb{R}$ is an interval. We next prove that $\mu \in I$.

Now the characterization of the non-negative real number μ as the least upper bound of the interval I ensures the existence of an infinite sequence $\rho_1, \rho_2, \rho_3, \ldots$ of real numbers belonging to I for which $\lim_{s \to +\infty} \rho_s = \mu$. Then $E_{\rho_s} \neq \emptyset$ for all positive integers s, and therefore there exists an infinite sequence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ of vectors belonging to the simplex Δ such that $\mathbf{v}_s \in E_{\rho_s}$ for all positive integers s. Then $T\mathbf{v}_s \geq \rho_s \mathbf{v}_s$ for all positive integers s.

Now the sequence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ of vectors belonging to the simplex Δ is a bounded sequence of vectors, because Δ is a bounded set. The multidimensional Bolzano-Weierstrass Theorem (Theorem 1.2) now ensures the existence of a subsequence $\mathbf{v}_{s_1}, \mathbf{v}_{s_2}, \mathbf{v}_{s_3}, \ldots$ of the sequence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ which converges to some vector $\mathbf{u}$. Moreover $\mathbf{u} \in \Delta$, because Δ is a closed set.
Now

$$T\mathbf{u} = \lim_{r \to +\infty} T\mathbf{v}_{s_r}.$$

Also

$$T\mathbf{v}_{s_r} - \rho_{s_r}\mathbf{v}_{s_r} \geq \mathbf{0}$$

for all positive integers r. Taking limits as $r \to +\infty$, we find that

$$T\mathbf{u} - \mu\mathbf{u} \geq 0,$$

and thus $T\mathbf{u} \geq \mu\mathbf{u}$.

The vector $\mathbf{u}$ is then a non-zero vector with non-negative coefficients, and $T\mathbf{u} \geq \rho\mathbf{u}$ for all real numbers ρ satisfying $0 \leq \rho \leq \mu$.

Now every non-zero n-dimensional vector with non-negative real coefficients is a scalar multiple of some vector belonging to the simplex Δ. We conclude therefore that if ρ is a non-negative real number, if $\mathbf{v}$ is a non-zero vector with non-negative coefficients, and if $T\mathbf{v} \geq \rho\mathbf{v}$ then $\rho \leq \mu$. The result follows. ∎

Definition

Let T be a non-negative square matrix. The *Perron root* (or *Perron-Frobenius eigenvalue*) of T is the unique non-negative real number μ of T that can be characterized as the greatest real number for which there exists a non-zero vector $\mathbf{v}$ with real coefficients satisfying the conditions $\mathbf{v} \geq \mathbf{0}$ and $T\mathbf{v} \geq \mu\mathbf{v}$.

Remark

Proposition 6.3 ensures that every non-negative square matrix has a well-defined Perron root. The alternative name *Perron-Frobenius eigenvalue* for the Perron root seems to imply that the Perron root of a non-negative square matrix is an eigenvalue of that matrix. This result is indeed true. It will be proved for positive square matrices in Proposition 6.4, and the result will be extended to non-negative square matrices in Proposition 6.5. The eigenvalues of a square matrix over the field of complex numbers are the roots of the characteristic polynomial of that matrix.

Proposition 6.4

Let T be a positive square matrix, and let μ be the Perron root of T. Then $\mu > 0$, and there exists $\mathbf{b} \in \mathbb{R}^n$ satisfying the conditions $\mathbf{b} >> \mathbf{0}$ and $T\mathbf{b} = \mu\mathbf{b}$. Moreover, given any $\mathbf{u} \in \mathbb{R}^n$ satisfying $T\mathbf{u} \geq \mu\mathbf{u}$, there exists a real number t for which $\mathbf{u} = t\mathbf{b}$, and thus $T\mathbf{u} = \mu\mathbf{u}$.

Proof

The definition of the Perron root μ of T ensures that there exists a non-zero vector $\mathbf{b}$ with the properties that $\mathbf{b} \geq \mathbf{0}$ and $T\mathbf{b} \geq \mu\mathbf{b}$. Suppose it were the case that $T\mathbf{b} \neq \mu\mathbf{b}$. Let $\mathbf{v} = T\mathbf{b}$. Then

$$T\mathbf{v} - \mu\mathbf{v} = T(T\mathbf{b} - \mu\mathbf{b}) >> \mathbf{0},$$

because $T\mathbf{b} - \mu\mathbf{b} \geq \mathbf{0}$, $T\mathbf{b} - \mu\mathbf{b} \neq \mathbf{0}$ and $T >> 0$ (see Lemma 6.2). But then there would exist real numbers ρ satisfying $\rho > \mu$ that were sufficiently close to μ to ensure that $T\mathbf{v} - \rho\mathbf{v} >> \mathbf{0}$ and thus $T\mathbf{v} \geq \rho\mathbf{v}$. This would contradict the condition on the statement of the proposition that characterizes the value of μ. We conclude therefore that $T\mathbf{b} = \mu\mathbf{b}$.

Moreover $T\mathbf{b} >> \mathbf{0}$, because $T >> 0$ and $\mathbf{b} \geq 0$. It follows that $\mu > 0$ and $\mathbf{b} >> \mathbf{0}$.

Next let $\mathbf{u}$ be an n-dimensional vector with real coefficients for which $T\mathbf{u} \geq \mu\mathbf{u}$. If s is positive and sufficiently large then $s\mathbf{b} - \mathbf{u} >> 0$. On the other hand if s is negative and $|s|$ is sufficiently large then $s\mathbf{b} - \mathbf{u} << \mathbf{0}$. It follows from this that there exists a well-defined real number t defined so that

$$t = \inf\{s \in \mathbb{R} : s\mathbf{b} - \mathbf{u} \geq \mathbf{0}\}.$$

Then $t\mathbf{b} - \mathbf{u} \geq 0$, and moreover there exists some integer j between 1 and n for which $t(\mathbf{b})_j - (\mathbf{u})_j = 0$. Now

$$T(t\mathbf{b} - \mathbf{u}) = \mu t\mathbf{b} - T\mathbf{u} \leq \mu(t\mathbf{b} - \mathbf{u}),$$

and therefore $(T(t\mathbf{b} - \mathbf{u}))_j \leq 0$. If it were the case that $t\mathbf{b} - \mathbf{u} \neq 0$ then the inequalities $t\mathbf{b} - \mathbf{u} \geq 0$ and $T >> 0$ would ensure that $T(t\mathbf{b} - \mathbf{u}) >> \mathbf{0}$ (Lemma 6.2), from which it would follow that $(T(t\mathbf{b} - \mathbf{u}))_j > 0$. Because this latter inequality does not hold, it must be the case that $t\mathbf{b} - \mathbf{u} = 0$, and thus $\mathbf{u} = t\mathbf{b}$. The result follows. ∎

Proposition 6.5

Let T be a non-negative square matrix, and let μ be the Perron root of T. Then μ is an eigenvalue of T, and there exists a non-negative eigenvector $\mathbf{b}$ *associated with the eigenvalue μ.*

Proof

Let T be an non-negative $n \times n$ matrix. Then there exists an infinite sequence $T_1, T_2, T_3, \ldots$ of positive $n \times n$ matrices such that $T_r >> T$ for all positive integers r and $T_r \to T$ as $r \to +\infty$. Let μ_r be the Perron root of T_r and let $\mathbf{b}_r$ be the associated positive eigenvector, normalized to satisfy the condition $\sum_{j=1}^{n} (\mathbf{b}_r)_j = 1$.

The multidimensional Bolzano-Weierstrass Theorem (Theorem 1.2) ensures the existence of an infinite subsequence $T_{r_1}, T_{r_2}, T_{r_3}, \ldots$, a real number μ' and a vector $\mathbf{b} \in \mathbb{R}^n$ such that $\mu_{r_s} \to \mu$ and $\mathbf{b}_{r_s} \to \mathbf{b}$. Replacing the original sequence T_1, T_2, T_3 by a subsequence, if necessary, we may assume, without loss of generality, that $\mu_r \to \mu'$ and $\mathbf{b}_r \to \mathbf{b}$ as $r \to +\infty$. Then $\mu' \geq 0$, $(\mathbf{b})_j \geq 0$ for $j = 1, 2, \ldots, n$ and $\sum_{j=1}^{n} (\mathbf{b})_j = 1$. Then

$$T\mathbf{b} - \mu'\mathbf{b} = \lim_{r \to +\infty} (T_r \mathbf{b}_r - \mu_r \mathbf{b}_r) = \mathbf{0}.$$

Thus μ' is an eigenvalue of T, and $\mathbf{b}$ is a non-zero non-negative eigenvector of T associated to the eigenvalue μ'.

It remains to show that $\mu' = \mu$. Let ρ be a non-negative real number. Suppose that there exists a non-zero vector $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{v} \geq \mathbf{0}$ and $T\mathbf{v} \geq \rho\mathbf{v}$. Then, for each integer r, $T_r\mathbf{v} \geq \rho\mathbf{v}$, because $T_r >> T$, and therefore $\rho \leq \mu_r$. It follows that $\rho \leq \mu'$, because $\mu' = \lim_{r \to +\infty} \mu_r$. Also $T\mathbf{b} = \mu'\mathbf{b}$. It follows that μ' is the largest real number for which there exists a non-zero vector $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{v} \geq \mathbf{0}$ and $T\mathbf{v} \geq \rho\mathbf{v}$. Thus $\mu' = \mu$. The result follows. $\blacksquare$

Lemma 6.6

Let T be a non-negative $n \times n$ (square) matrix, let λ be a complex number, let $\mathbf{u}$ be an non-zero n-dimensional vector with complex coefficients, and let $\mathbf{v}$ be the n-dimensional vector with non-negative real coefficients defined such that $(\mathbf{v})_j = |(\mathbf{u})_j|$ for $j = 1, 2, \ldots, n$. Suppose that $\mathbf{u}$ is an eigenvector of T with eigenvalue λ, so that $T\mathbf{u} = \lambda\mathbf{u}$. Then $T\mathbf{v} \geq |\lambda|\mathbf{v}$. Moreover if $T >> 0$, and if $T\mathbf{v} = |\lambda|\mathbf{v}$, then λ is a positive real number, and there exists some complex number ω satisfying $|\omega| = 1$ for which $\mathbf{u} = \omega\mathbf{v}$.

Proof

There exist real numbers $\theta_1, \theta_2, \ldots, \theta_n$ and φ such that $u_j = e^{i\theta_j} v_j$ for $j = 1, 2, \ldots, n$ and $\lambda = e^{i\varphi}|\lambda|$, where $i = \sqrt{-1}$. (Here $e^{i\alpha} = \cos\alpha + i \sin\alpha$ for all real numbers α.) The identity $T\mathbf{u} = \lambda\mathbf{u}$ ensures that

$$|\lambda| v_j = e^{-i\varphi - i\theta_j} \lambda u_j = e^{-i\varphi - i\theta_j} \sum_{k=1}^{n} T_{j,k} u_k = \sum_{k=1}^{n} e^{-i\varphi + i\theta_k - i\theta_j} T_{j,k} v_k.$$

Taking real parts, we see that

$$|\lambda| v_j = \sum_{k=1}^{n} \cos(-\varphi + \theta_k - \theta_j) T_{j,k} v_k \leq \sum_{k=1}^{n} T_{j,k} v_k.$$

It follows that $T\mathbf{v} \geq |\lambda|\mathbf{v}$. Moreover if $T\mathbf{v} = |\lambda|\mathbf{v}$ then $\cos(-\varphi + \theta_k - \theta_j) = 1$ for all integers j and k between 1 and n for which $v_k > 0$ and $T_{j,k} > 0$.

Now suppose that $T >> 0$ and $T\mathbf{v} = |\lambda|\mathbf{v}$. Then $\mathbf{v} \neq 0$, because $\mathbf{u} \neq 0$. Also $\mathbf{v} \geq \mathbf{0}$. Therefore $T\mathbf{v} >> \mathbf{0}$ (Lemma 6.2). It follows that $\lambda \neq 0$ and $\mathbf{v} >> \mathbf{0}$. Then $T_{j,k} > 0$ and $v_k > 0$ for all integers j and k between 1 and n, and therefore $\cos(-\varphi + \theta_k - \theta_j) = 1$ for all integers j and k. Applying this result with $j = k$, we find that $\cos(-\varphi) = 1$, and therefore φ is an integer multiple of 2π. It then follows that $\theta_j - \theta_k$ is an integer multiple of 2π for all j and k. But these real numbers φ, θ_j and θ_k are only determined up to addition of an integer multiple of 2π. Let $\omega = e^{i\theta_1}$. Then $e^{i\varphi} = 1$ and and $e^{i\theta_j} = \omega$ for $j = 1, 2, \ldots, n$. It follows that λ is real and positive, and $\mathbf{u} = \omega\mathbf{v}$, where ω is a complex number satisfying $|\omega| = 1$, as required. ∎

Proposition 6.7

Let T be a non-negative square matrix, and let μ be the Perron root of T. Then every eigenvalue λ of T satisfies the inequality $|\lambda| \leq \mu$.

Proof

Let λ be an eigenvalue of T, and let $\mathbf{u}$ be an eigenvector of T with eigenvalue λ. The number λ and the coefficients of the vector $\mathbf{u}$ may be real or complex. Let $\mathbf{v} \in \mathbb{R}^n$ be defined such that $(\mathbf{v})_j = |(\mathbf{u})_j)|$ for $j = 1, 2, \ldots, n$. Now $T\mathbf{u} = \lambda\mathbf{u}$. It follows from Lemma 6.6 that $T\mathbf{v} \geq |\lambda|\mathbf{v}$. The definition of the Perron root μ then ensures that $|\lambda| \leq \mu$, as required. ∎

Proposition 6.8

Let T be a non-negative $n \times n$ (square) matrix, and let μ denote the Perron root of T. Let I denote the identity $n \times n$ matrix. Then, given any σ is a non-negative real number satisfying $\mu\sigma < 1$, the matrix $I - \sigma T$ is invertible and $(1 - \sigma T)^{-1}$ is a non-negative matrix.

Proof

We use some basic results of linear algebra and complex analysis. Let z be a complex number. Then the eigenvectors of the matrix $I - zT$ are the same as those of the matrix T, and therefore the eigenvalues of $I - zT$ are of the form $1 - z\lambda$ as λ ranges of the eigenvales of T.

Now the modulus of any eigenvalue of the non-negative matrix T is bounded above by the Perron root of T (Proposition 6.7). Therefore the eigenvalues of $I - zT$ have real part not less than $1 - |z|\mu$. A square matrix is invertible if zero is not an eigenvalue of that matrix. It follows that the matrix $I - zT$ is invertible for all complex numbers z satisfying $\mu|z| < 1$.

The determinant of the matrix $I - zT$ is a polynomial function of z. It follows that if $\mu > 0$ then all coefficients of the matrix $(I - zT)^{-1}$ are holomorphic functions of the complex variable z throughout the disk $\{z \in \mathbb{C} : |z| < \mu^{-1}\}$, and if $\mu = 0$ then all coefficients of the matrix $(I - zT)^{-1}$ are holomorphic functions of the complex variable z throughout entire complex plane. A basic theorem of complex analysis therefore ensures that each coefficient of the matrix $(I - zT)^{-1}$ may be represented as a power series in the complex plane z that converges for all complex numbers z satisfying $\mu|z| < 1$.

Now

$$(1 - zT)(1 + zT + z^2T^2 + z^3T^3 + \cdots + z^kT^k) = 1 - z^{k+1}T^{k+1},$$

and thus

$$\begin{aligned}(1 - zT)^{-1} &= 1 + zT + z^2T^2 + z^3T^3 + \cdots + z^kT^k \\ &\quad + z^{k+1}T^{k+1}(I - zT)^{-1}\end{aligned}$$

when $\mu|z| < 1$.

Now it has already been shown that $(1 - zT)^{-1}$ can be represented by a power series in z that converges whenever $\mu|z| < 1$. we can therefore conclude that

$$(1 - zT)^{-1} = 1 + zT + z^2T^2 + z^3T^3 + \cdots$$

for all complex numbers z satisfying $\mu|z| < 1$.

In particular

$$(1 - \sigma T)^{-1} = 1 + \sigma T + \sigma^2 T^2 + \sigma^3 T^3 + \cdots$$

for all non-negative real numbers σ satisfying $\mu\sigma < 1$. But each summand on the right side of this power series representation of $(1 - \sigma T)^{-1}$ is a non-negative matrix. It follows that $I - \sigma T$ is invertible and $(1 - \sigma T)^{-1}$ is a non-negative matrix for all non-negative real numbers σ satisfying $\sigma\rho < 1$, as required. ∎

Proposition 6.9

Let T be a non-negative $n \times n$ (square) matrix, let μ denote the Perron root of T. Then the Perron root of the transpose T^T is equal to the Perron root μ of T, and there exists a non-zero vector $\mathbf{p} \in \mathbb{R}^n$ satisfying $\mathbf{p} \geq \mathbf{0}$ and $\mathbf{p}^T T = \mu \mathbf{p}^T$, where $\mathbf{p}^T$, the transpose of $\mathbf{p}$ is the row vector components are the components of the column vector $\mathbf{p}$.

Proof

The transpose T^T of the non-negative square matrix T is itself a non-negative square matrix with the same characteristic polynomial as T, and thus with the same eigenvalues as T. The Perron root of the transpose T^T of T is a non-negative real eigenvalue of T^T (Proposition 6.5), and moreover it is an upper bound on the modulus of every eigenvalue of T^T (Proposition 6.7. It follows that the non-negative square matrix T and its transpose T^T have the same Perron root. Moreover the Perron root is an eigenvalue of T^T, and therefore there exists a non-zero vector $\mathbf{p} \in \mathbb{R}^n$ for which $\mathbf{p} \geq 0$ and $T^T\mathbf{p} = \mu\mathbf{p}$. Taking the transpose of this equation, we find that $\mathbf{p}^T T = \mu\mathbf{p}^T$, as required. ∎

Proposition 6.10

Let T be a non-negative $n \times n$ (square) matrix, let μ denote the Perron root of T, and let σ is a non-negative real number. Then there exists a non-zero vector $\mathbf{w} \in \mathbb{R}^n$ satisfying $\mathbf{w} \geq \mathbf{0}$ and $\mathbf{w} >> \sigma T\mathbf{w}$ if and only if $\mu\sigma < 1$.

Proof

Let $\mathbf{v} \in \mathbb{R}^n$ satisfy $\mathbf{v} >> \mathbf{0}$, and let $\mathbf{w} = (I - \sigma T)^{-1}\mathbf{v}$, where I denotes the identity $n \times n$ matrix. It follows from Proposition 6.8 that if $\mu\sigma < 1$ then $(I - \sigma T)^{-1}$ a non-negative matrix, and therefore $\mathbf{w} \geq 0$. Also

$$\mathbf{w} - \sigma T\mathbf{w} = (I - \sigma T)\mathbf{w} = \mathbf{v} >> \mathbf{0},$$

and therefore $\mathbf{w} >> \sigma T\mathbf{w}$. We have thus shown that if $\mu\sigma < 1$ then there exists a vector $\mathbf{w}$ with the required properties.

Conversely suppose that σ is a non-negative real number and that $\mathbf{w} \in \mathbb{R}^n$ is a non-zero vector for which $\mathbf{w} \geq 0$ and $\mathbf{w} >> \sigma T\mathbf{w}$. It follows from Proposition 6.9 that there exists a non-zero vector $\mathbf{p} \in \mathbb{R}^n$ satisfying $\mathbf{p} \geq \mathbf{0}$ and $\mathbf{p}^T T = \mu \mathbf{p}^T$, where $\mathbf{p}^T$ denotes the transpose of $\mathbf{p}$. Then

$$(1 - \sigma\mu)\mathbf{p}^T\mathbf{w} = \mathbf{p}^T\mathbf{w} - \sigma\mu\mathbf{p}^T\mathbf{w} = \mathbf{p}^T(\mathbf{w} - \sigma T\mathbf{w}) > 0.$$

It follows that $\mathbf{p}^T\mathbf{w} > 0$ and $\sigma\mu < 1$, as required. This completes the proof. ∎

6.2. Perron's Theorem for Positive Matrices

In 1907 Oskar Perron (1880–1975) proved a fundamental theorem concerning the eigenvalues and eigenvectors of a positive square matrix, in particular showing that the positive real number now referred to as the *Perron root* (or *Perron-Frobenius eigenvalue*) of the matrix is a simple eigenvector, with a one-dimensional eigenspace spanned by a positive eigenvector, and that any other eigenvalues of the matrix has a modulus strictly less than the Perron root. In 1912, Georg Frobenius (1849-1917) generalized Perron's Theorem to a particular class of non-negative square matrices that are said to be *unzerlegbar* (i.e., "indecomposible" or "irreducible"). These discoveries initiated the development of a body of results concerning non-negative square matrices that is today referred to as *Perron-Frobenius Theory*

Before stating and proving Perron's Theorem, we review (without proof) some standard results from linear algebra, related to the Jordan normal form of a square matrix, that are relevant to the proof of Perron's Theorem.

Let T be a linear operator defined on a finite-dimensional complex vector space V. Then the vector space V can be decomposed as a direct sum of subspaces that are invariant under the action of T and cannot be further decomposed as direct sums of invariant subspaces. Then

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$$

where, for each integer r between 1 and m, the linear operator T maps the subspace V_r of V into itself. Moreover the subspace V_r has no proper non-zero vector subspace that is invariant under the action of T. Associated with each subspace V_r is a complex number λ_r that is the unique eigenvalue of the restriction of the linear operator T to V_r.

The *characteristic polynomial* χ of T on V is defined such that $\chi(z) = \det(zI_V - T)$, where I_V denotes the identity operator on V. It can be shown that

$$\chi(z) = \prod_{r=1}^{m} (z - \lambda_r)^{d_r},$$

where $d_r = \dim_{\mathbb{C}} V_r$ for $r = 1, 2, \ldots, m$. It follows that a complex number λ is a simple root of the characteristic polynomial χ of T if and only if the following two conditions are satisfied: there exists exactly one integer r between 1 and m for which $\lambda = \lambda_r$; for this value of r, $d_r = 1$.

The theory of the Jordan Normal Form ensures that each subspace V_r has a basis of the form

$$\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{d_r},$$

with the property that $T\mathbf{e}_1 = \lambda_r \mathbf{e}_s$ and $T\mathbf{e}_s = \lambda_r \mathbf{e}_s + \mathbf{e}_{s-1}$ for $1 < s \leq d_r$. All eigenvectors of T contained in V_r are scalar multiples of $\mathbf{e}_1$. Moreover if $d_r > 1$ then $(T - \lambda_r I_{V_r})^2 \mathbf{e}_2 = \mathbf{0}$ but $T\mathbf{e}_2 \neq \lambda_r \mathbf{e}_2$.

These results of linear algebra, summarized without detailed proof, yield the result stated in the following lemma.

Lemma 6.11

Let T be a linear operator acting on a finite-dimensional complex vector space V, and let λ be an eigenvalue of T. Then λ is a simple root of the characteristic polynomial of T if and only if the following two conditions are satisfied:

- *the eigenspace associated with the eigenvalue λ is one-dimensional;*
- *if $\mathbf{v} \in V$ satisfies the identity $(T - \lambda I_V)^2 \mathbf{v} = \mathbf{0}$ then $T\mathbf{v} = \lambda \mathbf{v}$.*

Theorem 6.12 (Perron)

Let T be a positive square matrix, and let μ be the Perron root of T. Then the following properties are satisfied:—

(i) *there exists an eigenvector of T with associated eigenvalue μ whose coefficients are all strictly positive;*

(ii) *the eigenvalue μ is a simple root of the characteristic polynomial of T, and the corresponding eigenspace is therefore one-dimensional;*

(iii) *all eigenvalues λ (real or complex) of T that are distinct from μ satisfy the inequality $|\lambda| < \mu$.*

Proof

Let the positive square matrix T be an $n \times n$ matrix, and let μ denote the Perron root of T. Proposition 6.4 establishes that the Perron root μ of T is well-defined and is an eigenvalue of T with which is associated an eigenvector $\mathbf{b}$ with positive coefficients. Moreover Proposition 6.4 ensures that the following properties are then satisfied:—

(iv) $\mathbf{b} >> \mathbf{0}$ and $T\mathbf{b} = \mu\mathbf{b}$;

(v) if ρ is a non-negative real number, if $\mathbf{v}$ is a non-zero n-dimensional vector with non-negative coefficients, and if $T\mathbf{v} \geq \rho\mathbf{v}$, then $\rho \leq \mu$.

(vi) given any n-dimensional vector $\mathbf{u}$ with real coefficients for which $T\mathbf{u} \geq \mu\mathbf{u}$, there exists a real number t for which $\mathbf{u} = t\mathbf{b}$, and thus $T\mathbf{u} = \mu\mathbf{u}$.

Now because the coefficients of the matrix T are all real, and μ is also a real number, the real and imaginary parts of any eigenvector of T with associated eigenvalue μ must themselves be eigenvectors with eigenvalue μ. The result just obtained therefore ensures that any convex eigenvector of T with eigenvalue μ must be a complex scalar multiple of the eigenvector $\mathbf{b}$. Thus the eigenspace of T associated with the eigenvalue μ is one-dimensional, when considered over the field of complex numbers.

Let I denote the identity $n \times n$ matrix, and let $\mathbf{v}$ be real n-dimensional vector for which $(T - \mu I)^2 \mathbf{v} = \mathbf{0}$. Then $T\mathbf{v} - \mu\mathbf{v}$ is an eigenvector of T with associated eigenvalue μ. It follows from property (vi) above that there must exist some real number α for which $T\mathbf{v} - \mu\mathbf{v} = \alpha\mathbf{b}$. Now $\mathbf{b} >> \mathbf{0}$. It follows that if $\alpha \geq 0$ then $T\mathbf{v} \geq \mu\mathbf{v}$. But property (vi) stated at the commencement of the proof then ensures that $\mathbf{v} = t\mathbf{b}$ for some real number t. But then $T\mathbf{v} = \mu\mathbf{v}$ and $\alpha = 0$. Similarly if $\alpha \leq 0$ then $T(-\mathbf{v}) \geq \mu(-\mathbf{v})$, and this also ensures that $\alpha = 0$. It follows that if $\mathbf{v}$ is a real n-dimensional vector satisfying $(T - \mu I)^2 \mathbf{v} = \mathbf{0}$ then $T\mathbf{v} = \mu\mathbf{v}$. The criterion stated in Lemma 6.11 therefore establishes that μ is a simple root of the characteristic polynomial of T.

We have now verified (i) and (ii). It remains to verify that all eigenvalues λ of T distinct from μ satisfy the inequality $|\lambda| < \mu$. Now it follows from Proposition 6.7 that all eigenvalues λ of T satisfy the inequality $|\lambda| \leq \mu$.

Now suppose that $|\lambda| = \mu$. It then follows from property (vi), stated at the commencement of the proof, that $T\mathbf{v} = \mu\mathbf{v} = |\lambda|\mathbf{v}$. It then follows from Lemma 6.6 that λ is a positive real number, and therefore $\lambda = \mu$. This completes the proof of (iii), and therefore completes the proof of the theorem. $\blacksquare$