MAU34804—Fixed Point Theorems and Economic Equilibria
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Section 7: Game Theory and Nash Equilibria

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7. Game Theory and Nash Equilibria

7.1. Zero-Sum Two-Person Games

Example

Consider the following hand game. This is a zero-sum two-person game. At each go, the two players present simultaneously either and open hand or a fist. If both players present fists, or if both players present open hands, then no money changes hands. If one player presents a fist and the other player presents an open hand then the player presenting the fist receives ten cents from the player presenting the open hand.

The payoff for the first player can be represented by the following payoff matrix:

$$\begin{pmatrix} 0 & -10 \\ 10 & 0 \end{pmatrix}.$$

In this matrix the entry in the first row represent the payoffs when the first player presents an open hand; those in the second row represent the payoffs when the first player presents a fist. The entries in the first column represent the payoff when the second player presents an open hand; those in the second column represent the payoffs when the second player presents a fist. In this game the second player, choosing the best strategy, is always going to play a fist, because that reduces the payoff for the first player, whatever the first player chooses to play. Similarly the first player, choosing the best strategy, is going to play a fist, because that maximizes the payoff for the first player whatever the second player does. Thus in this game, both players choosing the best strategies, play fists.

It should be noticed that, in this situation, if the second player always plays a fist, the first player would not be tempted to move from a strategy of always playing a fist in order get a better payoff. Similarly if the first player always plays a fist, then the second player would not be tempted to move from a strategy of always playing a fist in order to reduce the payoff to the first player. This is a very simple example of a *Nash Equilibrium*. This equilibrium arises because the element in the second row and second column of the payoff matrix is simultaneously the largest element in its column and the smallest element in its row. Matrix elements with this property as said to be *saddle points* of the matrix.

Example

Now consider the game of *Rock, Paper, Scissors*. This game has a long history, and versions of this game were well-established in China and Japan in particular for many centuries.

Two players simultaneously present hand symbols representing *Rock* (a closed fist), *Paper* (a flat hand), or *Scissors* (first two fingers outstretched in a 'V'). Paper beats Rock, Scissors beats Paper, Rock beats Scissors. If both players present the same hand symbol then that round is a draw.

Ordering the strategies for the playes in the order *Rock* (1st), *Paper* (2nd) and *Scissors* (3rd), the payoff matrix for the first player is the following:—

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

The entry in the ith row and jth column of this payoff matrix represents the return to the first player on a round of the game if the first player plays strategy i and the second player plays strategy j.

A *pure strategy* would be one in which a player presents the same hand symbol in every round. But it is not profitable for any player in this game to adopt a pure strategy. If the first player adopts a strategy of playing *Paper*, then the second player, on observing this, would adopt a strategy of always playing *Scissors*, and would beat the first player on every round. A preferable strategy, for each player, is the *mixed strategy* of playing *Rock*, *Paper* and *Scissors* with equal probability, and seeking to ensure that the sequence of plays is as random as possible.

Let us denote by M the payoff matrix above. A *mixed strategy* for the first player is one in which, on any given round *Rock* is played with probability p_1, *Paper* is played with probability p_2 and *Scissors* is played with probability p_3. The mixed strategies for the first player can therefore be represented by points of a triangle Δ_P, where

$$\begin{aligned}\Delta_P &= \Big\{(p_1, p_2, p_3) \in \mathbb{R}^3 : \\ &\qquad p_1 \geq 0, \;\; p_2 \geq 0, \;\; p_3 \geq 0, \;\; p_1 + p_2 + p_3 = 1\Big\}.\end{aligned}$$

A mixed strategy for the second player is one in which *Rock* is played with probability q_1, *Paper* with probability q_2 and *Scissors* with probability q_3. The mixed strategies for the second player can therefore be represented by points of a triangle Δ_Q, where

$$\begin{aligned}\Delta_Q &= \Big\{(q_1, q_2, q_3) \in \mathbb{R}^3 : \\ &\qquad q_1 \geq 0,\ \ q_2 \geq 0,\ \ q_3 \geq 0,\ \ q_1 + q_2 + q_3 = 1\Big\}.\end{aligned}$$

Let $\mathbf{p} \in \Delta_P$ represent the mixed strategy chosen by the first player, and $\mathbf{q} \in \Delta_Q$ the mixed strategy chosen by the second player, where

$$\mathbf{p} = (p_1, p_2, p_3), \quad \mathbf{q} = (q_1, q_2, q_3).$$

Let M_{ij} the payoff for the first player when the first player plays strategy i and the second player plays strategy j. Then M_{ij} is the entry in the ith row and jth column of the payoff matrix M. In matrix equations we consider $\mathbf{p}$ and $\mathbf{q}$ to be column vectors, denoting their transposes by the row matrices $\mathbf{p}^T$ and $\mathbf{q}^T$. The *expected payoff* for the first player is then $f(\mathbf{p}, \mathbf{q})$, where

$$f(\mathbf{p}, \mathbf{q}) = \mathbf{p}^T M \mathbf{q} = \sum_{i=1}^{3} \sum_{j=1}^{3} p_i M_{ij} q_j.$$

Let $\mathbf{p}^* = (p_1^*, p_2^*, p_3^*)$, where

$$p_1^* = p_2^* = p_3^* = \tfrac{1}{3}.$$

Then $\mathbf{p}^{*T} M = (0, 0, 0)$, and therefore

$$f(\mathbf{p}^*, \mathbf{q}) = 0$$

for all $\mathbf{q} \in \Delta_Q$. Similarly let $\mathbf{q}^* = (q_1^*, q_2^*, q_3^*)$, where

$$q_1^* = q_2^* = q_3^* = \tfrac{1}{3}.$$

Then

$$f(\mathbf{p}, \mathbf{q}^*) = 0$$

for all $\mathbf{p} \in \Delta_Q$. Thus the inequalities

$$f(\mathbf{p}, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q})$$

are satisfied for all $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_q$, because each of the quantities occurring is equal to zero.

Were the first player to adopt a mixed strategy $\mathbf{p}$, where $\mathbf{p} = (p_1, p_2, p_3)$, $p_i \geq 0$ for $i = 1, 2, 3$ and $p_1 + p_2 + p_3 = 1$, the second player could adopt mixed strategy $\mathbf{q}$, where $\mathbf{q} = (q_1, q_2, q_3) = (p_3, p_1, p_2)$. The payoff $f(\mathbf{p}, \mathbf{q})$ is then

$$\begin{aligned}
f(\mathbf{p}, \mathbf{q}) &= -p_1 q_2 + p_1 q_3 - p_2 q_3 + p_2 q_1 - p_3 q_1 + p_3 q_2 \\
&= -p_1^2 + p_1 p_2 - p_2^2 + p_2 p_3 - p_3^2 + p_3 p_1 \\
&= -\tfrac{1}{6}\Big((2p_1 - p_2 - p_3)^2 + (2p_2 - p_3 - p_1)^2 \\
&\qquad + (2p_3 - p_1 - p_2)^2 \Big) \\
&\leq 0.
\end{aligned}$$

Moreover if $f(\mathbf{p}, \mathbf{q}) = 0$, where $q_1 = p_3$, $q_2 = p_1$ and $q_3 = p_2$, then

$$(2p_1 - p_2 - p_3)^2 + (2p_2 - p_3 - p_1)^2 + (2p_3 - p_1 - p_2)^2 = 0$$

and therefore $2p_1 = p_2 + p_3$, $2p_2 = p_3 + p_1$ and $2p_3 = p_1 + p_2$. But then

$$3p_1 = 3p_2 = 3p_3 = p_1 + p_2 + p_3 = 1,$$

and thus $\mathbf{p} = \mathbf{p}^*$. It follows that if $\mathbf{p} \in \Delta_Q$ and $\mathbf{p} \neq \mathbf{p}^*$ then there exists $\mathbf{q} \in \Delta_Q$ for which $f(\mathbf{p}, \mathbf{q}) < 0$. Thus if the first player adopts a mixed strategy other than the strategy $\mathbf{p}^*$ in which *Rock*, *Paper*, *Scissors* are played with equal probability on each round, there is a mixed strategy for the second player that ensures that the average payoff for the first player is negative, and thus the first player will lose in the long run over many rounds. Thus strategy $\mathbf{p}^*$ is the only sensible mixed strategy that the first player can adopt. The corresponding strategy $\mathbf{q}^*$ is the only sensible mixed strategy that the second player can adopt. The average payoff for each player is then equal to zero.

7.2. Von Neumann's Minimax Theorem

In 1920, John Von Neumann published a paper entitled "Zur Theorie der Gesellschaftsspielle" (*Mathematische Annalen*, Vol. 100 (1928), pp. 295–320). The title translates as "On the Theory of Social Games". This paper included a proof of the following "Minimax Theorem", which made use of the Brouwer Fixed Point Theorem. An alternative proof using results concerning convexity was presented in the book *On the Theory of Games and Economic Behaviour* by John Von Neumann and Oskar Morgenstern (Princeton University Press, 1944). George Dantzig, in a paper published in 1951, showed how the theorem could be solved using linear programming methods (see Joel N. Franklin, *Methods of Mathematical Economics*, (Springer Verlag, 1980, republished by SIAM in 1982).

Theorem 7.1 (Von Neumann's Minimax Theorem)

Let M be an $m \times n$ matrix, let

$$\begin{aligned}
\Delta_P &= \Bigg\{(p_1, p_2, \ldots, p_m) \in \mathbb{R}^m : p_i \geq 0 \text{ for } i = 1, 2, \ldots, m, \text{ and} \\
&\qquad \sum_{i=1}^{m} p_i = 1\Bigg\}, \\
\Delta_Q &= \Bigg\{(q_1, q_2, \ldots, q_n) \in \mathbb{R}^n : q_i \geq 0 \text{ for } i = 1, 2, \ldots, n, \text{ and} \\
&\qquad \sum_{j=1}^{n} q_j = 1\Bigg\},
\end{aligned}$$

and let

$$f(\mathbf{p}, \mathbf{q}) = \mathbf{p}^T M \mathbf{q} = \sum_{i=1}^{m} \sum_{j=1}^{n} M_{i,j} p_i q_j$$

for all $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_Q$. Then there exist $\mathbf{p}^* \in \Delta_P$ and $\mathbf{q}^* \in \Delta_Q$ such that

$$f(\mathbf{p}, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q})$$

for all $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_Q$.

Proof

Let $f(\mathbf{p}, \mathbf{q}) = \mathbf{p}^T M \mathbf{q}$ for all $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_Q$. Given $\mathbf{q} \in \Delta_Q$, let

$$\mu_P(\mathbf{q}) = \sup\{f(\mathbf{p}, \mathbf{q}) : \mathbf{p} \in \Delta_P\}$$

and let

$$P(\mathbf{q}) = \{\mathbf{p} \in \Delta_P : f(\mathbf{p}, \mathbf{q}) = \mu_P(\mathbf{q})\}.$$

Similarly given $\mathbf{p} \in \Delta_P$, let

$$\mu_Q(\mathbf{p}) = \inf\{f(\mathbf{p}, \mathbf{q}) : \mathbf{q} \in \Delta_Q\}$$

and let

$$Q(\mathbf{p}) = \{\mathbf{q} \in \Delta_Q : f(\mathbf{p}, \mathbf{q}) = \mu_Q(\mathbf{q})\}.$$

An application of Berge's Maximum Theorem (Theorem 2.23) ensures that the functions $\mu_P\colon \Delta_P \to \mathbb{R}$ and $\mu_Q\colon \Delta_Q \to \mathbb{R}$ are continuous, and that the correspondences $P\colon \Delta_Q \rightrightarrows \Delta_P$ and $Q\colon \Delta_P \rightrightarrows \Delta_Q$ are non-empty, compact-valued and upper hemicontinuous. These correspondences therefore have closed graphs (see Proposition 2.11). Morever $P(\mathbf{q})$ is convex for all $\mathbf{q} \in \Delta_Q$ and $Q(\mathbf{p})$ is convex for all $\mathbf{p} \in \Delta_P$. Let $X = \Delta_P \times \Delta_Q$, and let $\Phi\colon X \rightrightarrows X$ be defined such that

$$\Phi(\mathbf{p}, \mathbf{q}) = P(\mathbf{q}) \times Q(\mathbf{p})$$

for all $(\mathbf{p}, \mathbf{q}) \in X$. Kakutani's Fixed Point Theorem (Theorem 5.4) then ensures that there exists $(\mathbf{p}^*, \mathbf{q}^*) \in X$ such that $(\mathbf{p}^*, \mathbf{q}^*) \in \Phi(\mathbf{p}^*, \mathbf{q}^*)$. Then $\mathbf{p}^* \in P(\mathbf{q}^*)$ and $\mathbf{q}^* \in Q(\mathbf{p}^*)$ and therefore

$$f(\mathbf{p}, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q})$$

for all $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_Q$, as required. ∎

7.3. Quasiconvex Functions

Definition

Let K be a convex set in some real vector space. A real-valued function $f\colon K \to \mathbb{R}$ is said to be *quasiconvex* if

$$f((1-t)\mathbf{u} + t\mathbf{v}) \leq \max\Big(f(\mathbf{u}), f(\mathbf{v})\Big)$$

for all $\mathbf{u}, \mathbf{v} \in K$ and for all real numbers t satisfying $0 \leq t \leq 1$.

Definition

Let K be a convex set in some real vector space. A real-valued function $f\colon K \to \mathbb{R}$ is said to be *quasiconcave* if

$$f((1-t)\mathbf{u} + t\mathbf{v}) \geq \min\Big(f(\mathbf{u}), f(\mathbf{v})\Big)$$

for all $\mathbf{u}, \mathbf{v} \in K$ and for all real numbers t satisfying $0 \leq t \leq 1$.

Linear functionals are quasiconvex and quasiconcave.

A function $f\colon K \to \mathbb{R}$ defined over a convex subset K of a real vector space is quasiconcave if and only if the function $-f$ is quasiconvex.

Lemma 7.2

Let K be a convex set in a real vector space, and let $f\colon K \to \mathbb{R}$ be a quasiconcave function. Then, for each real number s, the preimage $f^{-1}([s, +\infty))$ of the interval $[s, +\infty)$ is a convex subset of K, where

$$f^{-1}([s, +\infty)) = \{\mathbf{x} \in K : f(\mathbf{x}) \geq s\}.$$

Proof

Let $\mathbf{u}, \mathbf{v} \in f^{-1}([s, +\infty))$, and let t be a real number satisfying $0 \leq t \leq 1$. Then $f(\mathbf{u}) \geq s$ and $f(\mathbf{v}) \geq s$. It follows from the definition of quasiconcavity that

$$f((1-t)\mathbf{u} + t\mathbf{v}) \geq \min\Big(f(\mathbf{u}), f(\mathbf{v})\Big) \geq s,$$

and therefore $(1-t)\mathbf{u} + t\mathbf{v} \in f^{-1}([s, +\infty))$, as required. ∎

7.4. Nash Equilibria

We consider a *game* with n players. Each player choses a strategy from an appropriate *strategy sets*. The strategies chosen by the players in the game constitute a *strategy profile*. The *utility*, or *payoff*, of the game, for each player is determined by the strategy profile chosen by the players in the game. The technical details involved are explored and specified in more detail in the following discussion.

We suppose that, in an n-player game, the ith player choses strategies from a *strategy set* S_i, where S_i is represented as a non-empty compact convex set in $\mathbb{R}^{m_i}$ for some positive integer m_i. (The convexity requirement would typically be satisfied in games where players can adopt mixed strategies.) We let $S = S_1 \times S_1 \times \cdot \times S_n$. The elements of the set S are referred to as *strategy profiles*. The *strategy profile set* S is a compact convex subset of $\mathbb{R}^m$, where

$$m = m_1 + m_2 + \cdots + m_n.$$

For each integer i between 1 and n let us define

$$\begin{array}{rcl} S_{-1} & = & S_2 \times S_3 \times S_4 \times \cdots \times S_n, \\ S_{-2} & = & S_1 \times S_3 \times S_4 \times \cdots \times S_n, \\ S_{-3} & = & S_1 \times S_2 \times S_4 \times \cdots \times S_n, \\ & \vdots & \\ S_{-n} & = & S_1 \times S_2 \times S_3 \times \cdots \times S_{n-1}, \end{array}$$

so that

$$S_{-i} = S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$$

for all integers i between 1 and n (making the appropriate interpretation of the right hand side of this expression, as specified above, in the cases $i = 1$ and $i = n$). The set S_{-i} is then a compact convex subset of $\mathbb{R}^{m-m_i}$ for $i = 1, 2, \ldots, n$.

We define projections $\pi_i\colon S \to S_i$ and $\pi_{-i}\colon S \to S_{-i}$ for $i = 1, 2, \ldots, n$ in the obvious fashion so that

$$\pi_i(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = \mathbf{x}_i$$

and

$$\begin{array}{rcl}
\pi_{-1}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) & = & (\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \ldots, \mathbf{x}_n), \\
\pi_{-2}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) & = & (\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4, \ldots, \mathbf{x}_n), \\
\pi_{-3}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) & = & (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4, \ldots, \mathbf{x}_n), \\
 & \vdots & \\
\pi_{-n}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) & = & (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_{n-1}).
\end{array}$$

We now consider the utility, or payoff, of the game for the players. We suppose that, for each integer i between 1 and n, the *utility* of the game, from the perspective of the ith player, is determined by a utility function $u_i \colon S_i \times S_{-i} \to \mathbb{R}$ defined so that, for each element $\mathbf{x}_{-i}$ of S_{-i} representing a choice of strategies by players of the game other than the ith player, the real number $u_i(\mathbf{x}_i, \mathbf{x}_{-i})$ represents the utility, or payoff, for the ith player on adopting the strategy $\mathbf{i}$. We impose the following two requirements on these utility functions:

- the utility function $u_i \colon S_i \times S_{-i} \to \mathbb{R}$ is continuous on $S_i \times S_{-i}$;
- for fixed $\mathbf{x}_{-i}$, the function sending $\mathbf{x}_i$ to $u_i(\mathbf{x}_i, \mathbf{x}_{-i})$ is quasiconcave on S_i, and thus

$$u_i((1-t)\mathbf{x}_i' + t\mathbf{x}_i'', \mathbf{x}_{-i}) \geq \min\Big(u_i(\mathbf{x}_i', \mathbf{x}_{-i}), u_i(\mathbf{x}_i'', \mathbf{x}_{-i})\Big)$$

  for all $\mathbf{x}_i', \mathbf{x}_i'' \in S_i$, $\mathbf{x}_{-i} \in S_{-i}$ and real numbers t satisfying $0 \leq t \leq 1$.

Let $\mathbf{x}_i'$ and $\mathbf{x}_i''$ elements of the strategy set S_i, representing strategies for the ith player, and let $\mathbf{x}_{-i}$ be an element of S_{-i}, representing a profile of strategies adopted by the other players. Then the ith player actively prefers the outcome of strategy profile $\mathbf{x}_i''$ to that of strategy profile $\mathbf{x}_i'$ if and only if

$$u_i(\mathbf{x}_i', \mathbf{x}_{-i}) < u_i(\mathbf{x}_i'', \mathbf{x}_{-i}).$$

The ith player is indifferent between the outcomes of the strategy profiles $\mathbf{x}_i'$ and $\mathbf{x}_i''$ if and only if

$$u_i(\mathbf{x}_i', \mathbf{x}_{-i}) = u_i(\mathbf{x}_i'', \mathbf{x}_{-i}).$$

Definition

In an n-player game, let $S_1, S_2, \ldots, S_n$ denote the strategy sets for the players in the game, and let $u_i \colon S_i \times S_{-i} \to \mathbb{R}$ denote the utility function for the ith player in the game (where the set S_{-i} is defined for $i = 1, 2, \ldots, n$ as described above). A strategy profile

$$(\mathbf{x}_1^*, \mathbf{x}_2^*, \ldots, \mathbf{x}_n^*)$$

is said to be a *Nash equilibrium* for the game if

$$u_i(\mathbf{x}_i, \mathbf{x}_{-i}^*) \leq u_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*).$$

for all integers i between 1 and n and for all $\mathbf{x}_i \in S_i$.

Given any element $\mathbf{x}_{-i}$ of S_{-i} (representing a choice of strategies that might be adopted by the other players of the game), there will be a subset $B_i(\mathbf{x}_{-i})$ of S_i that represents the best strategies that the ith player can adopt when the other players are adopting the strategies represented by the element $\mathbf{x}_{-i}$ of S_{-i}. These best strategies are those strategies that maximize the utility function for the ith player, and we denote the value of the utility function u_i for those best strategies by $b_i(\mathbf{x}_{-i})$. Accordingly

$$\begin{aligned} b_i(\mathbf{x}_{-i}) &= \sup\{u_i(\mathbf{x}_i, \mathbf{x}_{-i}) : \mathbf{x}_i \in S_i\}, \\ B_i(\mathbf{x}_{-i}) &= \{\mathbf{x}_i \in S_i : u_i(\mathbf{x}_i, \mathbf{x}_{-i}) = b(\mathbf{x}_{-i})\}. \end{aligned}$$

We obtain in this fashion a single-valued function $b_i \colon S_{-i} \to S_i$ and a correspondence $B_i \colon S_{-i} \rightrightarrows S_i$.

Now, for each integer i between 1 and n, the constant correspondence that sends each element of S_{-i} to the strategy set S_i is clearly both upper hemicontinuous and lower hemicontinuous. The function $u_i \colon S_i \times S_{-i} \to \mathbb{R}$ is required to be continuous. Moreover, for each $\mathbf{x}_{i-1} \in S_{-i}$, the Extreme Value Theorem ensures that the set $B_i(\mathbf{x}_{-i})$ is non-empty, and the continuity of the utility function u_i ensures that $B_i(\mathbf{x}_{-i})$ is a closed subset of the compact set S_i. It follows that the the correspondence $B \colon S_{-i} \rightrightarrows S_i$ is both non-empty and compact. It therefore follows from Berge's Maximum Theorem (Theorem 2.23) that the function $b \colon S_{-i} \to \mathbb{R}$ is continuous on S_{-i}, $B_i(\mathbf{x}_{-i})$ is a compact subset of S_i for all $\mathbf{x}_{-i} \in S_{-i}$, and the correspondence $B \colon S_{-i} \rightrightarrows S_i$ is upper hemicontinuous in S_{-i}. Now every upper hemicontinuous closed-valued correspondence has a closed graph (Proposition 2.11). We conclude therefore that the correspondence $B \colon S_{-i} \rightrightarrows S_i$ has a closed graph.

Now, for each i, and for each $\mathbf{x}_{-i} \in S_{-i}$, the quasiconcavity requirement imposed on the utility function i ensures that the non-empty compact set $B_i(\mathbf{x}_{-i})$ is convex. Indeed the definition of $b_i(\mathbf{x}_{-i})$ and $B_i(\mathbf{x}_{-i})$ ensures that $u_i(\mathbf{z}, \mathbf{x}_{-i}) \leq b_i(\mathbf{x}_{-i})$ for all $\mathbf{z} \in S_i$, and $u_i(\mathbf{z}, \mathbf{x}_{-i}) = b_i(\mathbf{x}_{-i})$ for all $\mathbf{z} \in B_i(\mathbf{x}_{-i})$. It follows that

$$B_i(\mathbf{x}_{-i}) = \{\mathbf{z} \in S_i : u_i(\mathbf{z}, \mathbf{x}_{-i}) \geq b(\mathbf{x}_{-i})\}.$$

The quasiconcavity condition on the function u_i ensures that, for all $\mathbf{z}, \mathbf{z}' \in B_i(\mathbf{x}_{-i})$ and for all real numbers t satisfing $0 \leq t \leq 1$,

$$u_i((1-t)\mathbf{z}' + t\mathbf{z}'', \mathbf{x}_{-i}) \geq \min\Big(u_i(\mathbf{z}', \mathbf{x}_{-i}), u_i(\mathbf{z}'', \mathbf{x}_{-i})\Big) \geq b(\mathbf{x}_{-i}),$$

and therefore $(1-t)\mathbf{z}' + t\mathbf{z}'' \in B_i(\mathbf{x}_{-i})$. (This justification of the convexity of $B_i(\mathbf{x}_{-i})$ essentially repeats the argument presented in the proof of Lemma 7.2.)

We have now shown that, for each integer i between 1 and n, the correspondence $B_i\colon S_{-i} \to S_i$ that assigns to each element $\mathbf{x}_{-i}$ of S_{-i} the set of best strategies that the ith player can adopt in the event that the other players adopt the strategies represented by $\mathbf{x}_{-i}$ has closed graph, and maps each element $\mathbf{x}_{-i}$ of S_{-i} to a subset $B_i(\mathbf{x}_{-i})$ that is non-empty, compact and convex.

Now the Kakutani Fixed Point Theorem (Theorem 5.4) applies to correspondences with closed graph that map elements of a non-empty, compact and convex subset to non-empty convex subsets of that set. Thus in order to obtain a proof of the existence of Nash equilibria that utilizes the Kakutani Fixed Point Theorem, we must construct such a correspondence from a non-empty compact convex set to itself.

We recall that the *strategy profile set* S is defined to be the Cartesian product $S_1 \times S_2 \times \cdots \times S_n$ of the strategy sets for the players of the game. Let $\Phi\colon S \rightrightarrows S$ be the correspondence from the strategy profile set S to itself defined so that

$$\Phi(\mathbf{x}) = \Big(B_1(\pi_{-1}(\mathbf{x})), B_2(\pi_{-2}(\mathbf{x})), \cdots B_n(\pi_{-n}(\mathbf{x}))\Big)$$

for $i = 1, 2, \ldots, n$. Then

$$\{(\mathbf{x}, \mathbf{y}) \in S \times S : \mathbf{y} \in \Phi(\mathbf{x})\} = \bigcap_{i=1}^{n} G_i,$$

where

$$G_i = \{(\mathbf{x}, \mathbf{y}) \in S \times S : \pi_i(\mathbf{y}) \in B_i(\pi_{-i}(\mathbf{x}))\}$$

for $i = 1, 2, \ldots, n$.

Now, for each i, the set

$$\{(\mathbf{x}_{-i}, \mathbf{y}_i) \in S_{-i} \times S_i : \mathbf{y}_i \in B_i(\mathbf{x}_{-i})\}$$

is closed in $S_{-i} \times S_i$, because the correspondence $B_i \colon S_{-i} \rightrightarrows S_i$ has closed graph. It follows that each set G_i is closed in $S \times S$, because the set G_i is the preimage of a closed set under the continuous mapping from $S \times S$ to $S_{-i} \times S_i$ that sends each ordered pair $(\mathbf{x}, \mathbf{y})$ in $S \times S$ to $(\pi_{-i}(\mathbf{x}), \pi_i(\mathbf{y}))$. The graph of the correspondence Φ is the intersection of the closed sets $G_1, G_2, \ldots, G_n$. It is therefore itself a closed set. Thus the correspondence $\Phi \colon S \rightrightarrows S$ has closed graph. Moreover S is a non-empty compact convex set, and $\Phi(\mathbf{x})$ is a non-empty convex subset of S for all $\mathbf{x} \in S$. It follows from the Kakutani Fixed Point Theorem (Theorem 5.4) that there exists a fixed point $\mathbf{x}^*$ for the correspondence Φ. This fixed point is strategy profile that satisfies $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$.

Let $\mathbf{x}_i^* = \pi_i(\mathbf{x}^*)$ and $\mathbf{x}_{-i}^* = \pi_{-i}(\mathbf{x}^*)$ for $i = 1, 2, \ldots, n$. Then $\mathbf{x}_i^* \in B_i(\mathbf{x}_{-i}^*)$ for $i = 1, 2, \ldots, n$, because $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$. It follows from the definition of $B_i(\mathbf{x}_{-i}^*$ that

$$u_i(\mathbf{x}_i, \mathbf{x}_{-i}^*) \leq u_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*)$$

for all integers i between 1 and n and for all $\mathbf{x}_i \in S_i$. The strategy profile $(\mathbf{x}_1^*, \mathbf{x}_2^*, \ldots, \mathbf{x}_n^*)$ therefore represents a Nash equilibrium for the game.

Theorem 7.3 (Existence of Nash Equilibria)

Consider an n-person game in which, for each integer i between 1 and n, the strategy set S_i is a compact convex subset of a Euclidean space, and in which the utility function $u_i \colon S_i \times S_{i-1} \to \mathbb{R}$ that determines the utility for the ith player, given a strategy profile $\mathbf{x}_{-i}$ representing strategies chosen by the other players, is a continuous function that, for any fixed $\mathbf{x}_{-i} \in S_{-i}$, determines a quasiconcave function mapping $\mathbf{x}_i$ to $u_i(\mathbf{x}_i, \mathbf{x}_{-i})$ as $\mathbf{x}_i$ varies over the strategy set S_i. Then there exists a Nash equilibrium $(\mathbf{x}_1^*, \mathbf{x}_2^*, \ldots, \mathbf{x}_n^*)$ for the game. Accordingly

$$u_i(\mathbf{x}_i, \mathbf{x}_{-i}^*) \leq u_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*)$$

for all integers i between 1 and n and for all $\mathbf{x}_i \in S_i$.