MAU34804—Fixed Point Theorems and Economic Equilibria
School of Mathematics, Trinity College
Hilary Term 2022
Section 1: Review of Basic Results of Analysis in Euclidean Spaces

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1. Review of Basic Results of Analysis in Euclidean Spaces

1.1. Basic Properties of Vectors and Norms

We denote by $\mathbb{R}^n$ the set consisting of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers. The set $\mathbb{R}^n$ represents n-dimensional *Euclidean space* (with respect to the standard Cartesian coordinate system). Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$, where

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \quad \mathbf{y} = (y_1, y_2, \ldots, y_n),$$

and let λ be a real number. We define

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n),\\
\mathbf{x} - \mathbf{y} &= (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n),\\
\lambda\mathbf{x} &= (\lambda x_1, \lambda x_2, \ldots, \lambda x_n),\\
\mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n,\\
|\mathbf{x}| &= \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.
\end{aligned}$$

The quantity $\mathbf{x} \cdot \mathbf{y}$ is the *scalar product* (or *inner product*) of $\mathbf{x}$ and $\mathbf{y}$, and the quantity $|\mathbf{x}|$ is the *Euclidean norm* of $\mathbf{x}$. Note that $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$. The *Euclidean distance* between two points $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^n$ is defined to be the Euclidean norm $|\mathbf{y} - \mathbf{x}|$ of the vector $\mathbf{y} - \mathbf{x}$.

Let $\mathbf{x}$ and $\mathbf{y}$ be elements in $\mathbb{R}^n$, Let $p(t) = |t\mathbf{x} + \mathbf{y}|^2$ for all real numbers t. Then

$$\begin{aligned} p(t) &= (t\mathbf{x} + \mathbf{y}) \,.\, (t\mathbf{x} + \mathbf{y}) \\ &= t^2|\mathbf{x}|^2 + 2t\mathbf{x} \,.\, \mathbf{y} + |\mathbf{y}|^2 \end{aligned}$$

for all real numbers t. But $p(t) \geq 0$ for all real numbers t. It follows that $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$. This inquality is known as *Schwarz's Inequality*.

Moreover, given any elements $\mathbf{x}$ and $\mathbf{y}$ of $\mathbf{R}^n$,

$$\begin{aligned} |\mathbf{x}+\mathbf{y}|^2 &= (\mathbf{x}+\mathbf{y}).(\mathbf{x}+\mathbf{y}) = |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\mathbf{x}\cdot\mathbf{y} \\ &\leq |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2|\mathbf{x}||\mathbf{y}| = (|\mathbf{x}| + |\mathbf{y}|)^2. \end{aligned}$$

It follows that $|\mathbf{x}+\mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$. It follows from this inequality that

$$|\mathbf{x}-\mathbf{z}| \leq |\mathbf{x}-\mathbf{y}| + |\mathbf{y}-\mathbf{z}|$$

for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$. This identity is known as the *Triangle Inequality*. It expresses the geometric result that the length of any side of a triangle in a Euclidean space of any dimension is the sum of the lengths of the other two sides of that triangle.

Definition

A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to *converge* to a point $\mathbf{p}$ if and only if the following criterion is satisfied:—

> *given any real number ε satisfying $\varepsilon > 0$ there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$.*

We refer to $\mathbf{p}$ as the *limit* $\lim\limits_{j \to +\infty} \mathbf{x}_j$ of the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$.

Lemma 1.1

Let $\mathbf{p}$ *be a point of* $\mathbb{R}^n$, *where* $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. *Then a sequence* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ *of points in* $\mathbb{R}^n$ *converges to* $\mathbf{p}$ *if and only if the* i*th components of the elements of this sequence converge to* p_i *for* $i = 1, 2, \ldots, n$.

A proof of Lemma 1.1 is to be found in Appendix A.

Definition

Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be an infinite sequence of points in n-dimensional Euclidean space $\mathbb{R}^n$. A *subsequence* of this infinite sequence is a sequence of the form $\mathbf{x}_{j_1}, \mathbf{x}_{j_2}, \mathbf{x}_{j_3}, \ldots$ where $j_1, j_2, j_3, \ldots$ is an infinite sequence of positive integers with

$$j_1 < j_2 < j_3 < \cdots .$$

Theorem 1.2 (Multidimensional Bolzano-Weierstrass Theorem)

Every bounded sequence of points in a Euclidean space has a convergent subsequence.

A proof of Theorem 1.2 is to be found in Appendix A.

Definition

Let X be a subset of $\mathbb{R}^n$. Given a point $\mathbf{p}$ of X and a non-negative real number r, the *open ball* $B_X(\mathbf{p}, r)$ in X of *radius* r about $\mathbf{p}$ is defined to be the subset of X defined so that

$$B_X(\mathbf{p}, r) = \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < r\}.$$

(Thus $B_X(\mathbf{p}, r)$ is the set consisting of all points of X that lie within a sphere of radius r centred on the point $\mathbf{p}$.)

Definition

Let X be a subset of $\mathbb{R}^n$. A subset V of X is said to be *open* in X if, given any point $\mathbf{p}$ of V, there exists some strictly positive real number δ such that $B_X(\mathbf{p}, \delta) \subset V$, where $B_X(\mathbf{p}, \delta)$ is the open ball in X of radius δ about on the point $\mathbf{p}$. The empty set $\emptyset$ is also defined to be an open set in X.

Lemma 1.3

Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any positive real number r, the open ball $B_X(\mathbf{p}, r)$ in X of radius r about $\mathbf{p}$ is open in X.

A proof of Lemma 1.3 is to be found in Appendix A.

Proposition 1.4

Let X be a subset of $\mathbb{R}^n$. The collection of open sets in X has the following properties:—

(i) *the empty set $\emptyset$ and the whole set X are both open in X;*

(ii) *the union of any collection of open sets in X is itself open in X;*

(iii) *the intersection of any finite collection of open sets in X is itself open in X.*

A proof of Proposition 1.4 is to be found in Appendix A.

Proposition 1.5

Let X be a subset of $\mathbb{R}^n$, and let U be a subset of X. Then U is open in X if and only if there exists some open set V in $\mathbb{R}^n$ for which $U = V \cap X$.

A proof of Proposition 1.5 is to be found in Appendix A.

Lemma 1.6

A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ *of points in* $\mathbb{R}^n$ *converges to a point* $\mathbf{p}$ *if and only if, given any open set* U *which contains* $\mathbf{p}$*, there exists some positive integer* N *such that* $\mathbf{x}_j \in U$ *for all* j *satisfying* $j \geq N$.

A proof of Lemma 1.6 is to be found in Appendix A.

Definition

Let X be a subset of $\mathbb{R}^n$. A subset F of X is said to be *closed* in X if and only if its complement $X \setminus F$ in X is open in X. (Recall that $X \setminus F = \{\mathbf{x} \in X : \mathbf{x} \notin F\}$.)

Proposition 1.7

Let X be a subset of $\mathbb{R}^n$. The collection of closed sets in X has the following properties:—

(i) *the empty set $\emptyset$ and the whole set X are both closed in X;*

(ii) *the intersection of any collection of closed sets in X is itself closed in X;*

(iii) *the union of any finite collection of closed sets in X is itself closed in X.*

A proof of Proposition 1.7 is to be found in Appendix A.

Lemma 1.8

Let X be a subset of $\mathbb{R}^n$, and let F be a subset of X which is closed in X. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of F which converges to a point $\mathbf{p}$ of X. Then $\mathbf{p} \in F$.

A proof of Lemma 1.8 is to be found in Appendix A.

Definition

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. A function $f\colon X \to Y$ from X to Y is said to be *continuous* at a point $\mathbf{p}$ of X if and only if the following criterion is satisfied:—

> *given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$.*

The function $f\colon X \to Y$ is said to be continuous on X if and only if it is continuous at every point $\mathbf{p}$ of X.

Lemma 1.9

Let X, Y and Z be subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively, and let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions satisfying $f(X) \subset Y$. Suppose that f is continuous at some point $\mathbf{p}$ of X and that g is continuous at $f(\mathbf{p})$. Then the composition function $g \circ f\colon X \to Z$ is continuous at $\mathbf{p}$.

A proof of Lemma 1.9 is to be found in Appendix A.

Lemma 1.10

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f\colon X \to Y$ be a continuous function from X to Y. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of X which converges to some point $\mathbf{p}$ of X. Then the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$.

A proof of Lemma 1.10 is to be found in Appendix A.

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f\colon X \to Y$ be a function from X to Y. Then

$$f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$$

for all $\mathbf{x} \in X$, where $f_1, f_2, \ldots, f_n$ are functions from X to $\mathbb{R}$, referred to as the *components* of the function f.

Proposition 1.11

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $\mathbf{p} \in X$. A function $f\colon X \to Y$ is continuous at the point $\mathbf{p}$ if and only if its components are all continuous at $\mathbf{p}$.

A proof of Proposition 1.11 is to be found in Appendix A.

Proposition 1.12

Let X be a subset of $\mathbb{R}^n$, and let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be continuous functions from X to $\mathbb{R}$. Then the functions $f + g$, $f - g$ and $f \cdot g$ are continuous. If in addition $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ then the quotient function f/g is continuous.

A proof of Proposition 1.12 is to be found in Appendix A.

Lemma 1.13

Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a continuous function mapping X into $\mathbb{R}^n$, and let $|f|\colon X \to \mathbb{R}$ be defined such that $|f|(\mathbf{x}) = |f(\mathbf{x})|$ for all $\mathbf{x} \in X$. Then the real-valued function $|f|$ is continuous on X.

A proof of Proposition 1.13 is to be found in Appendix A.

Given any function $f\colon X \to Y$, we denote by $f^{-1}(V)$ the *preimage* of a subset V of Y under the map f, defined by $f^{-1}(V) = \{\mathbf{x} \in X : f(\mathbf{x}) \in V\}$.

Proposition 1.14

Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f\colon X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(V)$ is open in X for every open subset V of Y.

A proof of Proposition 1.14 is to be found in Appendix A.

Let X be a subset of $\mathbb{R}^n$, let $f\colon X \to \mathbb{R}$ be continuous, and let c be some real number. Proposition 1.14 ensures that the sets $\{\mathbf{x} \in X : f(\mathbf{x}) > c\}$ and $\{\mathbf{x} \in X : f(\mathbf{x}) < c\}$ are open in X. Moreover given real numbers a and b satisfying $a < b$, the set $\{\mathbf{x} \in X : a < f(\mathbf{x}) < b\}$ is open in X.

Corollary 1.15

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi\colon X \to Y$ be a continuous function from X to Y. Then $\varphi^{-1}(F)$ is closed in X for every subset F of Y that is closed in Y.

A proof of Corollary 1.15 is to be found in Appendix A.

Lemma 1.16

Let X be a closed subset of n-dimensional Euclidean space $\mathbb{R}^n$. Then a subset of X is closed in X if and only if it is closed in $\mathbb{R}^n$.

A proof of Lemma 1.16 is to be found in Appendix A.

1.3. The Multidimensional Extreme Value Theorem

Theorem 1.17 (The Multidimensional Extreme Value Theorem)

Let X be a non-empty closed bounded set in $\mathbb{R}^m$, and let $f: X \to \mathbb{R}$ be a continuous real-valued function defined on X. Then there exist points $\mathbf{u}$ and $\mathbf{v}$ of X such that $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$.

A proof of Theorem 1.17 is to be found in Appendix A.

1.4. The Glueing Lemma

The following result, together with its generalizations, is sometimes referred to as the *Glueing Lemma*.

Lemma 1.18 (Glueing Lemma)

Let $\varphi\colon X \to \mathbb{R}^n$ be a function mapping a subset X of $\mathbb{R}^m$ into $\mathbb{R}^n$. Let $F_1, F_2, \ldots, F_k$ be a finite collection of subsets of X such that F_i is closed in X for $i = 1, 2, \ldots, k$ and

$$F_1 \cup F_2 \cup \cdots \cup F_k = X.$$

Then the function φ is continuous on X if and only if the restriction of φ to F_i is continuous on F_i for $i = 1, 2, \ldots, k$.

Proof

Suppose that $\varphi\colon X \to \mathbb{R}^n$ is continuous. Then it follows directly from the definition of continuity that the restriction of φ to each subset of X is continuous on that subset. Therefore the restriction of φ to F_i is continuous on F_i for $i = 1, 2, \ldots, k$.

Conversely we must prove that if the restriction of the function φ to F_i is continuous on F_i for $i = 1, 2, \ldots, k$ then the function $\varphi\colon X \to \mathbb{R}^m$ is continuous. Let $\mathbf{p}$ be a point of X, and let some positive real number ε be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots \delta_k$ satisfying the following conditions:—

(i) if $\mathbf{p} \in F_i$, where $1 \leq i \leq k$, and if $\mathbf{x} \in F_i$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_i$ then $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$;

(ii) if $\mathbf{p} \notin F_i$, where $1 \leq i \leq k$, and if $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_i$ then $\mathbf{x} \notin F_i$.

Indeed the continuity of the function φ on each set F_i ensures that δ_i may be chosen to satisfy (i) for each integer i between 1 and k for which $\mathbf{p} \in F_i$. Also the requirement that F_i be closed in X ensures that $X \setminus F_i$ is open in X and therefore δ_i may be chosen to to satisfy (ii) for each integer i between 1 and k for which $\mathbf{p} \notin F_i$.

Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. Let $\mathbf{x} \in X$ satisfy $|\mathbf{x} - \mathbf{p}| < \delta$. If $\mathbf{p} \notin F_i$ then the choice of δ_i ensures that if $\mathbf{x} \notin F_i$. But X is the union of the sets $F_1, F_2, \ldots, F_k$, and therefore there must exist some integer i between 1 and k for which $\mathbf{x} \in F_i$. Then $\mathbf{p} \in F_i$, and the choice of δ_i ensures that $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$. We have thus shown that $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$ for all points $\mathbf{x}$ of X that satisfy $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $\varphi\colon X \to \mathbb{R}^n$ is continuous, as required. ∎

1.5. Lebesgue Numbers

Definition

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. A collection of subsets of $\mathbb{R}^n$ is said to *cover* X if and only if every point of X belongs to at least one of these subsets.

Definition

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. An *open cover* of X is a collection of subsets of X that are open in X and cover the set X.

Proposition 1.19

Let X be a closed bounded set in n-dimensional Euclidean space, and let $\mathcal{V}$ be an open cover of X. Then there exists a positive real number δ_L with the property that, given any point $\mathbf{u}$ of X, there exists a member V of the open cover $\mathcal{V}$ for which

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta_L\} \subset V.$$

Proof

Let

$$B_X(\mathbf{u}, \delta) = \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta\}$$

for all $\mathbf{u} \in X$ and for all positive real numbers δ. Suppose that there did not exist any positive real number δ_L with the stated property.

Then, given any positive number δ, there would exist a point $\mathbf{u}$ of X for which the set $B_X(\mathbf{u}, \delta)$ would not be wholly contained within any open set V belonging to the open cover $\mathcal{V}$. Consequently there would exist an infinite sequence

$$\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$$

of points of X with the property that, for each positive integer j, the set $B_X(\mathbf{u}_j, 1/j)$ would not be wholly contained within any open set V belonging to the open cover $\mathcal{V}$. The multidimensional Bolzano-Weierstrass Theorem (Theorem 1.2) would then ensure the existence of a convergent subsequence

$$\mathbf{u}_{j_1}, \mathbf{u}_{j_2}, \mathbf{u}_{j_3}, \ldots$$

of this infinite sequence.

Let $\mathbf{p}$ be the limit of this convergent subsequence. Then the point $\mathbf{p}$ would then belong to X, because X is closed (see Lemma 1.8). But then the point $\mathbf{p}$ would belong to an open set V belonging to the open cover $\mathcal{V}$. It would then follow from the definition of open sets that there would exist a positive real number δ for which $B_X(\mathbf{p}, 2\delta) \subset V$. Let $j = j_k$ for a positive integer k large enough to ensure that both $1/j < \delta$ and $\mathbf{u}_j \in B_X(\mathbf{p}, \delta)$. The Triangle Inequality would then ensure that every point of X within a distance $1/j$ of the point $\mathbf{u}_j$ would lie within a distance 2δ of the point $\mathbf{p}$, and therefore

$$B_X(\mathbf{u}_j, 1/j) \subset B_X(\mathbf{p}, 2\delta) \subset V.$$

But we supposed that the point $\mathbf{u}_j$ was chosen so as to ensure that the set $B_X(\mathbf{u}_j, 1/j)$ was not wholly contained within any open set V belonging to the open cover $\mathcal{V}$. Thus a logical contradiction as resulted from the assumption that there is no positive real number δ_L with the property that, given any point $\mathbf{u}$ of X, the set $B_X(\mathbf{u}, \delta_L)$ is not wholly contained within any open set belonging to the open cover $\mathcal{V}$. Consequently some positive real number δ_L satisfying this property must exist, and thus the required result has been proved. $\blacksquare$

Definition

Let X be a subset of n-dimensional Euclidean space, and let $\mathcal{V}$ be an open cover of X. A positive real number δ_L is said to be a *Lebesgue number* for the open cover $\mathcal{V}$ if, given any point $\mathbf{p}$ of X, there exists some member V of the open cover $\mathcal{V}$ for which

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta_L\} \subset V.$$

Proposition 1.19 ensures that, given any open cover of a closed bounded subset of n-dimensional Euclidean space, there exists a positive real number that is a Lebesgue number for that open cover.

Definition

The *diameter* $\mathrm{diam}(A)$ of a bounded subset A of n-dimensional Euclidean space is defined so that

$$\mathrm{diam}(A) = \sup\{|\mathbf{x} - \mathbf{y}| : \mathbf{x}, \mathbf{y} \in A\}.$$

It follows from this definition that $\mathrm{diam}(A)$ is the smallest real number K with the property that $|\mathbf{x} - \mathbf{y}| \leq K$ for all $\mathbf{x}, \mathbf{y} \in A$.

Lemma 1.20

Let X be a bounded subset of n-dimensional Euclidean space, and let δ be a positive real number. Then there exists a finite collection $A_1, A_2, \ldots, A_s$ of subsets of X such that the $\mathrm{diam}(A_i) < \delta$ for $i = 1, 2, \ldots, s$ and

$$X = A_1 \cup A_2 \cup \cdots \cup A_k.$$

Proof

Let b be a real number satisfying $0 < \sqrt{n}\, b < \delta$ and, for each n-tuple $(j_1, j_2, \ldots, j_n)$ of integers, let $H_{(j_1, j_2, \ldots, j_n)}$ denote the hypercube in $\mathbb{R}^n$ defined such that

$$\begin{aligned} H_{(j_1, j_2, \ldots, j_n)} \quad = \quad & \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \\ & \quad j_i b \leq x_i \leq (j_i + 1) b \text{ for } i = 1, 2, \ldots, n\}. \end{aligned}$$

Note that if $\mathbf{u}$ and $\mathbf{v}$ are points of $H_{(j_1,j_2,\ldots,j_n)}$ for some n-tuple $(j_1, j_2, \ldots, j_n)$ of integers then $|u_i - v_i| < b$ for $i = 1, 2, \ldots, n$, and therefore $|\mathbf{u} - \mathbf{v}| \leq \sqrt{n}\, b < \delta$. Therefore the diameter of each hypercube $H_{(j_1,j_2,\ldots,j_n)}$ is less than δ.

The boundedness of the set X ensures that there are only finitely many n-tuples $(j_1, j_2, \ldots, j_n)$ of integers for which $X \cap H_{(j_1,j_2,\ldots,j_n)}$ is non-empty. It follows that X is covered by a finite collection $A_1, A_2, \ldots, A_k$ of subsets of X, where each of these subsets is of the form $X \cap H_{(j_1,j_2,\ldots,j_n)}$ for some n-tuple $(j_1, j_2, \ldots, j_n)$ of integers. These subsets all have diameter less than δ. The result follows. ∎

Definition

Let $\mathcal{V}$ and $\mathcal{W}$ be open covers of some subset X of a Euclidean space. Then $\mathcal{W}$ is said to be a *subcover* of $\mathcal{V}$ if and only if every open set belonging to $\mathcal{W}$ also belongs to $\mathcal{V}$.

Definition

A subset X of a Euclidean space is said to be *compact* if and only if every open cover of X possesses a finite subcover.

Theorem 1.21 (The Multidimensional Heine-Borel Theorem)

A subset of n-dimensional Euclidean space $\mathbb{R}^n$ is compact if and only if it is both closed and bounded.

Proof

Let X be a compact subset of $\mathbb{R}^n$ and let

$$V_j = \{\mathbf{x} \in X : |\mathbf{x}| < j\}$$

for all positive integers j. Then the sets $V_1, V_2, V_3, \ldots$ constitute an open cover of X. This open cover has a finite subcover, and therefore there exist positive integers $j_1, j_2, \ldots, j_k$ such that

$$X \subset V_{j_1} \cup V_{j_2} \cup \cdots \cup V_{j_k}.$$

Let M be the largest of the positive integers $j_1, j_2, \ldots, j_k$. Then $|\mathbf{x}| \leq M$ for all $\mathbf{x} \in X$. Thus the set X is bounded.

Let $\mathbf{q}$ be a point of $\mathbb{R}^n$ that does not belong to X, and let

$$W_j = \left\{ \mathbf{x} \in X : |\mathbf{x} - \mathbf{q}| > \frac{1}{j} \right\}$$

for all positive integers j. Then the sets $W_1, W_2, W_3, \ldots$ constitute an open cover of X. This open cover has a finite subcover, and therefore there exist positive integers $j_1, j_2, \ldots, j_k$ such that

$$X \subset W_{j_1} \cup W_{j_2} \cup \cdots \cup W_{j_k}.$$

Let $\delta = 1/M$, where M is the largest of the positive integers $j_1, j_2, \ldots, j_k$. Then $|\mathbf{x} - \mathbf{q}| \geq \delta$ for all $\mathbf{x} \in X$ and thus the open ball of radius δ about the point $\mathbf{q}$ does not intersect the set X. We conclude that the set X is closed. We have now shown that every compact subset of $\mathbb{R}^n$ is both closed and bounded.

We now prove the converse. Let X be a closed bounded subset of $\mathbb{R}^n$, and let $\mathcal{V}$ be an open cover of X. It follows from Proposition 1.19 that there exists a Lebesgue number δ_L for the open cover $\mathcal{V}$. It then follows from Lemma 1.20 that there exist subsets $A_1, A_2, \ldots, A_s$ of X such that $\operatorname{diam}(A_i) < \delta_L$ for $i = 1, 2, \ldots, s$ and

$$X = A_1 \cup A_2 \cup \cdots \cup A_s.$$

We may suppose that A_i is non-empty for $i = 1, 2, \ldots, s$ (because if $A_i = \emptyset$ then A_i could be deleted from the list). Choose $\mathbf{p}_i \in A_i$ for $i = 1, 2, \ldots, s$. Then $A_i \subset B_X(\mathbf{p}_i, \delta_L)$ for $i = 1, 2, \ldots, s$. The definition of the Lebesgue number δ_L then ensures that there exist members $V_1, V_2, \ldots, V_s$ of the open cover $\mathcal{V}$ such that $B_X(\mathbf{p}_i, \delta_L) \subset V_i$ for $i = 1, 2, \ldots, s$. Then $A_i \subset V_i$ for $i = 1, 2, \ldots, s$, and therefore

$$X \subset V_1 \cup V_2 \cup \cdots \cup V_s.$$

Thus $V_1, V_2, \ldots, V_s$ constitute a finite subcover of the open cover $\mathcal{U}$. We have therefore proved that every closed bounded subset of n-dimensional Euclidean space is compact, as required. $\blacksquare$