MAU34804—Fixed Point Theorems and Economic Equilibria
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Section 2: Correspondences and Hemicontinuity

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2. Correspondences and Hemicontinuity

2.1. Correspondences

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A *correspondence* $\Phi\colon X \rightrightarrows Y$ assigns to each point $\mathbf{x}$ of X a subset $\Phi(\mathbf{x})$ of Y.

The *power set* $\mathcal{P}(Y)$ of Y is the set whose elements are the subsets of Y. A correspondence $\Phi\colon X \rightrightarrows Y$ may be regarded as a function from X to $\mathcal{P}(Y)$.

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Then the following definitions apply:–

- the correspondence $\Phi\colon X \to Y$ is said to be *non-empty-valued* if $\Phi(\mathbf{x})$ is a non-empty subset of Y for all $\mathbf{x} \in X$;
- the correspondence $\Phi\colon X \to Y$ is said to be *closed-valued* if $\Phi(\mathbf{x})$ is a closed subset of Y for all $\mathbf{x} \in X$;
- the correspondence $\Phi\colon X \to Y$ is said to be *compact-valued* if $\Phi(\mathbf{x})$ is a compact subset of Y for all $\mathbf{x} \in X$.

The multidimensional Heine-Borel Theorem (Theorem 1.21) ensures that the correspondence $\Phi\colon X \to Y$ is compact-valued if and only if $\Phi(\mathbf{x})$ is a closed bounded subset of $\mathbb{R}^m$ for all $\mathbf{x} \in X$.

Definition

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi\colon X \rightrightarrows Y$ is said to be *upper hemicontinuous* at a point $\mathbf{p}$ of X if, given any set V in Y that is open in Y and satisfies $\Phi(\mathbf{p}) \subset V$, there exists some positive real number δ such that $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. The correspondence Φ is upper hemicontinuous on X if it is upper hemicontinuous at each point of X.

Example

Let $F\colon \mathbb{R} \rightrightarrows \mathbb{R}$ and $G\colon \mathbb{R} \rightrightarrows \mathbb{R}$ be the correspondences from $\mathbb{R}$ to $\mathbb{R}$ defined such that

$$F(x) = \begin{cases} [1,2] & \text{if } x < 0, \\ [0,3] & \text{if } x \geq 0, \end{cases}$$

and

$$G(x) = \begin{cases} [1,2] & \text{if } x \leq 0, \\ [0,3] & \text{if } x > 0, \end{cases}$$

The correspondences F and G are upper hemicontinuous at x for all non-zero real numbers x. The correspondence F is also upper hemicontinuous at 0, for if V is an open set in $\mathbb{R}$ and if $F(0) \subset V$ then $[0,3] \subset V$ and therefore $F(x) \in V$ for all real numbers x.

However the correspondence G is not upper hemicontinuous at 0. Indeed let

$$V = \{y \in \mathbb{R} : \tfrac{1}{2} < y < \tfrac{5}{2}\}.$$

Then $G(0) \subset V$, but $G(x)$ is not contained in V for any positive real number x. Therefore there cannot exist any positive real number δ such that $G(x) \subset V$ whenever $|x| < \delta$.

Let

$$\mathrm{Graph}(F) = \{(x, y) \in \mathbb{R}^2 : y \in F(x)\}$$

and

$$\mathrm{Graph}(G) = \{(x, y) \in \mathbb{R}^2 : y \in G(x)\}.$$

Then $\mathrm{Graph}(F)$ is a closed subset of $\mathbb{R}^2$ but $\mathrm{Graph}(G)$ is not a closed subset of $\mathbb{R}^2$.

Example

Let S^1 be the unit circle in $\mathbb{R}^2$, defined such that

$$S^1 = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 = 1\},$$

let Z be the closed square with corners at $(1, 1)$, $(-1, 1)$, $(-1, -1)$ and $(1, -1)$, so that

$$Z = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1\}.$$

Let $g_{(u,v)} \colon \mathbb{R}^2 \to \mathbb{R}$ be defined for all $(u, v) \in S^1$ such that

$$g_{(u,v)}(x, y) = ux + vy,$$

and let $\Phi \colon S^1 \rightrightarrows \mathbb{R}^2$ be defined such that, for all $(u, v) \in S^1$, $\Phi(u, v)$ is the subset of $\mathbb{R}^2$ consisting of the point of points of Z at which the linear functional $g_{(u,v)}$ attains its maximum value on Z.

Thus a point (x, y) of Z belongs to $\Phi(u, v)$ if and only if $g_{(u,v)}(x, y) \geq g_{(u,v)}(x', y')$ for all $(x', y') \in Z$. Then

$$\Phi(u, v) = \begin{cases} \{(1, 1)\} & \text{if } u > 0 \text{ and } v > 0; \\ \{(x, 1) : -1 \leq x \leq 1\} & \text{if } u = 0 \text{ and } v > 0; \\ \{(-1, 1)\} & \text{if } u < 0 \text{ and } v > 0; \\ \{(-1, y) : -1 \leq y \leq 1\} & \text{if } u < 0 \text{ and } v = 0; \\ \{(-1, -1)\} & \text{if } u < 0 \text{ and } v < 0; \\ \{(x, -1) : -1 \leq x \leq 1\} & \text{if } u = 0 \text{ and } v < 0; \\ \{(1, -1)\} & \text{if } u > 0 \text{ and } v < 0; \\ \{(1, y) : -1 \leq y \leq 1\} & \text{if } u > 0 \text{ and } v = 0. \end{cases}$$

It is a straightforward exercise to verify that the correspondence $\Phi\colon S^1 \rightrightarrows \mathbb{R}^2$ is upper hemicontinuous.

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence between X and Y. Given any subset V of Y, we denote by $\Phi^+(V)$ the subset of X defined such that

$$\Phi^+(V) = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset V\}.$$

Lemma 2.1

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous on X if and only if, given any set V in Y that is open in Y, the set $\Phi^+(V)$ is open in X.

Proof

First suppose that $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at each point of X. Let V be an open set in Y and let $\mathbf{p} \in \Phi^+(V)$. Then $\Phi(\mathbf{p}) \subset V$. It then follows from the definition of upper hemicontinuity that there exists some positive real number δ such that $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then $\mathbf{x} \in \Phi^+(V)$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $\Phi^+(V)$ is open in X.

Conversely suppose that $\Phi\colon X \rightrightarrows Y$ is a correspondence with the property that, for all subsets V of Y that are open in Y, $\Phi^+(V)$ is open in X. Let $\mathbf{p} \in X$, and let V be an open set in Y satisfying $\Phi(\mathbf{p}) \subset V$. Then $\Phi^+(V)$ is open in X and $\mathbf{p} \in \Phi^+(V)$, and therefore there exists some positive number δ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\} \subset \Phi^+(V).$$

Then $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$. The result follows. ■

Definition

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi\colon X \rightrightarrows Y$ is said to be *lower hemicontinuous* at a point $\mathbf{p}$ of X if, given any set V in Y that is open in Y and satisfies $\Phi(\mathbf{p}) \cap V \neq \emptyset$, there exists some positive real number δ such that $\Phi(\mathbf{x}) \cap V \neq \emptyset$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. The correspondence Φ is lower hemicontinuous on X if it is lower hemicontinuous at each point of X.

Example

Let $F\colon \mathbb{R} \rightrightarrows \mathbb{R}$ and $G\colon \mathbb{R} \rightrightarrows \mathbb{R}$ be the correspondences from $\mathbb{R}$ to $\mathbb{R}$ defined such that

$$F(x) = \begin{cases} [1,2] & \text{if } x < 0, \\ [0,3] & \text{if } x \geq 0, \end{cases}$$

and

$$G(x) = \begin{cases} [1,2] & \text{if } x \leq 0, \\ [0,3] & \text{if } x > 0, \end{cases}$$

The correspondences F and G are lower hemicontinuous at x for all non-zero real numbers x. The correspondence G is also lower hemicontinuous at 0, for if V is an open set in $\mathbb{R}$ and if $G(0) \cap V \neq \emptyset$ then $[1,2] \cap V \neq \emptyset$ and therefore $G(x) \cap V \neq \emptyset$ for all real numbers x.

However the correspondence F is not lower hemicontinuous at 0. Indeed let

$$V = \{y \in \mathbb{R} : 0 < y < \tfrac{1}{2}\}.$$

Then $F(0) \cap V \neq \emptyset$, but $F(x) \cap V = \emptyset$ for all negative real numbers x. Therefore there cannot exist any positive real number δ such that $F(x) \cap V \neq \emptyset$ whenever $|x| < \delta$.

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence between X and Y. Given any subset V of Y, we denote by $\Phi^-(V)$ the subset of X defined such that

$$\Phi^-(V) = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \cap V \neq \emptyset\}.$$

Lemma 2.2

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi\colon X \rightrightarrows Y$ is lower hemicontinuous on X if and only if, given any set V in Y that is open in Y, the set $\Phi^-(V)$ is open in X.

Proof

First suppose that $\Phi\colon X \rightrightarrows Y$ is lower hemicontinuous at each point of X. Let V be an open set in Y and let $\mathbf{p} \in \Phi^-(V)$. Then $\Phi(\mathbf{p}) \cap V$ is non-empty. It then follows from the definition of lower hemicontinuity that there exists some positive real number δ such that $\Phi(\mathbf{x}) \cap V$ is non-empty for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then $\mathbf{x} \in \Phi^-(V)$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $\Phi^-(V)$ is open in X.

Conversely suppose that $\Phi\colon X \rightrightarrows Y$ is a correspondence with the property that, for all subsets V of Y that are open in Y, $\Phi^{-}(V)$ is open in X. Let $\mathbf{p} \in X$, and let V be an open set in Y satisfying $\Phi(\mathbf{p}) \cap V \neq \emptyset$. Then $\Phi^{-}(V)$ is open in X and $\mathbf{p} \in \Phi^{-}(V)$, and therefore there exists some positive number δ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\} \subset \Phi^{-}(V).$$

Then $\Phi(\mathbf{x}) \cap V \neq \emptyset$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus $\Phi\colon X \rightrightarrows Y$ is lower hemicontinuous at $\mathbf{p}$. The result follows. ∎

Definition

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi\colon X \rightrightarrows Y$ is said to be *continuous* at a point $\mathbf{p}$ of X if it is both upper hemicontinuous and lower hemicontinuous at $\mathbf{p}$. The correspondence Φ is continuous on X if it is continuous at each point of X.

Lemma 2.3

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let $\varphi\colon X \to Y$ be a function from X to Y, and let $\Phi\colon X \rightrightarrows Y$ be the correspondence defined such that $\Phi(\mathbf{x}) = \{\varphi(\mathbf{x})\}$ for all $\mathbf{x} \in X$. Then $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous if and only if $\varphi\colon X \to Y$ is continuous. Similarly $\Phi\colon X \rightrightarrows Y$ is lower hemicontinuous if and only if $\varphi\colon X \to Y$ is continuous.

Proof

The function $\varphi\colon X \to Y$ is continuous if and only if

$$\{\mathbf{x} \in X : \varphi(\mathbf{x}) \in V\}$$

is open in X for all subsets V of Y that are open in Y (see Proposition 1.14). Let V be a subset of Y that is open in Y. Then $\Phi(\mathbf{x}) \subset V$ if and only if $\varphi(\mathbf{x}) \in V$. Also $\Phi(\mathbf{x}) \cap V \neq \emptyset$ if and only if $\varphi(\mathbf{x}) \in V$. The result therefore follows from the definitions of upper and lower hemicontinuity. ∎

2.2. The Graph of a Correspondence

Let m and n be integers. Then the Cartesian product $\mathbb{R}^n \times \mathbb{R}^m$ of the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ of dimensions n and m is itself a Euclidean space of dimension $n + m$ whose Euclidean norm is characterized by the property that

$$|(\mathbf{x}, \mathbf{y})|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2$$

for all $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$.

Lemma 2.4

Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ *and* $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ *be infinite sequences of points in* $\mathbb{R}^n$ *and* $\mathbb{R}^m$ *respectively, and let* $\mathbf{p} \in \mathbb{R}^n$ *and* $\mathbf{q} \in \mathbb{R}^m$*. Then the infinite sequence*

$$(\mathbf{x}_1, \mathbf{y}_1),\ (\mathbf{x}_2, \mathbf{y}_2),\ (\mathbf{x}_3, \mathbf{y}_3), \ldots$$

converges in $\mathbb{R}^n \times \mathbb{R}^m$ *to the point* $(\mathbf{p}, \mathbf{q})$ *if and only if the infinite sequence Let* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ *converges to the point* $\mathbf{p}$ *and the infinite sequence* $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ *converges to the point* $\mathbf{q}$*.*

Proof

Suppose that the infinite sequence

$$(\mathbf{x}_1, \mathbf{y}_1),\ (\mathbf{x}_2, \mathbf{y}_2),\ (\mathbf{x}_3, \mathbf{y}_3), \ldots$$

converges in $\mathbb{R}^n \times \mathbb{R}^m$ to the point $(\mathbf{p}, \mathbf{q})$. Let some strictly positive real number ε be given. Then there exists some positive integer N such that

$$|\mathbf{x}_j - \mathbf{p}|^2 + |\mathbf{y}_j - \mathbf{q}|^2 < \varepsilon^2$$

whenever $j \geq N$. But then

$$|\mathbf{x}_j - \mathbf{p}| < \varepsilon \quad \text{and} \quad |\mathbf{y}_j - \mathbf{q}| < \varepsilon$$

whenever $j \geq N$. It follows that $\mathbf{x}_j \to \mathbf{p}$ and $\mathbf{y}_j \to \mathbf{q}$ as $j \to +\infty$.

Conversely suppose that $\mathbf{x}_j \to \mathbf{p}$ and $\mathbf{y}_j \to \mathbf{q}$ as $j \to +\infty$. Let some positive real number ε be given. Then there exist positive integers N_1 and N_2 such that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon/\sqrt{2}$ whenever $j \geq N_1$ and $|\mathbf{y}_j - \mathbf{q}| < \varepsilon/\sqrt{2}$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. Then

$$|\mathbf{x}_j - \mathbf{p}|^2 + |\mathbf{y}_j - \mathbf{q}|^2 < \varepsilon^2$$

whenever $j \geq N$. It follows that $(\mathbf{x}_j, \mathbf{y}_j) \to (\mathbf{p}, \mathbf{q})$ as $j \to +\infty$, as required. ∎

Lemma 2.5

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let V be a subset of $X \times Y$. Then V is open in $X \times Y$ if and only if, given any point $(\mathbf{p}, \mathbf{q})$ of V, where $\mathbf{p} \in X$ and $\mathbf{q} \in Y$, there exist subsets W_X and W_Y of X and Y respectively such that $\mathbf{p} \in W_X$, $\mathbf{q} \in W_Y$, W_X is open in X, W_Y is open in Y and $W_X \times W_Y \subset V$.

Proof

Let V be a subset of $X \times Y$ and let $(\mathbf{p}, \mathbf{q}) \in V$, where $\mathbf{p} \in X$ and $\mathbf{q} \in Y$.

Suppose that V is open in $X \times Y$. Then there exists a positive real number δ such that $(\mathbf{x}, \mathbf{y}) \in V$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfying

$$|\mathbf{x} - \mathbf{p}|^2 + |\mathbf{y} - \mathbf{q}|^2 < \delta^2.$$

Let

$$W_X = \left\{ \mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \frac{\delta}{\sqrt{2}} \right\}$$

and

$$W_Y = \left\{ \mathbf{y} \in Y : |\mathbf{y} - \mathbf{q}| < \frac{\delta}{\sqrt{2}} \right\}$$

If $\mathbf{x} \in W_X$ and $\mathbf{y} \in W_Y$ then

$$|\mathbf{x} - \mathbf{p}|^2 + |\mathbf{y} - \mathbf{q}|^2 < 2 \left(\frac{\delta}{\sqrt{2}} \right)^2 = \delta^2$$

and therefore $(\mathbf{x}, \mathbf{y}) \in V$. It follows that $W_X \times W_Y \subset V$.

Conversely suppose that there exist open sets W_X and W_Y in X and Y respectively such that $\mathbf{p} \in W_X$, $\mathbf{q} \in W_Y$ and $W_X \times W_Y \subset V$. Then there exists some positive real number δ such that $\mathbf{x} \in W_X$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$ and also $\mathbf{y} \in W_Y$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - \mathbf{q}| < \delta$. If $(\mathbf{x}, \mathbf{y})$ is a point of $X \times Y$ that lies within a distance δ of $(\mathbf{p}, \mathbf{q})$ then $|\mathbf{x} - \mathbf{p}| < \delta$ and $|\mathbf{y} - \mathbf{q}| < \delta$, and therefore $(\mathbf{x}, \mathbf{y}) \in W_X \times W_Y$. But $W_X \times W_Y \subset V$. It follows that the open ball of radius δ about the point $(\mathbf{p}, \mathbf{q})$ is wholly contained within the subset V of $X \times Y$. The result follows. ∎

Proposition 2.6

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let G be a subset of $X \times Y$. Then G is closed in $X \times Y$ if and only if

$$(\lim_{j \to \infty} \mathbf{x}_j, \lim_{j \to \infty} \mathbf{y}_j) \in G$$

for all convergent infinite sequences $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ in X and for all convergent infinite sequences $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ in Y with the property that $(\mathbf{x}_j, \mathbf{y}_j) \in G$ for all positive integers j.

Proof

Suppose that G is closed in $X \times Y$. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be an infinite sequence in X converging to some point $\mathbf{p}$ of X and let $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ be an infinite sequence in Y converging to a point $\mathbf{q}$ of Y, where $(\mathbf{x}_j, \mathbf{y}_j) \in G$ for all positive integers j. We must prove that $(\mathbf{p}, \mathbf{q}) \in G$. Now the infinite sequence consisting of the ordered pairs $(\mathbf{x}_j, \mathbf{y}_j)$ converges in $X \times Y$ to $(\mathbf{p}, \mathbf{q})$ (see Lemma 2.4). Now every infinite sequence contained in G that converges to a point of $X \times Y$ must converge to a point of G, because G is closed in $X \times Y$ (see Lemma 1.8). It follows that $(\mathbf{p}, \mathbf{q}) \in G$.

Now suppose that G is not closed in $X \times Y$. Then the complement of G in $X \times Y$ is not open, and therefore there exists a point $(\mathbf{p}, \mathbf{q})$ of $X \times Y$ that does not belong to G though every open ball of positive radius about the point $(\mathbf{p}, \mathbf{q})$ intersects G. It follows that, given any positive integer j, the open ball of radius $1/j$ about the point $(\mathbf{p}, \mathbf{q})$ intersects G and therefore there exist $\mathbf{x}_j \in X$ and $\mathbf{y}_j \in Y$ for which $|\mathbf{x}_j - \mathbf{p}| < 1/j$, $|\mathbf{y}_j - \mathbf{q}| < 1/j$ and $(\mathbf{x}_j, \mathbf{y}_j) \in G$. Then $\lim_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$ and $\lim_{j \to +\infty} \mathbf{y}_j = \mathbf{q}$ and therefore

$$(\lim_{j \to \infty} \mathbf{x}_j, \lim_{j \to \infty} \mathbf{y}_j) \notin G.$$

The result follows. ∎

Definition

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi\colon X \to Y$ be a function from X and Y. The *graph* $\mathrm{Graph}(\varphi)$ of the function φ is the subset of $\mathbb{R}^n \times \mathbb{R}^m$ defined so that

$$\mathrm{Graph}(\varphi) = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m : \mathbf{x} \in X \text{ and } \mathbf{y} = \varphi(\mathbf{x})\}.$$

Definition

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence between X and Y. The *graph* $\mathrm{Graph}(\Phi)$ of the correspondence Φ is the subset of $\mathbb{R}^n \times \mathbb{R}^m$ defined so that

$$\mathrm{Graph}(\Phi) = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m : \mathbf{x} \in X \text{ and } \mathbf{y} \in \Phi(\mathbf{x})\}.$$

Lemma 2.7

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi\colon X \to Y$ be a function from X to Y. Suppose that $\varphi\colon X \to Y$ is continuous. Then the graph $\mathrm{Graph}(\varphi)$ of the function φ is closed in $X \times Y$.

Proof

Let $\psi\colon X \times Y \to Y$ be the function defined such that

$$\psi(\mathbf{x}, \mathbf{y}) = \mathbf{y} - \varphi(\mathbf{x})$$

for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$. Then $\mathrm{Graph}(\varphi) = \psi^{-1}(\{\mathbf{0}\})$, and $\{\mathbf{0}\}$ is closed in $\mathbb{R}^m$. It follows that $\mathrm{Graph}(\varphi)$ is closed in $X \times Y$ (see Corollary 1.15). ∎

Example

Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined such that

$$f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

Then the graph $\mathrm{Graph}(f)$ of the function f satisfies $\mathrm{Graph}(f) = Z \cup H$, where

$$Z = \{(x, y) \in \mathbb{R}^2 : x \leq 0 \text{ and } y = 0\}$$

and

$$H = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } xy = 1\}.$$

Each of the sets Z and H is a closed set in $\mathbb{R}^2$. It follows that $\mathrm{Graph}(f)$ is a closed set in $\mathbb{R}^2$. However the function $f\colon \mathbb{R} \to \mathbb{R}$ is not continuous at 0.

Lemma 2.8

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$, let S be a non-empty subset of X, and let

$$d(\mathbf{x}, S) = \inf\{|\mathbf{x} - \mathbf{s}| : \mathbf{s} \in S\}$$

for all $\mathbf{x} \in X$. Then the function sending $\mathbf{x}$ to $d(\mathbf{x}, S)$ for all $\mathbf{x} \in X$ is a continuous function on X.

Proof

Let $f(\mathbf{x}) = d(\mathbf{x}, S) = \inf\{|\mathbf{x} - \mathbf{s}| : \mathbf{s} \in S\}$ for all $\mathbf{x} \in X$.
Let $\mathbf{x}$ and $\mathbf{x}'$ be points of X. It follows from the Triangle Inequality that

$$f(\mathbf{x}) \leq |\mathbf{x} - \mathbf{s}| \leq |\mathbf{x} - \mathbf{x}'| + |\mathbf{x}' - \mathbf{s}|$$

for all $\mathbf{s} \in S$, and therefore

$$|\mathbf{x}' - \mathbf{s}| \geq f(\mathbf{x}) - |\mathbf{x} - \mathbf{x}'|$$

for all $\mathbf{s} \in S$. Thus $f(\mathbf{x}) - |\mathbf{x} - \mathbf{x}'|$ is a lower bound for the quantities $|\mathbf{x}' - \mathbf{s}|$ as $\mathbf{s}$ ranges over the set S, and therefore cannot exceed the greatest lower bound of these quantities. It follows that

$$f(\mathbf{x}') = \inf\{|\mathbf{x}' - \mathbf{s}| : \mathbf{s} \in S\} \geq f(\mathbf{x}) - |\mathbf{x} - \mathbf{x}'|,$$

and thus

$$f(\mathbf{x}) - f(\mathbf{x}') \leq |\mathbf{x} - \mathbf{x}'|.$$

Interchanging $\mathbf{x}$ and $\mathbf{x}'$, it follows that

$$f(\mathbf{x}') - f(\mathbf{x}) \leq |\mathbf{x} - \mathbf{x}'|.$$

Thus

$$|f(\mathbf{x}) - f(\mathbf{x}')| \leq |\mathbf{x} - \mathbf{x}'|$$

for all $\mathbf{x}, \mathbf{x}' \in X$. It follows that the function $f\colon X \to \mathbb{R}$ is continuous, as required. ∎

The multidimensional Heine-Borel Theorem (Theorem 1.21) ensures that a subset of a Euclidean space is compact if and only if it is both closed and bounded.

Proposition 2.9

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$, let V be a subset of X that is open in X, and let K be a compact subset of $\mathbb{R}^n$ satisfying $K \subset V$. Then there exists some positive real number ε with the property that $B_X(K, \varepsilon) \subset V$, where $B_X(K, \varepsilon)$ denotes the subset of X consisting of those points of X that lie within a distance less than ε of some point of K.

Proof of Proposition 2.9 using the Extreme Value Theorem
Let $f\colon K \to \mathbb{R}$ be defined such that

$$f(\mathbf{x}) = \inf\{|\mathbf{z} - \mathbf{x}| : \mathbf{z} \in X \setminus V\}.$$

for all $\mathbf{x} \in K$. It follows from Lemma 2.8 that the function f is continuous on K.

Now $K \subset V$ and therefore, given any point $\mathbf{x} \in K$, there exists some positive real number δ such that the open ball of radius δ about the point $\mathbf{x}$ is contained in V, and therefore $f(\mathbf{x}) \geq \delta$. It follows that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in K$.

It follows from the Extreme Value Theorem for continuous real-valued functions on closed bounded subsets of Euclidean spaces (Theorem 1.17) that the function $f\colon K \to \mathbb{R}$ attains its minimum value at some point of K. Let that minimum value be ε. Then $f(\mathbf{x}) \geq \varepsilon > 0$ for all $\mathbf{x} \in K$, and therefore $|\mathbf{x} - \mathbf{z}| \geq \varepsilon > 0$ for all $\mathbf{x} \in K$ and $\mathbf{z} \in X \setminus V$. It follows that $B_X(K, \varepsilon) \subset V$, as required. ∎

Example

Let

$$F = \{(x, y) \in \mathbb{R}^2 : x \geq 0, \;\; y \geq 0 \text{ and } xy \geq 1\}.$$

and let

$$V = \{(x, y) \in \mathbb{R}^2 : y > 0\}.$$

Note that if $(x, y) \in F$ then $x > 0$ and $y > 0$, because $xy = 1$. It follows that $F \subset V$. Also F is a closed set in $\mathbb{R}^2$ and V is an open set in $\mathbb{R}^2$. However F is not a compact subset of $\mathbb{R}^2$ because F is not bounded.

We now show that there does not exist any positive real number ε with the property that $B_{\mathbb{R}^2}(F, \varepsilon) \subset V$, where $B_{\mathbb{R}^2}(F, \varepsilon)$ denotes the set of points of $\mathbb{R}^2$ that lie within a distance ε of some point of F. Indeed let some positive real number ε be given, let x be a positive real number satisfying $x > 2\varepsilon^{-1}$, and let $y = x^{-1} - \frac{1}{2}\varepsilon$. Then $y < 0$, and therefore $(x, y) \notin V$. But $(x, y + \frac{1}{2}\varepsilon) \in F$, and therefore $(x, y) \in B_{\mathbb{R}^2}(F, \varepsilon)$. This shows that there does not exist any positive real number ε for which $B_{\mathbb{R}^2}(F, \varepsilon) \subset V$.

Proposition 2.10

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let K be a non-empty compact subset of Y, and let U be a subset in $X \times Y$ that is open in $X \times Y$. Let

$$d_Y(\mathbf{y}, K) = \inf\{|\mathbf{y} - \mathbf{z}| : \mathbf{z} \in K\}$$

for all $\mathbf{y} \in Y$. Let $\mathbf{p}$ be a point of X with the property that $(\mathbf{p}, \mathbf{z}) \in U$ for all $\mathbf{z} \in K$. Then there exists some positive number δ such that $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$ and $d(\mathbf{y}, K) < \delta$.

Proof
Let

$$\tilde{K} = \{(\mathbf{p}, \mathbf{z}) : \mathbf{z} \in K\}.$$

Then $\tilde{K}$ is a closed bounded subset of $\mathbb{R}^n \times \mathbb{R}^m$. It follows from Proposition 2.9 that there exists some positive real number ε such that

$$B_{X \times Y}(\tilde{K}, \varepsilon) \subset U$$

where $B_{X \times Y}(\tilde{K}, \varepsilon)$ denotes that subset of $X \times Y$ consisting of those points $(\mathbf{x}, \mathbf{y})$ of $X \times Y$ that lie within a distance ε of a point of $\tilde{K}$. Now a point $(\mathbf{x}, \mathbf{y})$ of $X \times Y$ belongs to $B_{X \times Y}(\tilde{K}, \varepsilon)$ if and only if there exists some point $\mathbf{z}$ of K for which

$$|\mathbf{x} - \mathbf{p}|^2 + |\mathbf{y} - \mathbf{z}|^2 < \varepsilon^2.$$

Let $\delta = \varepsilon/\sqrt{2}$. If $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfy $|\mathbf{x} - \mathbf{p}| < \delta$ and $d_Y(\mathbf{y}, K) < \delta$ then there exists some point $\mathbf{z}$ of K for which $|\mathbf{y} - \mathbf{z}| < \delta$. But then

$$|\mathbf{x} - \mathbf{p}|^2 + |\mathbf{y} - \mathbf{z}|^2 < 2\delta^2 = \varepsilon^2,$$

and therefore $(\mathbf{x}, \mathbf{y}) \in U$, as required. ∎

Proposition 2.11

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that $\Phi(\mathbf{x})$ is closed in Y for every $\mathbf{x} \in X$. Suppose also that $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous. Then the graph $\mathrm{Graph}(\Phi)$ of $\Phi\colon X \rightrightarrows Y$ is closed in $X \times Y$.

Proof

Let $(\mathbf{p}, \mathbf{q})$ be a point of the complement $X \times Y \setminus \mathrm{Graph}(\Phi)$ of the graph $\mathrm{Graph}(\Phi)$ of Φ in $X \times Y$. Then $\Phi(\mathbf{p})$ is closed in Y and $\mathbf{q} \notin \Phi(\mathbf{p})$. It follows that there exists some positive real number δ_Y such that $|\mathbf{y} - \mathbf{q}| > \delta_Y$ for all $\mathbf{y} \in \Phi(\mathbf{p})$.

Let

$$V = \{\mathbf{y} \in Y : |\mathbf{y} - \mathbf{q}| > \delta_Y\}$$

and

$$W = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset V\}.$$

Then V is open in Y and $\Phi(\mathbf{p}) \subset V$. Now the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous. It therefore follows from the definition of upper hemicontinuity that the subset W of X is open in X. Moreover $\mathbf{p} \in W$. It follows that there exists some positive real number δ_X such that $\mathbf{x} \in W$ for all points $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta_X$. Then $\Phi(\mathbf{x}) \subset V$ for all points $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta_X$.

Let δ be the minimum of δ_X and δ_Y, and let $(\mathbf{x}, \mathbf{y})$ be a point of $X \times Y$ whose distance from the point $(\mathbf{p}, \mathbf{q})$ is less than δ. Then $|\mathbf{x} - \mathbf{p}| < \delta_X$ and therefore $\Phi(\mathbf{x}) \subset V$. Also $|\mathbf{y} - \mathbf{q}| < \delta_Y$, and therefore $\mathbf{y} \notin V$. It follows that $\mathbf{y} \notin \Phi(\mathbf{x})$, and therefore $(\mathbf{x}, \mathbf{y}) \notin \mathrm{Graph}(\Phi)$. We conclude from this that the complement of $\mathrm{Graph}(\Phi)$ is open in $X \times Y$. It follows that $\mathrm{Graph}(\Phi)$ itself is closed in $X \times Y$, as required. ∎

Proposition 2.12

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that the graph $\mathrm{Graph}(\Phi)$ of the correspondence Φ is closed in $X \times Y$. Suppose also that Y is a compact subset of $\mathbb{R}^m$. Then the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous.

Proof of Proposition 2.12 using Proposition 2.10

Let $\mathbf{p}$ be a point of X, let V be an open set satisfying $\Phi(\mathbf{p}) \subset V$, and let $K = Y \setminus V$. The compact set Y is closed and bounded in $\mathbb{R}^m$. Also K is closed in Y. It follows that K is a closed bounded subset of $\mathbb{R}^m$ (see Lemma 1.16). Let U be the complement of $\mathrm{Graph}(\Phi)$ in $X \times Y$. Then U is open in $X \times Y$, because $\mathrm{Graph}(\Phi)$ is closed in $X \times Y$. Also $(\mathbf{p}, \mathbf{y}) \in U$ for all $\mathbf{y} \in K$, because $\Phi(\mathbf{p}) \cap K = \emptyset$. It follows from Proposition 2.10 that there exists some positive number δ such that $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in K$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus if $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then $\mathbf{y} \notin \Phi(\mathbf{x})$ for all $\mathbf{y} \in K$, and therefore $\Phi(\mathbf{x}) \subset V$, where $V = Y \setminus K$. Thus the correspondence Φ is upper hemicontinuous at $\mathbf{p}$, as required. ∎

Corollary 2.13

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi\colon X \to Y$ be a function from X to Y. Suppose that the graph $\mathrm{Graph}(\varphi)$ of the function φ is closed in $X \times Y$. Suppose also that Y is a compact subset of $\mathbb{R}^m$. Then the function $\varphi\colon X \to Y$ is continuous.

Proof

Let $\Phi\colon X \rightrightarrows Y$ be the correspondence defined such that $\Phi(\mathbf{x}) = \{\varphi(\mathbf{x})\}$ for all $\mathbf{x} \in X$. Then $\mathrm{Graph}(\Phi) = \mathrm{Graph}(\varphi)$, and therefore $\mathrm{Graph}(\Phi)$ is closed in $X \times Y$. The subset Y of $\mathbb{R}^m$ is compact. It therefore follows from Proposition 2.12 that the correspondence Φ is upper hemicontinuous. It then follows from Lemma 2.3 that the function $\varphi\colon X \to Y$ is continuous, as required. ∎

2.3. Compact-Valued Upper Hemicontinuous Correspondences

Lemma 2.14

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous. Then

$$\{\mathbf{x} \in X : \Phi(\mathbf{x}) \neq \emptyset\}$$

is closed in X.

Proof

Given any open set V in Y, let

$$\Phi^+(V) = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset V\}.$$

It follows from the upper hemicontinuity of Φ that $\Phi^+(V)$ is open in X for all open sets V in Y (see Lemma 2.1). Now the empty set $\emptyset$ is open in Y. It follows that $\Phi^+(\emptyset)$ is open in X. But

$$\Phi^+(\emptyset) = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset \emptyset\} = \{\mathbf{x} \in X : \Phi(\mathbf{x}) = \emptyset\}.$$

It follows that the set of point $\mathbf{x}$ in X for which $\Phi(\mathbf{x}) = \emptyset$ is open in X, and therefore the set of points $\mathbf{x} \in X$ for which $\Phi(\mathbf{x}) \neq \emptyset$ is closed in X, as required. ∎

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Given any subset S of X, we define the *image* $\Phi(S)$ of S under the correspondence Φ to be the subset of Y defined such that

$$\Phi(S) = \bigcup_{\mathbf{x} \in S} \Phi(\mathbf{x})$$

Lemma 2.15

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is compact-valued and upper hemicontinuous. Let K be a compact subset of X. Then $\Phi(K)$ is a compact subset of Y.

Proof

Let $\mathcal{V}$ be collection of open sets in Y that covers $\Phi(K)$. Given any point $\mathbf{p}$ of K, there exists a finite subcollection $\mathcal{W}_\mathbf{p}$ of $\mathcal{V}$ that covers the compact set $\Phi(\mathbf{p})$. Let $U_\mathbf{p}$ be the union of the open sets belonging to this subcollection $\mathcal{W}_\mathbf{p}$. Then $\Phi(\mathbf{p}) \subset U_\mathbf{p}$. Now it follows from the upper hemicontinuity of $\Phi\colon X \rightrightarrows Y$ that there exists an open set $N_\mathbf{p}$ in X such that $\Phi(\mathbf{x}) \subset U_\mathbf{p}$ for all $\mathbf{x} \in N_\mathbf{p}$. Moreover, given any $\mathbf{p} \in K$, the finite collection $\mathcal{W}_\mathbf{p}$ of open sets in Y covers $\Phi(N_\mathbf{p})$.

It then follows from the compactness of K that there exist points

$$\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k$$

of K such that

$$K \subset N_{\mathbf{p}_1} \cup N_{\mathbf{p}_2} \cup \cdots \cup N_{\mathbf{p}_k}.$$

Let

$$\mathcal{W} = \mathcal{W}_{\mathbf{p}_1} \cup \mathcal{W}_{\mathbf{p}_2} \cup \cdots \cup \mathcal{W}_{\mathbf{p}_k}.$$

Then $\mathcal{W}$ is a finite subcollection of $\mathcal{V}$ that covers $\Phi(K)$. The result follows. ∎

Proposition 2.16

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a compact-valued correspondence from X to Y. Let $\mathbf{p}$ be a point of X for which $\Phi(\mathbf{p})$ is non-empty. Then the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$ if and only if, given any positive real number ε, there exists some positive real number δ such that

$$\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon)$$

for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, where $B_Y(\Phi(\mathbf{p}), \varepsilon)$ denotes the subset of Y consisting of all points of Y that lie within a distance ε of some point of $\Phi(\mathbf{p})$.

Proof

Let $\Phi\colon X \rightrightarrows Y$ is a compact-valued correspondence, and let $\mathbf{p}$ be a point of X for which $\Phi(\mathbf{p}) \neq \emptyset$.

First suppose that, given any positive real number ε, there exists some positive real number δ such that

$$\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon)$$

for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. We must prove that $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$.

Let V be an open set in Y that satisfies $\Phi(\mathbf{p}) \subset V$. Now $\Phi(\mathbf{p})$ is a compact subset of Y, because $\Phi\colon X \to Y$ is compact-valued. It follows that there exists some positive real number ε such that $B_Y(\Phi(\mathbf{p}), \varepsilon) \subset V$ (see Proposition 2.9). There then exists some positive number δ such that

$$\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon) \subset V$$

whenever $|\mathbf{x} - \mathbf{p}| < \delta$. Thus $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$.

Conversely suppose that the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at the point $\mathbf{p}$. Now $\Phi(\mathbf{p})$ is a non-empty subset of Y. Let some positive number ε be given. Then $B_Y(\Phi(\mathbf{p}), \varepsilon)$ is open in Y and $\Phi(\mathbf{p}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon)$. It follows from the upper hemicontinuity of Φ at $\mathbf{p}$ that there exists some positive number δ such that $\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon)$ whenever $|\mathbf{x} - \mathbf{p}| < \delta$. The result follows. ∎

Proposition 2.17

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Then the correspondence is both compact-valued and upper hemicontinuous at a point $\mathbf{p} \in X$ if and only if, given any infinite sequences

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

in X and Y respectively, where $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim_{j\to+\infty} \mathbf{x}_j = \mathbf{p}$, there exists a subsequence of

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

which converges to a point of $\Phi(\mathbf{p})$.

Proof

Throughout this proof, let us say that the correspondence Φ satisfies the *constrained convergent subsequence criterion* if (and only if), given any infinite sequences

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

in X and Y respectively, where $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$, there exists a subsequence of

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

which converges to a point of $\Phi(\mathbf{p})$.

We must prove that the correspondence $\Phi\colon X \rightrightarrows Y$ satisfies the constrained convergent subsequence criterion if and only if it is compact-valued and upper hemicontinuous.

Suppose first that the correspondence $\Phi\colon X \rightrightarrows Y$ satisfies the constrained convergent subsequence criterion. Applying this criterion when $\mathbf{x}_j = \mathbf{p}$ for all positive integers j, we conclude that every infinite sequence $(\mathbf{y}_j : j \in \mathbb{N})$ of points of $\Phi(\mathbf{p})$ has a convergent subsequence, and therefore $\Phi(\mathbf{x})$ is compact.

Let

$$D = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \neq \emptyset\}.$$

We show that D is closed in X. Let

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

be a sequence of points of D converging to some point of $\mathbf{p}$ of X. Then $\Phi(\mathbf{x}_j)$ is non-empty for all positive integers j, and therefore there exists an infinite sequence

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

of points of Y such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j. The constrained convergent subsequence criterion ensures that this infinite sequence in Y must have a subsequence that converges to some point of $\Phi(\mathbf{p})$. It follows that $\phi(\mathbf{p})$ is non-empty, and thus $\mathbf{p} \in D$.

Let $\mathbf{p}$ be a point of the complement of D. Then $\Phi(\mathbf{p}) = \emptyset$. There then exists $\delta > 0$ such that $\Phi(\mathbf{x}) = \emptyset$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then $\Phi(\mathbf{x}) \subset V$ for all open sets V in Y. It follows that the correspondence Φ is upper hemicontinuous at those points $\mathbf{p}$ for which $\Phi(\mathbf{p}) = \emptyset$.

Now consider the situation in which $\Phi\colon X \rightrightarrows Y$ satisfies the constrained convergent subsequence criterion and $\mathbf{p}$ is some point of X for which $\Phi(\mathbf{p}) \neq \emptyset$. Let $K = \Phi(\mathbf{p})$. Then K is a compact non-empty subset of Y. Let V be an open set in Y that satisfies $\Phi(\mathbf{p}) \subset V$. Suppose that there did not exist any positive real number δ with the property that $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It would then follow that there would exist infinite sequences

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

in X and Y respectively for which $|\mathbf{x}_j - \mathbf{p}| < 1/j$, $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ and $\mathbf{y}_j \notin V$.

Then $\lim_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$, and thus the constrained convergent subsequence criterion satisfied by the correspondence Φ would ensure the existence of a subsequence

$$\mathbf{y}_{k_1}, \mathbf{y}_{k_2}, \mathbf{y}_{k_3}, \ldots$$

of $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ converging to some point $\mathbf{q}$ of $\Phi(\mathbf{p})$. But then $\mathbf{q} \notin V$, because $\mathbf{y}_{k_j} \notin V$ for all positive integers j, and the complement $Y \setminus V$ of V is closed in Y. But $\Phi(\mathbf{p}) \subset V$, and $\mathbf{q} \in \Phi(\mathbf{p})$, and therefore $\mathbf{q} \in V$. Thus a contradiction would arise were there not to exist a positive real number δ with the property that $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus such a real number δ must exist, and thus the constrained convergent subsequence criterion ensures that the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$.

It remains to show that any compact-valued upper hemicontinuous correspondence $\Phi\colon X \rightrightarrows Y$ satisfies the constrained convergent subsequence criterion. Let $\Phi\colon X \rightrightarrows Y$ be compact-valued and upper hemicontinuous. It follows from Lemma 2.14 that

$$\{\mathbf{x} \in X : \Phi(\mathbf{x}) \neq \emptyset\}$$

is closed in X.

Let

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

be infinite sequences in X and Y respectively, where $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$. Then $\Phi(\mathbf{p})$ is non-empty, because

$$\{\mathbf{x} \in X : \Phi(\mathbf{x}) \neq \emptyset\}$$

is closed in X (see Lemma 2.14). Let $K = \Phi(\mathbf{p})$. Then K is compact, because $\Phi\colon X \rightrightarrows Y$ is compact-valued by assumption.

For each integer j let $d(\mathbf{y}_j, K)$ denote the greatest lower bound on the distances from $\mathbf{y}_j$ to points of K. There then exists an infinite sequence

$$\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots$$

of points of K such that $|\mathbf{y}_j - \mathbf{z}_j| \leq 2d(\mathbf{y}_j, K)$. for all positive integers j. (Indeed if $d(\mathbf{y}_j, K) = 0$ then $\mathbf{y}_j \in K$, because the compact set K is closed, and in that case we can take $\mathbf{z}_j = \mathbf{y}_j$. Otherwise $2d(\mathbf{y}, K)$ is strictly greater than the greatest lower bound on the distances from $\mathbf{y}_j$ to points of K, and we can therefore find $\mathbf{z}_j \in K$ with $|\mathbf{y}_j - \mathbf{z}_j| \leq 2d(\mathbf{y}_j, K)$.)

Now the upper hemicontinuity of $\Phi\colon X \rightrightarrows Y$ ensures that $d(\mathbf{y}_j, K) \to 0$ as $j \to +\infty$. Indeed, given any positive real number ε, the set $B_Y(K, \varepsilon)$ of points of Y that lie within a distance ε of a point of K is an open set containing $\Phi(\mathbf{p})$. It follows from the upper hemicontinuity of Φ that there exists some positive real number δ such that $\Phi(\mathbf{x}) \subset B_Y(K, \varepsilon)$ whenever $|\mathbf{x} - \mathbf{p}| < \delta$. Now $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$. It follows that there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \delta$ whenever $j \geq N$. But then $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ and therefore $d(\mathbf{y}_j, K) < \varepsilon$ whenever $j \geq N$.

Now the compactness of K ensures that the infinite sequence

$$\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots$$

of points of K has a subsequence

$$\mathbf{z}_{k_1}, \mathbf{z}_{k_2}, \mathbf{z}_{k_3}, \ldots$$

that converges to some point $\mathbf{q}$ of K. Now $|\mathbf{y}_j - \mathbf{z}_j| \leq 2d(\mathbf{y}_j, K)$ for all positive integers j, and $d(\mathbf{y}_j, K) \to 0$ as $j \to +\infty$. It follows that $\mathbf{y}_{k_j} \to \mathbf{q}$ as $j \to +\infty$. Morever $\mathbf{q} \in \Phi(\mathbf{p})$. We have therefore verified that the constrained convergent subsequence criterion is satisfied by any correspondence $\Phi\colon X \rightrightarrows Y$ that is compact-valued and upper hemicontinuous. This completes the proof. ∎

Proposition 2.18

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is both upper hemicontinuous and compact-valued. Let U be an open set in $X \times Y$. Then

$$\{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X.

Proof of Proposition 2.18 using Proposition 2.17

Let

$$W = \{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\},$$

and let $\mathbf{p} \in W$. Suppose that there did not exist any strictly positive real number δ with the property that $\mathbf{x} \in W$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Then, given any positive real number δ, there would exist points $\mathbf{x}$ of X and $\mathbf{y}$ of Y such that $|\mathbf{x} - \mathbf{p}| < \delta$, $\mathbf{y} \in \Phi(\mathbf{x})$ and $(\mathbf{x}, \mathbf{y}) \notin U$. Therefore there would exist infinite sequences

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

in X and Y respectively such that $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ and $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ and $(\mathbf{x}_j, \mathbf{y}_j) \notin U$ for all positive integers j.

The correspondence $\Phi\colon X \rightrightarrows Y$ is compact-valued and upper hemicontinuous. Proposition 2.17 would therefore ensure the existence of a subsequence

$$\mathbf{y}_{k_1}, \mathbf{y}_{k_2}, \mathbf{y}_{k_3}, \ldots$$

of Y converging to some point $\mathbf{q}$ of $\Phi(\mathbf{p})$. Now the complement of U in $X \times Y$ is closed in $X \times Y$, because U is open in $X \times Y$ and $(\mathbf{x}_j, \mathbf{y}_j) \not\in U$. It would therefore follow that $(\mathbf{p}, \mathbf{q}) \not\in U$ (see Proposition 2.6). But this gives rise to a contradiction, because $\mathbf{q} \in \Phi(\mathbf{p})$ and $(\mathbf{p}, \mathbf{y}) \in U$ for all $\mathbf{y} \in \Phi(\mathbf{p})$. In order to avoid the contradiction, there must exist some positive real number δ with the property that with the property that $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$ and $\mathbf{y} \in \Phi(\mathbf{x})$. The result follows. ∎

Remark

It should be noted that other results proved in this section do not necessarily generalize to correspondences $\Phi\colon X \rightrightarrows Y$ mapping the topological space X into an arbitrary topological space Y. For example all closed-valued upper hemicontinuous correspondences between metric spaces have closed graphs. The appropriate generalization of this result states that any closed-valued upper hemicontinuous correspondence $\Phi\colon X \rightrightarrows Y$ from a topological space X to a regular topological space Y has a closed graph. To interpret this, one needs to know the definition of what is meant by saying that a topological space is *regular*. A topological space Y is said to be *regular* if, given any closed subset F of Y, and given any point p of the complement $Y \setminus F$ of F, there exist open sets V and W in Y such that $F \subset V$, $p \in W$ and $V \cap W = \emptyset$. Metric spaces are regular. Also compact Hausdorff spaces are regular.

2.4. A Criterion characterizing Lower Hemicontinuity

Proposition 2.19

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi\colon X \rightrightarrows Y$ is lower hemicontinuous at a point $\mathbf{p}$ of X if and only if given any infinite sequence

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

in X for which $\lim_{j\to+\infty} \mathbf{x}_j = \mathbf{p}$ and given any point $\mathbf{q}$ of $\Phi(\mathbf{p})$, there exists an infinite sequence

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

of points of Y such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim_{j\to+\infty} \mathbf{y}_j = \mathbf{q}$.

Proof

First suppose that $\Phi\colon X \to Y$ is lower hemicontinuous at some point $\mathbf{p}$ of X. Let $\mathbf{q} \in \Phi(\mathbf{p})$, and let some positive number ε be given. Then the open ball $B_Y(\mathbf{q}, \varepsilon)$ in Y of radius ε centred on the point $\mathbf{q}$ is an open set in Y. It follows from the lower hemicontinuity of $\Phi\colon X \to Y$ that there exists some positive real number δ such that $\Phi(\mathbf{x}) \cap B_Y(\mathbf{q}, \varepsilon)$ is non-empty whenever $|\mathbf{x} - \mathbf{p}| < \delta$. Then, given any point $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta$ there exists some $\mathbf{y} \in \Phi(\mathbf{x})$ that satisfies $|\mathbf{y} - \mathbf{q}| < \varepsilon$. In particular, given any positive integer s, there exists some positive integer δ_s such that, given any point $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta_s$, there exists some $\mathbf{y} \in \Phi(\mathbf{x})$ that satisfies $|\mathbf{y} - \mathbf{q}| < 1/s$.

Now $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$. It follows that there exist positive integers $k(1), k(2), k(3), \ldots$, where

$$k(1) < k(2) < k(3) < \cdots$$

such that $|\mathbf{x}_j - \mathbf{p}| < \delta_s$ for all positive integers j satisfying $j \geq k(s)$. There then exists an infinite sequence

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $|\mathbf{y}_j - \mathbf{q}| < 1/s$ for all positive integers j and s satisfying $k(s) \leq j < k(s+1)$. Then $\lim_{j \to +\infty} \mathbf{y}_j = \mathbf{q}$. We have thus shown that if $\Phi \colon X \to Y$ is lower hemicontinuous at the point $\mathbf{p}$, if $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is a sequence in X converging to the point $\mathbf{p}$, and if $\mathbf{q} \in \Phi(\mathbf{p})$, then there exists an infinite sequence $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ in Y such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integer j and $\lim_{j \to +\infty} \mathbf{y}_j = \mathbf{q}$.

Next suppose that the correspondence $\Phi\colon X \rightrightarrows Y$ is not lower hemicontinuous at $\mathbf{p}$. Then there exists an open set V in Y such that $\Phi(\mathbf{p}) \cap V$ is non-empty but there does not exist any positive real number δ with the property that $\Phi(\mathbf{x}) \cap V \neq \emptyset$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{p} - \mathbf{x}| < \delta$. Let $\mathbf{q} \in \Phi(\mathbf{p}) \cap V$. There then exists an infinite sequence

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

converging to the point $\mathbf{p}$ with the property that $\Phi(\mathbf{x}_j) \cap V = \emptyset$ for all positive integers j. It is not then possible to construct an infinite sequence

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim\limits_{j \to +\infty} \mathbf{y}_j = \mathbf{q}$.

The result follows. ∎

2.5. Intersections of Correspondences

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ and $\Psi\colon X \to Y$ be correspondences between X and Y. The *intersection* $\Phi \cap \Psi$ of the correspondences Φ and Ψ is defined such that

$$(\Phi \cap \Psi)(\mathbf{x}) = \Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$$

for all $\mathbf{x} \in X$.

Proposition 2.20

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let $\Phi\colon X \rightrightarrows Y$ and $\Psi\colon X \rightrightarrows Y$ be correspondences from X to Y, where the correspondence $\Phi\colon X \rightrightarrows Y$ is compact-valued and upper hemicontinuous and the correspondence $\Psi\colon X \rightrightarrows Y$ has closed graph. Let $\Phi \cap \Psi\colon X \rightrightarrows Y$ be the correspondence defined such that

$$(\Phi \cap \Psi)(\mathbf{x}) = \Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$$

for all $\mathbf{x} \in X$. Then the correspondence Let $\Phi \cap \Psi\colon X \rightrightarrows Y$ is compact-valued and upper hemicontinuous.

Proof

Let

$$W = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : \mathbf{y} \notin \Psi(\mathbf{x})\}.$$

Then W is the complement of the graph $\mathrm{Graph}(\Psi)$ of Ψ in $X \times Y$. The graph of Ψ is closed in $X \times Y$, by assumption. It follows that W is open in $X \times Y$.

Let $\mathbf{x} \in X$. The subset $\Psi(\mathbf{x})$ of Y is closed in Y, because the graph of the correspondence Ψ is closed. It follows from the compactness of $\Phi(\mathbf{x})$ that $\Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$ is a closed subset of the compact set $\Phi(\mathbf{x})$, and must therefore be compact. Thus the correspondence $\Phi \cap \Psi$ is compact-valued.

Now let $\mathbf{p}$ be a point of X, and let V be any open set in Y for which $\Phi(\mathbf{p}) \cap \Psi(\mathbf{p}) \subset V$. In order to prove that $\Phi \cap \Psi$ is upper hemicontinuous we must show that there exists some positive real number δ such that $\Phi(\mathbf{x}) \cap \Psi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Let

$$U = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : \text{either } \mathbf{y} \in V \text{ or else } \mathbf{y} \notin \Psi(\mathbf{x})\}.$$

Then U is the union of the subsets $X \times V$ and W of $X \times Y$, where both these subsets are open in $X \times Y$. It follows that U is open in $X \times Y$. Moreover if $\mathbf{y} \in \Phi(\mathbf{p})$ then either $\mathbf{y} \in \Phi(\mathbf{p}) \cap \Psi(\mathbf{p})$, in which case $\mathbf{y} \in V$, or else $\mathbf{y} \notin \Psi(\mathbf{p})$. It follows that $(\mathbf{p}, \mathbf{y}) \in U$ for all $\mathbf{y} \in \Phi(\mathbf{p})$.

Now it follows from Proposition 2.18 that

$$\{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X. Therefore there exists some positive real number δ such that $(\mathbf{x}, \mathbf{y}) \in U$ for all $(\mathbf{x}, \mathbf{y}) \in X \times Y$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$ and $\mathbf{y} \in \Phi(\mathbf{x})$. Now if $(\mathbf{x}, \mathbf{y})$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ and $\mathbf{y} \in \Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$ then $(\mathbf{x}, \mathbf{y}) \in U$ but $(\mathbf{x}, \mathbf{y}) \notin W$. It follows from the definition of U that $\mathbf{y} \in V$. Thus $\Phi(\mathbf{x}) \cap \Psi(\mathbf{x}) \subset V$ whenever $|\mathbf{x} - \mathbf{p}| < \delta$. The result follows. ■

2.6. Berge's Maximum Theorem

Lemma 2.21

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is both upper hemicontinuous and compact-valued. Let $f\colon X \times Y \to \mathbb{R}$ be a continuous real-valued function on $X \times Y$, and let c be a real number. Then

$$\{\mathbf{x} \in X : f(\mathbf{x}, \mathbf{y}) < c \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X.

Proof

Let

$$U = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : f(\mathbf{x}, \mathbf{y}) < c\}.$$

It follows from the continuity of the function f that U is open in $X \times Y$. It then follows from Proposition 2.18 that

$$\{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X. The result follows. ∎

Lemma 2.22

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is lower hemicontinuous. Let $f\colon X \times Y \to \mathbb{R}$ be a continuous real-valued function on $X \times Y$, and let c be a real number. Then

$$\{\mathbf{x} \in X : \text{ there exists } \mathbf{y} \in \Phi(\mathbf{x}) \text{ for which } f(\mathbf{x}, \mathbf{y}) > c\}$$

is open in X.

Proof

Let

$$U = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : f(\mathbf{x}, \mathbf{y}) > c\},$$

and let

$$W = \{\mathbf{x} \in X : \text{ there exists } \mathbf{y} \in \Phi(\mathbf{x}) \text{ for which } f(\mathbf{x}, \mathbf{y}) > c\},$$

Let $\mathbf{p} \in W$. Then there exists $\mathbf{y} \in \Phi(\mathbf{p})$ for which $(\mathbf{p}, \mathbf{y}) \in U$. There then exist subsets W_X of X and W_Y of Y, where W_X is open in X and W_Y is open in Y, such that $\mathbf{p} \in W_X$, $\mathbf{y} \in W_Y$ and $W_X \times W_Y \subset U$ (see Lemma 2.5). There then exists some positive real number δ_1 such that $\mathbf{x} \in W_X$ whenever $|\mathbf{x} - \mathbf{p}| < \delta_1$.

Now $\Phi(\mathbf{p}) \cap W_Y \neq \emptyset$, because $\mathbf{y} \in \Phi(\mathbf{p}) \cap W_Y$. It follows from the lower hemicontinuity of the correspondence Φ that there exists some positive real number δ_2 such that $\Phi(\mathbf{x}) \cap W_Y \neq \emptyset$ whenever $|\mathbf{x} - \mathbf{p}| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then there exists $\mathbf{y} \in \Phi(\mathbf{x})$ for which $\mathbf{y} \in W_Y$. But then $(\mathbf{x}, \mathbf{y}) \in W_X \times W_Y$ and therefore $(\mathbf{x}, \mathbf{y}) \in U$, and thus $f(\mathbf{x}, \mathbf{y}) > c$. The result follows. ∎

Theorem 2.23 (Berge's Maximum Theorem)

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let $f\colon X \times Y \to \mathbb{R}$ be a continuous real-valued function on $X \times Y$, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that $\Phi(\mathbf{x})$ is both non-empty and compact for all $\mathbf{x} \in X$ and that the correspondence $\Phi\colon X \to Y$ is both upper hemicontinuous and lower hemicontinuous. Let

$$m(\mathbf{x}) = \sup\{f(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in \Phi(\mathbf{x})\}$$

for all $\mathbf{x} \in X$, and let

$$M(\mathbf{x}) = \{\mathbf{y} \in \Phi(\mathbf{x}) : f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}$$

for all $\mathbf{x} \in X$. Then $m\colon X \to \mathbb{R}$ is continuous, $M(\mathbf{x})$ is a non-empty compact subset of Y for all $\mathbf{x} \in X$, and the correspondence $M\colon X \rightrightarrows Y$ is upper hemicontinuous.

Proof

Let $\mathbf{x} \in X$. Then $\Phi(\mathbf{x})$ is a non-empty compact subset of Y. It is thus a closed bounded subset of $\mathbb{R}^m$. It follows from the Extreme Value Theorem (Theorem 1.17) that there exists at least one point $\mathbf{y}^*$ of $\Phi(\mathbf{x})$ with the property that $f(\mathbf{x}, \mathbf{y}^*) \geq f(\mathbf{x}, \mathbf{y})$ for all $\mathbf{y} \in \Phi(\mathbf{x})$. Then $m(\mathbf{x}) = f(\mathbf{x}, \mathbf{y}^*)$ and $\mathbf{y}^* \in M(\mathbf{x})$. Moreover

$$M(\mathbf{x}) = \{\mathbf{y} \in \Phi(\mathbf{x}) : f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}.$$

It follows from the continuity of f that the set $M(\mathbf{x})$ is closed in Y (see Corollary 1.15). It is thus a closed subset of the compact set $\Phi(\mathbf{x})$ and must therefore itself be compact.

Let some positive number ε be given. Then $f(\mathbf{p}, \mathbf{y}) < m(\mathbf{p}) + \varepsilon$ for all $\mathbf{y} \in \Phi(\mathbf{p})$. It follows from Lemma 2.21 that

$$\{\mathbf{x} \in X : f(\mathbf{x}, \mathbf{y}) < m(\mathbf{p}) + \varepsilon \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X, and thus there exists some positive real number δ_1 such that $f(\mathbf{x}, \mathbf{y}) < m(\mathbf{p}) + \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_1$ and $\mathbf{y} \in \Phi(\mathbf{x})$ Then $m(\mathbf{x}) < m(\mathbf{p}) + \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_1$.

The correspondence $\Phi\colon X \rightrightarrows Y$ is also lower hemicontinuous. It therefore follows from Lemma 2.22 that there exists some positive real number δ_2 such that, given any $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_2$, there exists some $\mathbf{y} \in \Phi(\mathbf{x})$ for which $f(\mathbf{x}, \mathbf{y}) > m(\mathbf{p}) - \varepsilon$. It follows that $m(\mathbf{x}) > m(\mathbf{p}) - \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and

$$m(\mathbf{p}) - \varepsilon < m(\mathbf{x}) < m(\mathbf{p}) + \varepsilon$$

whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$. Thus the function $m\colon X \to \mathbb{R}$ is continuous on X.

It only remains to prove that the correspondence $M\colon X \rightrightarrows Y$ is upper hemicontinuous. Let

$$\Psi(\mathbf{x}) = \{\mathbf{y} \in Y : f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}$$

for all $\mathbf{x} \in X$. Then

$$\mathrm{Graph}(\Psi) = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}$$

Thus $\mathrm{Graph}(\Psi)$ is the preimage of zero under the continuous real-valued function that sends $(\mathbf{x}, \mathbf{y}) \in X \times Y$ to $f(\mathbf{x}, \mathbf{y}) - m(\mathbf{x})$. It follows that $\mathrm{Graph}(\Psi)$ is a closed subset of $X \times Y$.

Now $M(\mathbf{x}) = \Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$ for all $\mathbf{x} \in X$, where the correspondence Φ is compact-valued and upper hemicontinuous and the correspondence Ψ has closed graph. It follows from Proposition 2.20 that the correspondence M must itself be both compact-valued and upper hemicontinuous. This completes the proof of Berge's Maximum Theorem. ∎