MAU34804—Fixed Point Theorems and Economic Equilibria
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Appendix B: Alternative Proofs of Results concerning Correspondences

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Proof of Proposition 2.9 using the Bolzano-Weierstrass Theorem

Suppose that the proposition were false. Then there would exist infinite sequences $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ and $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \ldots$ such that $\mathbf{x}_j \in K$, $\mathbf{w}_j \in X \setminus V$ and $|\mathbf{w}_j - \mathbf{x}_j| < 1/j$ for all positive integers j. The set K is both closed and bounded in $\mathbb{R}^n$. The multidimensional Bolzano-Weierstrass Theorem (Theorem 1.2) would then ensure the existence of a subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$ of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converging to some point $\mathbf{q}$ of K. Moreover $\lim_{j \to +\infty} (\mathbf{w}_j - \mathbf{x}_j) = \mathbf{0}$, and therefore

$$\lim_{j \to \infty} \mathbf{w}_{k_j} = \lim_{j \to \infty} \mathbf{x}_{k_j} = \mathbf{q}.$$

But $\mathbf{w}_j \in X \setminus V$. Moreover $X \setminus V$ is closed in X, and therefore any sequence of points in $X \setminus V$ that converges in X must converge to a point of $X \setminus V$ (see Lemma 1.8). It would therefore follow that $\mathbf{q} \in K \cap (X \setminus V)$. But this is impossible, because $K \subset V$ and therefore $K \cap (X \setminus V) = \emptyset$. Thus a contradiction would follow were the proposition false. The result follows. ∎

Proof of Proposition 2.9 using the Heine-Borel Theorem
It follows from the multidimensional Heine-Borel Theorem (Theorem 1.21) that the set K is compact, and thus every open cover of K has a finite subcover. Given point $\mathbf{x}$ of K let $\varepsilon_{\mathbf{x}}$ be a positive real number with the property that

$$B_X(\mathbf{x}, 2\varepsilon_{\mathbf{x}}) \subset V,$$

where

$$B_X(\mathbf{x}, r) = \{\mathbf{x}' \in X : |\mathbf{x}' - \mathbf{x}| < r\}$$

for all positive integers r. The collection of open balls $B_X(\mathbf{x}, \varepsilon_{\mathbf{x}})$ determined by the points $\mathbf{x}$ of K covers K. By compactness this open cover of K has a finite subcover. Therefore there exist points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ of K such that

$$K \subset B(\mathbf{x}_1, \varepsilon_{\mathbf{x}_1}) \cup B(\mathbf{x}_2, \varepsilon_{\mathbf{x}_2}) \cup \cdots \cup B(\mathbf{x}_k, \varepsilon_{\mathbf{x}_k}).$$

Let ε be the minimum of $\varepsilon_{\mathbf{x}_1}, \varepsilon_{\mathbf{x}_2}, \ldots, \varepsilon_{\mathbf{x}_k}$. If $\mathbf{x}$ is a point of K then $\mathbf{x} \in B_X(\mathbf{x}_j, \varepsilon_{\mathbf{x}_j}$ for some integer j between 1 and k. But it then follows from the Triangle Inequality that

$$B(\mathbf{x}, \varepsilon) \subset B_X(\mathbf{x}_j, 2\varepsilon_{\mathbf{x}_j}) \subset V.$$

It follows from this that

$$B_X(K, \varepsilon) \subset V,$$

as required. ∎

Proof of Proposition 2.12 using the Bolzano-Weierstrass Theorem

Let V be a subset of Y that is open in Y, and let $\mathbf{p}$ be a point of X for which $\Phi(\mathbf{p}) \subset V$. Let $F = Y \setminus V$. Then the set F is a subset of Y that is closed in Y, and $\Phi(\mathbf{p}) \cap F = \emptyset$. Now Y is a closed bounded subset of $\mathbb{R}^m$, because it is compact (Theorem 1.21). It follows that F is closed in $\mathbb{R}^m$ (Lemma 1.16).

Suppose that there did not exist any positive number δ such that $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Then there would exist an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X converging to the point $\mathbf{p}$ with the property that $\Phi(\mathbf{x}_j) \cap F \neq \emptyset$ for all positive integers j. There would then exist an infinite sequence $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ of elements of Y such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j) \cap F$ for all positive integers j. Then $(\mathbf{x}_j, \mathbf{y}_j) \in \mathrm{Graph}(\Phi)$ for all positive integers j. Moreover the infinite sequence $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ would be bounded, because the set Y is bounded.

It would therefore follow from the multidimensional Bolzano-Weierstrass Theorem (Theorem 1.2) that there would exist a convergent subsequence

$$\mathbf{y}_{k_1}, \mathbf{y}_{k_2}, \mathbf{y}_{k_3}, \ldots$$

of the sequence $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$. Let $\mathbf{q} = \lim_{j \to +\infty} \mathbf{y}_{k_j}$. Then $\mathbf{q} \in F$, because the set F is closed in Y and $\mathbf{y}_{k_j} \in F$ for all positive integers j (see Lemma 1.8). Similarly $(\mathbf{p}, \mathbf{q}) \in \mathrm{Graph}(\Phi)$, because the set $\mathrm{Graph}(\Phi)$ is closed in $X \times Y$, $(\mathbf{x}_{k_j}, \mathbf{y}_{k_j}) \in \mathrm{Graph}(\Phi)$ for all positive integers j, and

$$(\mathbf{p}, \mathbf{q}) = \lim_{j \to +\infty} (\mathbf{x}_{k_j}, \mathbf{y}_{k_j}).$$

But were there to exist $(\mathbf{p}, \mathbf{q}) \in X \times Y$ for which $\mathbf{q} \in F$ and $(\mathbf{p}, \mathbf{q}) \in \mathrm{Graph}(\Phi)$, it would follow that $\mathbf{q} \in \Phi(\mathbf{p}) \cap F$. But this is impossible, because $\Phi(\mathbf{p}) \cap F = \emptyset$. Thus a contradiction would arise were there to exist an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X for which $\Phi(\mathbf{x}_j) \cap F \neq \emptyset$ and $\lim_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$. Therefore no such infinite sequence can exist, and therefore there must exist some positive real number δ such that $\Phi(\mathbf{x}) \subset V$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$. We conclude that

$$\{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset V\}$$

is open in X. The result follows. ∎

Proof of Proposition 2.18 using Proposition 2.10

Let

$$W = \{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\},$$

and let $\mathbf{p} \in W$. If $\Phi(\mathbf{p}) = \emptyset$ then it follows from Lemma 2.14 that there exists some positive real number δ such that $\Phi(\mathbf{x}) = \emptyset$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Then $\mathbf{x} \in W$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$.

Suppose that $\Phi(\mathbf{p}) \neq 0$. Let $K = \Phi(\mathbf{p})$. Then K is a compact subset of Y, because the correspondence Φ is compact-valued. Also $(\mathbf{p}, \mathbf{y}) \in U$ for all $\mathbf{y} \in K$. It follows from Proposition 2.10 that there exists some positive real number δ_1 such that $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_1$ and $d_Y(\mathbf{y}, K) < \delta_1$, where

$$d_Y(\mathbf{y}, K) = \inf\{|\mathbf{y} - \mathbf{z}| : \mathbf{z} \in K\}.$$

Let

$$V = \{\mathbf{y} \in Y : d_Y(\mathbf{y}, K) < \delta_1\}.$$

Then V is open in Y because the function sending $\mathbf{y} \in Y$ to $d(\mathbf{y}, K)$ is continuous on Y (see Lemma 2.8). Also $\Phi(\mathbf{p}) \subset V$. It follows from the upper hemicontinuity of the correspondence Φ that there exists some positive number δ_2 such that $\Phi(\mathbf{x}) \subset V$ whenever $|\mathbf{x} - \mathbf{p}| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then $\Phi(\mathbf{x}) \subset V$. But then $d(\mathbf{y}, K) < \delta_1$ for all $\mathbf{y} \in \Phi(\mathbf{x})$. Moreover $|\mathbf{x} - \mathbf{p}| < \delta_1$. It follows that $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{y} \in \Phi(\mathbf{x})$, and therefore $\mathbf{x} \in W$. This shows that W is an open subset of X, as required. ■

Proof of Proposition 2.18 using the Heine-Borel Theorem

Let $\Phi\colon X \to Y$ be a compact-valued upper hemicontinuous correspondence, and let U be a subset of $X \times Y$ that is open in $X \times Y$. Let

$$W = \{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}.$$

We must prove that W is open in X.

Let $K = \Phi(\mathbf{p})$. Then, given any point $\mathbf{y}$ of K, there exists an open set $M_{\mathbf{p},\mathbf{y}}$ in X and an open set $V_{\mathbf{p},\mathbf{y}}$ in Y such that $M_{\mathbf{p},\mathbf{y}} \times V_{\mathbf{p},\mathbf{y}} \subset U$ (see Lemma 2.5). Now every open cover of K has a finite subcover, because K is compact. Therefore there exist points $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k$ of K such that

$$K \subset V_{\mathbf{p},\mathbf{y}_1} \cup V_{\mathbf{p},\mathbf{y}_2} \cup \cdots \cup V_{\mathbf{p},\mathbf{y}_k}.$$

Let

$$M_{\mathbf{p}} = M_{\mathbf{p},\mathbf{y}_1} \cap M_{\mathbf{p},\mathbf{y}_2} \cap \cdots \cap M_{\mathbf{p},\mathbf{y}_k}$$

and

$$V_{\mathbf{p}} = V_{\mathbf{p},\mathbf{y}_1} \cup V_{\mathbf{p},\mathbf{y}_2} \cup \cdots \cup V_{\mathbf{p},\mathbf{y}_k}.$$

Then

$$M_{\mathbf{p}} \times V_{\mathbf{p}} \subset \bigcup_{j=1}^{k} (M_{\mathbf{p}} \times V_{\mathbf{p},\mathbf{y}_j}) \subset \bigcup_{j=1}^{k} (M_{\mathbf{p},\mathbf{y}_j} \times V_{\mathbf{p},\mathbf{y}_j}) \subset U.$$

Now $M_{\mathbf{p}}$ is open in X, because it is the intersection of a finite number of subsets of X that are open in X. Also it follows from the upper hemicontinuity of the correspondence Φ that $\Phi^+(V_{\mathbf{p}})$ is open in X, where

$$\Phi^+(V_{\mathbf{p}}) = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset V_{\mathbf{p}}\}$$

(see Lemma 2.1). Let $N_{\mathbf{p}} = M_{\mathbf{p}} \cap \Phi^+(V_{\mathbf{p}})$. Then $N_{\mathbf{p}}$ is open in X and $\mathbf{p} \in N_{\mathbf{p}}$. Now if $\mathbf{x} \in N_{\mathbf{p}}$ then $\mathbf{x} \in M_{\mathbf{p}}$ and $\Phi(\mathbf{x}) \subset V_{\mathbf{p}}$, and therefore $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{y} \in \Phi(\mathbf{x})$. We have thus shown that $N_{\mathbf{p}} \subset W$ for all $\mathbf{p} \in W$, where

$$W = \{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}.$$

Thus W is the union of the subsets $N_{\mathbf{p}}$ as $\mathbf{p}$ ranges over the points of W. Moreover the set $N_{\mathbf{p}}$ is open in X for each $\mathbf{p} \in W$. It follows that W must itself be open in X. Indeed, given any point $\mathbf{p}$ of W, there exists some positive real number δ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\} \subset N_{\mathbf{p}} \subset W.$$

The result follows. ∎

Remark

The various proofs of Proposition 2.18 were presented in the contexts of correspondences between subsets of Eucldean spaces. All these proofs generalize easily so as to apply to correspondence between subsets of metric spaces. The last of the proofs can be generalized without difficulty so as to apply to correspondences between topological spaces. Inded the notion of *correspondences* between topological spaces is defined so that a correspondence $\Phi\colon X \rightrightarrows Y$ between topological spaces X and Y associates to each point of X a subset $\Phi(\mathbf{x})$ of Y. Such a correspondence is said to be upper hemicontinuous at a point p of X if, given any open subset V of Y for which $\Phi(p) \subset V$, there exists an open set $N(p)$ in X such that $\Phi(x) \subset V$ for all $x \in N$.

The proof of Proposition 2.18 using the Heine-Borel Theorem presented above can be generalized to show that, given a compact-valued correspondence $\Phi\colon X \rightrightarrows Y$ between topological spaces X and Y, and given a subset U of Y, the set

$$\{x \in X : (x, y) \in U \text{ for all } y \in \Phi(x)\}$$

is open in X.

We describe another proof of the Berge Maximum Theorem using the characterization of compact-valued upper hemicontinuous correspondences using sequences established in Proposition 2.17 and the characterization of lower hemicontinuous correspondences using sequences established in Proposition 2.19. First we introduce some terminology.

Definition

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Let $(\mathbf{x}_j : j \in \mathbb{N})$ be a sequence of points of the domain X of the correspondence. We say that an infinite sequence $(\mathbf{y}_j : j \in \mathbb{N})$ in the codomain of the correspondence is a *companion sequence* for $(\mathbf{x}_j)$ with respect to the correspondence Φ if $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j.

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Then the continuity properties of $\Phi\colon X \rightrightarrows Y$ can be characterized in terms of companion sequences with respect to Φ as follows:—

- the correspondence $\Phi\colon X \rightrightarrows Y$ is compact-valued and upper hemicontinuous at a point $\mathbf{p}$ of X if and only if, given any infinite sequence $(\mathbf{x}_j : j \in \mathbb{N})$ in X converging to the point $\mathbf{p}$, and given any companion sequence $(\mathbf{y}_j : j \in \mathbb{N})$ in Y, that companion sequence has a subsequence that converges to a point of $\Phi(\mathbf{p})$ (Proposition 2.17);
- the correspondence $\Phi\colon X \rightrightarrows Y$ is lower hemicontinuous at a point $\mathbf{p}$ of X if and only if, given any infinite sequence $(\mathbf{x}_j : j \in \mathbb{N})$ in X converging to the point $\mathbf{p}$, and given any point $\mathbf{q}$ of $\Phi(\mathbf{p})$, there exists a companion sequence $(\mathbf{y}_j : j \in \mathbb{N})$ in Y converging to the point $\mathbf{q}$. (Proposition 2.19).

Proof of Theorem 2.23 using Companion Sequences
Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let $f\colon X \times Y \to \mathbb{R}$ be a continuous real-valued function on $X \times Y$, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is both upper and lower hemicontinuous and that also has the property that $\Phi(\mathbf{x})$ is non-empty and compact for all $\mathbf{x} \in X$. Let

$$m(\mathbf{x}) = \sup\{f(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in \Phi(\mathbf{x})\}$$

for all $\mathbf{x} \in X$, and let the correspondence $M\colon X \rightrightarrows Y$ be defined such that

$$M(\mathbf{x}) = \{\mathbf{y} \in \Phi(\mathbf{x}) : f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}$$

for all $\mathbf{x} \in X$. We must prove that $m\colon X \to \mathbb{R}$ is continuous, $M(\mathbf{x})$ is a non-empty compact subset of Y for all $\mathbf{x} \in X$, and the correspondence $M\colon X \rightrightarrows Y$ is upper hemicontinuous.

It follows from the continuity of $f\colon X \times Y \to \mathbb{R}$ that $M(\mathbf{x})$ is closed in $\Phi(\mathbf{x})$ for all $\mathbf{x} \in X$. It also follows from the Extreme Value Theorem (Theorem 1.17) that $M(\mathbf{x})$ is non-empty for all $\mathbf{x}$. Let $(\mathbf{x}_j, j \in \mathbb{N})$ be a sequence in X which converges to a point $\mathbf{p}$ of X, and let $(\mathbf{y}_j^* : j \in \mathbb{N})$ be a companion sequence of $(\mathbf{x}_j)$ with respect to the correspondence M. Then, for each positive integer j, $\mathbf{y}_j^* \in \Phi(\mathbf{x}_j)$ and

$$f(\mathbf{x}_j, \mathbf{y}_j^*) \geq f(\mathbf{x}_j, \mathbf{y})$$

for all $\mathbf{y} \in \Phi(\mathbf{x}_j)$. Now the correspondence Φ is compact-valued and upper hemicontinuous. It follows from Proposition 2.17 that there exists a subsequence of $(\mathbf{y}_j^* : j \in \mathbb{N})$ that converges to an element $\mathbf{q}$ of $\Phi(\mathbf{q})$. Let that subsequence be the sequence $(\mathbf{y}_{k_j}^* : j \in \mathbb{N})$ whose members are

$$\mathbf{y}_{k_1}^*, \mathbf{y}_{k_2}^*, \mathbf{y}_{k_3}^*, \ldots,$$

where $k_1 < k_2 < k_3 < \cdots$. Then $\mathbf{q} = \lim_{j \to +\infty} \mathbf{y}_{k_j}^*$.

We show that $\mathbf{q} \in M(\mathbf{p})$. Let $\mathbf{r} \in \Phi(\mathbf{p})$. The correspondence $\Phi\colon X \to Y$ is lower hemicontinuous. It follows that there exists a companion sequence $(\mathbf{z}_j : j \in N)$ to $(\mathbf{x}_j : j \in N)$ with respect to the correspondence Φ that converges to $\mathbf{r}$ (Proposition 2.19). Then

$$\lim_{j\to+\infty} \mathbf{y}^*_{k_j} = \mathbf{q} \quad \text{and} \quad \lim_{j\to+\infty} \mathbf{z}_{k_j} = \mathbf{r}.$$

It follows from the continuity of $f\colon X \times Y \to \mathbb{R}$ that

$$\lim_{j\to+\infty} f(\mathbf{x}_{k_j}, \mathbf{y}^*_{k_j}) = f(\mathbf{p}, \mathbf{q}) \quad \text{and} \quad \lim_{j\to+\infty} f(\mathbf{x}_{k_j}, \mathbf{z}_{k_j}) = f(\mathbf{p}, \mathbf{r}).$$

Now

$$f(\mathbf{x}_{k_j}, \mathbf{y}^*_{k_j}) \geq f(\mathbf{x}_{k_j}, \mathbf{z}_{k_j})$$

for all positive integers j, because $\mathbf{y}^*_{k_j} \in M(\mathbf{x}_{k_j})$. It follows that

$$f(\mathbf{p}, \mathbf{q}) = \lim_{j\to+\infty} f(\mathbf{x}_{k_j}, \mathbf{y}^*_{k_j}) \geq \lim_{j\to+\infty} f(\mathbf{x}_{k_j}, \mathbf{z}_{k_j}) = f(\mathbf{p}, \mathbf{r}).$$

Thus $f(\mathbf{p}, \mathbf{q}) \geq f(\mathbf{p}, \mathbf{r})$ for all $\mathbf{r} \in \Phi(\mathbf{p})$. It follows that $\mathbf{q} \in M(\mathbf{p})$.

We have now shown that, given any sequence $(\mathbf{x}_j : j \in \mathbb{R})$ in X converging to the point $\mathbf{p}$, and given any companion sequence $(\mathbf{y}_j^* : j \in \mathbb{R})$ with respect to the correspondence M, there exists a subsequence of $(\mathbf{y}_j^* : j \in \mathbb{R})$ that converges to a point of $M(\mathbf{x})$. It follows that the correspondence $M\colon X \to Y$ is compact-valued and upper hemicontinuous at the point $\mathbf{p}$ (Proposition 2.17).

It remains to show that the function $m\colon X \to \mathbb{R}$ is continuous at the point $\mathbf{p}$, where $m(\mathbf{x}) = f(\mathbf{x}, \mathbf{y}^*)$ for all $\mathbf{x} \in X$ and $\mathbf{y}^* \in M(\mathbf{x})$. Let $(\mathbf{x}_j : j \in \mathbb{R})$ be an infinite sequence converging to the point $\mathbf{p}$, and let $v_j = m(\mathbf{x}_j)$ for all positive integers j. Then there exists an infinite sequence Let $(\mathbf{y}_j^* : j \in \mathbb{R})$ in Y that is a companion sequence to $(\mathbf{x}_j)$ with respect to the correspondence M. Then $\mathbf{y}_j^* \in M(\mathbf{x}_j)$ and therefore $v_j = f(\mathbf{x}_j, \mathbf{y}_j^*)$ for all positive integers j. Now the correspondence $M\colon X \rightrightarrows Y$ has been shown to be compact-valued and upper hemicontinuous. There therefore exists a subsequence $(\mathbf{y}_{k_j}^* : j \in \mathbb{N})$ of $(\mathbf{y}_j)$ that converges to a point $\mathbf{q}$ of $M(\mathbf{p})$. It then follows from the continuity of the function $f\colon X \times Y \to \mathbb{R}$ that

$$\lim_{j\to+\infty} m(\mathbf{x}_{k_j}) = \lim_{j\to+\infty} v_{k_j} = \lim_{j\to+\infty} f(\mathbf{x}_{k_j}, \mathbf{y}_{k_j}^*) = f(\mathbf{p}, \mathbf{q}) = m(\mathbf{p}).$$

Now the result just proved can be applied with any subsequence of $(\mathbf{x}_j : j \in \mathbb{N})$ in place of the original sequence. It follows that *every subsequence* of of $(v_j : j \in \mathbb{R})$ itself has a subsequence that converges to $m(\mathbf{p})$.

Let some positive real number ε be given. Suppose that there did not exist any positive integer N with the property that $|v_j - m(\mathbf{p})| < \varepsilon$ whenever $j \geq N$. Then there would exist infinitely many positive integers j for which $|v_j - m(\mathbf{p})| \geq \varepsilon$. It follows that there would exist some subsequence

$$v_{l_1}, v_{l_2}, v_{l_3}, \ldots$$

of $v_1, v_2, v_3, \ldots$ with the property that $|v_{l_j} - m(\mathbf{p})| \geq \varepsilon$ for all positive integers j. This subsequence would not in turn contain any subsequences converging to the point $m(\mathbf{p})$.

But we have shown that every subsequence of $(v_j : j \in \mathbb{N})$ contains a subsequence converging to $m(\mathbf{p})$. It follows that there must exist some positive integer N with the property that $|v_j - m(\mathbf{p})| < \varepsilon$ whenever $j \geq N$. We conclude from this that $\lim_{j \to +\infty} m(\mathbf{x}_j) = m(\mathbf{p})$.

We have shown that if $(\mathbf{x}_j : j \in \mathbb{N})$ is an infinite sequence in X and if $\lim_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$ then $\lim_{j \to +\infty} m(\mathbf{x}_j) = m(\mathbf{p})$. It follows that the function $m \colon X \to \mathbb{R}$ is continuous at $\mathbf{p}$. This completes the proof of Berge's Maximum Theorem. ∎