MAU34804—Fixed Point Theorems and Economic Equilibria
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Appendix A: Proofs of Basic Results of Real Analysis

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A. Proofs of Basic Results of Real Analysis

Lemma 1.1

Let $\mathbf{p}$ be a point of $\mathbb{R}^n$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. Then a sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to $\mathbf{p}$ if and only if the ith components of the elements of this sequence converge to p_i for $i = 1, 2, \ldots, n$.

Proof of Lemma 1.1

Let $(\mathbf{x}_j)_i$ denote the ith components of $\mathbf{x}_j$. Then $|(\mathbf{x}_j)_i - p_i| \leq |\mathbf{x}_j - \mathbf{p}|$ for $i = 1, 2, \ldots, n$ and for all positive integers j. It follows directly from the definition of convergence that if $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ then $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$.

Conversely suppose that, for each integer i between 1 and n, $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$. Let $\varepsilon > 0$ be given. Then there exist positive integers $N_1, N_2, \ldots, N_n$ such that $|(\mathbf{x}_j)_i - p_i| < \varepsilon/\sqrt{n}$ whenever $j \geq N_i$. Let N be the maximum of $N_1, N_2, \ldots, N_n$. If $j \geq N$ then $j \geq N_i$ for $i = 1, 2, \ldots, n$, and therefore

$$|\mathbf{x}_j - \mathbf{p}|^2 = \sum_{i=1}^{n} ((\mathbf{x}_j)_i - p_i)^2 < n \left(\frac{\varepsilon}{\sqrt{n}}\right)^2 = \varepsilon^2.$$

Thus $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$, as required. ∎

The real number system satisfies the *Least Upper Bound Principle*:

> *Any set of real numbers which is non-empty and bounded above has a least upper bound.*

Let S be a set of real numbers which is non-empty and bounded above. The least upper bound, or *supremum*, of the set S is denoted by $\sup S$, and is characterized by the following two properties:

(i) $x \leq \sup S$ for all $x \in S$;

(ii) if u is a real number, and if $x \leq u$ for all $x \in S$, then $\sup S \leq u$.

A straightforward application of the Least Upper Bound guarantees that any set of real numbers that is non-empty and bounded below has a greatest lower bound, or *infimum*. The greatest lower bound of such a set S of real numbers is denoted by $\inf S$.

A.0.

An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to be *strictly increasing* if $x_{j+1} > x_j$ for all positive integers j, *strictly decreasing* if $x_{j+1} < x_j$ for all positive integers j, *non-decreasing* if $x_{j+1} \geq x_j$ for all positive integers j, *non-increasing* if $x_{j+1} \leq x_j$ for all positive integers j. A sequence satisfying any one of these conditions is said to be *monotonic*; thus a monotonic sequence is either non-decreasing or non-increasing.

Theorem A.1

Any non-decreasing sequence of real numbers that is bounded above is convergent. Similarly any non-increasing sequence of real numbers that is bounded below is convergent.

Proof

Let $x_1, x_2, x_3, \ldots$ be a non-decreasing sequence of real numbers that is bounded above. It follows from the Least Upper Bound Principle that there exists a least upper bound p for the set $\{x_j : j \in \mathbb{N}\}$. We claim that the sequence converges to p.

Let some strictly positive real number ε be given. We must show that there exists some positive integer N such that $|x_j - p| < \varepsilon$ whenever $j \geq N$. Now $p - \varepsilon$ is not an upper bound for the set $\{x_j : j \in \mathbb{N}\}$ (since p is the least upper bound), and therefore there must exist some positive integer N such that $x_N > p - \varepsilon$. But then $p - \varepsilon < x_j \leq p$ whenever $j \geq N$, since the sequence is non-decreasing and bounded above by p. Thus $|x_j - p| < \varepsilon$ whenever $j \geq N$. Therefore $x_j \to p$ as $j \to +\infty$, as required.

If the sequence $x_1, x_2, x_3, \ldots$ is non-increasing and bounded below then the sequence $-x_1, -x_2, -x_3, \ldots$ is non-decreasing and bounded above, and is therefore convergent. It follows that the sequence $x_1, x_2, x_3, \ldots$ is also convergent. ∎

Theorem A.2 (Bolzano-Weierstrass Theorem in One Dimension)

Every bounded sequence of real numbers has a convergent subsequence.

Proof

Let $a_1, a_2, a_3, \ldots$ be a bounded sequence of real numbers. We define a *peak index* to be a positive integer j with the property that $a_j \geq a_k$ for all positive integers k satisfying $k \geq j$. Thus a positive integer j is a peak index if and only if the jth member of the infinite sequence $a_1, a_2, a_3, \ldots$ is greater than or equal to all succeeding members of the sequence. Let S be the set of all peak indices. Then

$$S = \{j \in \mathbb{N} : a_j \geq a_k \text{ for all } k \geq j\}.$$

First let us suppose that the set S of peak indices is infinite. Arrange the elements of S in increasing order so that $S = \{j_1, j_2, j_3, j_4, \ldots\}$, where $j_1 < j_2 < j_3 < j_4 < \cdots$. It follows from the definition of peak indices that $a_{j_1} \geq a_{j_2} \geq a_{j_3} \geq a_{j_4} \geq \cdots$. Thus $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a non-increasing subsequence of the original sequence $a_1, a_2, a_3, \ldots$. This subsequence is bounded below (since the original sequence is bounded). It follows from Theorem A.1 that $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a convergent subsequence of the original sequence.

Now suppose that the set S of peak indices is finite. Choose a positive integer j_1 which is greater than every peak index. Then j_1 is not a peak index. Therefore there must exist some positive integer j_2 satisfying $j_2 > j_1$ such that $a_{j_2} > a_{j_1}$. Moreover j_2 is not a peak index (because j_2 is greater than j_1 and j_1 in turn is greater than every peak index). Therefore there must exist some positive integer j_3 satisfying $j_3 > j_2$ such that $a_{j_3} > a_{j_2}$. We can continue in this way to construct (by induction on j) a strictly increasing subsequence $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ of our original sequence. This increasing subsequence is bounded above (since the original sequence is bounded) and thus is convergent, by Theorem A.1. This completes the proof of the Bolzano-Weierstrass Theorem. ∎

Theorem 1.2

Every bounded sequence of points in a Euclidean space has a convergent subsequence.

Proof of Theorem 1.2

The theorem is proved by induction on the dimension n of the space $\mathbb{R}^n$ within which the points reside. When $n = 1$, the required result is the one-dimensional case of the Bolzano-Weierstrass Theorem, and the result has already been established in this case (see Theorem A.2).

When $n > 1$, the result is proved in dimension n asssuming the result in dimensions $n - 1$ and 1. Consequently the result is established successively in dimensions $2, 3, 4, \ldots$, and therefore is valid for bounded sequences in $\mathbb{R}^n$ for all positive integers n.

It has been shown that every bounded infinite sequence of real numbers has a convergent subsequence (Theorem A.2). Let n be an integer greater than one, and suppose, as an induction hypothesis, that, in cases where $n > 2$, all bounded sequences of points in $\mathbb{R}^{n-1}$ have convergent subsequences. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a bounded infinite sequence in $\mathbf{R}^n$ and, for each positive integer j, let $\mathbf{s}_j$ denote the point of $\mathbb{R}^{n-1}$ whose ith component is equal to the ith component $x_{j,i}$ of $\mathbf{x}_j$ for each integer i between 1 and $n - 1$.

Let some strictly positive real number ε be given. Now the infinite sequence

$$\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \ldots$$

of points of $\mathbb{R}^{n-1}$ is a bounded infinite sequence. In the case when $n = 2$ we can apply the one-dimensional Bolzano-Weierstrass Theorem (Theorem A.2) to conclude that this sequence of real numbers has a convergent subsequence. In cases where $n > 2$, we are supposing as our induction hypothesis that any bounded sequence in $\mathbb{R}^{n-1}$ has a convergent subsequence. Thus, assuming this induction hypothesis in cases where $n > 2$, we can conclude, in all cases with $n > 1$, that the bounded infinite sequence $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \ldots$ of points in $\mathbb{R}^{n-1}$ has a convergent subsequence.

Let that convergent subsequence be

$$\mathbf{s}_{m_1}, \mathbf{s}_{m_2}, \mathbf{s}_{m_3}, \ldots,$$

where $m_1, m_2, m_3, \ldots$ is a strictly increasing infinite sequence of positive integers, and let $\mathbf{q} = \lim\limits_{j \to +\infty} \mathbf{s}_{m_j}$. There then exists some positive integer L such that

$$|\mathbf{s}_{m_j} - \mathbf{q}| < \tfrac{1}{2}\varepsilon$$

for all positive integers j for which $m_j \geq L$. (Indeed the definition of convergence ensures the existence of a positive integer N that is large enough to ensure that $|\mathbf{s}_{m_j} - \mathbf{q}| < \frac{1}{2}\varepsilon$ whenever $j \geq N$. Taking $L = m_N$ then ensures that $j \geq N$ whenever $m_j \geq L$.)

Let t_j denote the nth component of the point $\mathbf{x}_j$ of $\mathbb{R}^n$ for each positive integer j. The one-dimensional Bolzano-Weierstrass Theorem ensures that the bounded infinite sequence

$$t_{m_1}, t_{m_2}, t_{m_3}, \ldots$$

of real numbers has a convergent subsequence. It follows that there is a strictly increasing infinite sequence $k_1, k_2, k_3, \ldots$ of positive integers, where each k_j is equal to one of the positive integers $m_1, m_2, m_3, \ldots$, such that the infinite sequence

$$t_{k_1}, t_{k_2}, t_{k_3}, \ldots$$

is convergent.

Let $r = \lim_{j \to +\infty} t_{k_j}$. There then exists some positive integer M such that $M \geq L$ and

$$|t_{k_j} - r| < \tfrac{1}{2}\varepsilon$$

for all positive integers j for which $k_j \geq M$. It follows that if $k_j \geq M$ then

$$|\mathbf{s}_{k_j} - \mathbf{q}| < \tfrac{1}{2}\varepsilon \quad \text{and} \quad |t_{k_j} - r| < \tfrac{1}{2}\varepsilon.$$

Now there is a point $\mathbf{p}$ of $\mathbb{R}^n$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, determined so that the ith components of the point $\mathbf{p}$ of $\mathbb{R}^n$ is equal to the ith component of the point $\mathbf{q}$ of $\mathbb{R}^{n-1}$ for each integer i between 1 and $n - 1$ and also the nth component of the point $\mathbf{p}$ is equal to the real number t.

Also it follows from the definition of the Euclidean norm that

$$|\mathbf{x}_{k_j} - \mathbf{p}|^2 = |\mathbf{s}_{k_j} - \mathbf{q}|^2 + |t_{k_j} - r|^2 < \tfrac{1}{2}\varepsilon^2$$

whenever $k_j \geq M$. But then $|\mathbf{x}_{k_j} - \mathbf{p}| < \varepsilon$ for all positive integers j for which $k_j \geq M$. It follows that $\lim_{j \to +\infty} \mathbf{x}_{k_j} = \mathbf{p}$. We conclude therefore that the bounded infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ does indeed have a convergent subsequence. This completes the proof of the Bolzano-Weierstrass Theorem in dimension n for all positive integers n. ∎

Lemma 1.3

Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any positive real number r, the open ball $B_X(\mathbf{p}, r)$ in X of radius r about $\mathbf{p}$ is open in X.

Proof of Lemma 1.3

Let $\mathbf{x}$ be an element of $B_X(\mathbf{p}, r)$. We must show that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. Let $\delta = r - |\mathbf{x} - \mathbf{p}|$. Then $\delta > 0$, since $|\mathbf{x} - \mathbf{p}| < r$. Moreover if $\mathbf{y} \in B_X(\mathbf{x}, \delta)$ then

$$|\mathbf{y} - \mathbf{p}| \leq |\mathbf{y} - \mathbf{x}| + |\mathbf{x} - \mathbf{p}| < \delta + |\mathbf{x} - \mathbf{p}| = r,$$

by the Triangle Inequality, and hence $\mathbf{y} \in B_X(\mathbf{p}, r)$. Thus $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. This shows that $B_X(\mathbf{p}, r)$ is an open set, as required. ∎

Proposition 1.4

Let X be a subset of $\mathbb{R}^n$. The collection of open sets in X has the following properties:—

(i) the empty set $\emptyset$ and the whole set X are both open in X;

(ii) the union of any collection of open sets in X is itself open in X;

(iii) the intersection of any *finite* collection of open sets in X is itself open in X.

Proof of Proposition 1.4

The empty set $\emptyset$ is an open set by convention. Moreover the definition of an open set is satisfied trivially by the whole set X. This proves (i).

Let $\mathcal{A}$ be any collection of open sets in X, and let U denote the union of all the open sets belonging to $\mathcal{A}$. We must show that U is itself open in X. Let $\mathbf{x} \in U$. Then $\mathbf{x} \in V$ for some set V belonging to the collection $\mathcal{A}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset V$. But $V \subset U$, and thus $B_X(\mathbf{x}, \delta) \subset U$. This shows that U is open in X. This proves (ii).

Finally let $V_1, V_2, V_3, \ldots, V_k$ be a *finite* collection of subsets of X that are open in X, and let V denote the intersection $V_1 \cap V_2 \cap \cdots \cap V_k$ of these sets. Let $\mathbf{x} \in V$. Now $\mathbf{x} \in V_j$ for $j = 1, 2, \ldots, k$, and therefore there exist strictly positive real numbers $\delta_1, \delta_2, \ldots, \delta_k$ such that $B_X(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. (This is where we need the fact that we are dealing with a finite collection of sets.) Now $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$, and thus $B_X(\mathbf{x}, \delta) \subset V$. Thus the intersection V of the sets $V_1, V_2, \ldots, V_k$ is itself open in X. This proves (iii). ∎

Proposition 1.5

Let X be a subset of $\mathbb{R}^n$, and let U be a subset of X. Then U is open in X if and only if there exists some open set V in $\mathbb{R}^n$ for which $U = V \cap X$.

Proof of Proposition 1.5

First suppose that $U = V \cap X$ for some open set V in $\mathbb{R}^n$. Let $\mathbf{u} \in U$. Then the definition of open sets in $\mathbb{R}^n$ ensures that there exists some positive real number δ such that

$$\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{u}| < \delta\} \subset V.$$

Then

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta\} \subset U.$$

This shows that U is open in X.

Conversely suppose that the subset U of X is open in X. For each point $\mathbf{u}$ of U there exists some positive real number $\delta_{\mathbf{u}}$ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta_{\mathbf{u}}\} \subset U.$$

For each $\mathbf{u} \in U$, let $B(\mathbf{u}, \delta_{\mathbf{u}})$ denote the open ball in $\mathbb{R}^n$ of radius $\delta_{\mathbf{u}}$ about the point $\mathbf{u}$, so that

$$B(\mathbf{u}, \delta_{\mathbf{u}}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{u}| < \delta_{\mathbf{u}}\}$$

for all $\mathbf{u} \in U$, and let V be the union of all the open balls $B(\mathbf{u}, \delta_{\mathbf{u}})$ as $\mathbf{u}$ ranges over all the points of U. Then V is an open set in $\mathbb{R}^n$. Indeed every open ball in $\mathbb{R}^n$ is an open set (Lemma 1.3), and any union of open sets in $\mathbb{R}^n$ is itself an open set (Proposition 1.4). The set V is a union of open balls. It is therefore a union of open sets, and so must itself be an open set.

Now $B(\mathbf{u}, \delta_\mathbf{u}) \cap X \subset U$. for all $\mathbf{u} \in U$. Also every point of V belongs to $B(\mathbf{u}, \delta_\mathbf{u})$ for at least one point $\mathbf{u}$ of U. It follows that $V \cap X \subset U$. But $\mathbf{u} \in B(\mathbf{u}, \delta_\mathbf{u})$ and $B(\mathbf{u}, \delta_\mathbf{u}) \subset V$ for all $\mathbf{u} \in U$, and therefore $U \subset V$, and thus $U \subset V \cap X$. It follows that $U = V \cap X$, as required. ∎

Lemma 1.6

A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to a point $\mathbf{p}$ if and only if, given any open set U which contains $\mathbf{p}$, there exists some positive integer N such that $\mathbf{x}_j \in U$ for all j satisfying $j \geq N$.

Proof of Lemma 1.6

Suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ has the property that, given any open set U which contains $\mathbf{p}$, there exists some positive integer N such that $\mathbf{x}_j \in U$ whenever $j \geq N$. Let $\varepsilon > 0$ be given. The open ball $B(\mathbf{p}, \varepsilon)$ of radius ε about $\mathbf{p}$ is an open set by Lemma 1.3. Therefore there exists some positive integer N such that $\mathbf{x}_j \in B(\mathbf{p}, \varepsilon)$ whenever $j \geq N$. Thus $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$. This shows that the sequence converges to $\mathbf{p}$.

Conversely, suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Let U be an open set which contains $\mathbf{p}$. Then there exists some $\varepsilon > 0$ such that the open ball $B(\mathbf{p}, \varepsilon)$ of radius ε about $\mathbf{p}$ is a subset of U. Thus there exists some $\varepsilon > 0$ such that U contains all points $\mathbf{x}$ of X that satisfy $|\mathbf{x} - \mathbf{p}| < \varepsilon$. But there exists some positive integer N with the property that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$, since the sequence converges to $\mathbf{p}$. Therefore $\mathbf{x}_j \in U$ whenever $j \geq N$, as required. ∎

Lemma 1.8

Let X be a subset of $\mathbb{R}^n$, and let F be a subset of X which is closed in X. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of F which converges to a point $\mathbf{p}$ of X. Then $\mathbf{p} \in F$.

Proof of Lemma 1.8

The complement $X \setminus F$ of F in X is open, since F is closed. Suppose that $\mathbf{p}$ were a point belonging to $X \setminus F$. It would then follow from Lemma 1.6 that $\mathbf{x}_j \in X \setminus F$ for all values of j greater than some positive integer N, contradicting the fact that $\mathbf{x}_j \in F$ for all j. This contradiction shows that $\mathbf{p}$ must belong to F, as required. ∎

Lemma 1.9

Let X, Y and Z be subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively, and let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions satisfying $f(X) \subset Y$. Suppose that f is continuous at some point $\mathbf{p}$ of X and that g is continuous at $f(\mathbf{p})$. Then the composition function $g \circ f\colon X \to Z$ is continuous at $\mathbf{p}$.

Proof of Lemma 1.9

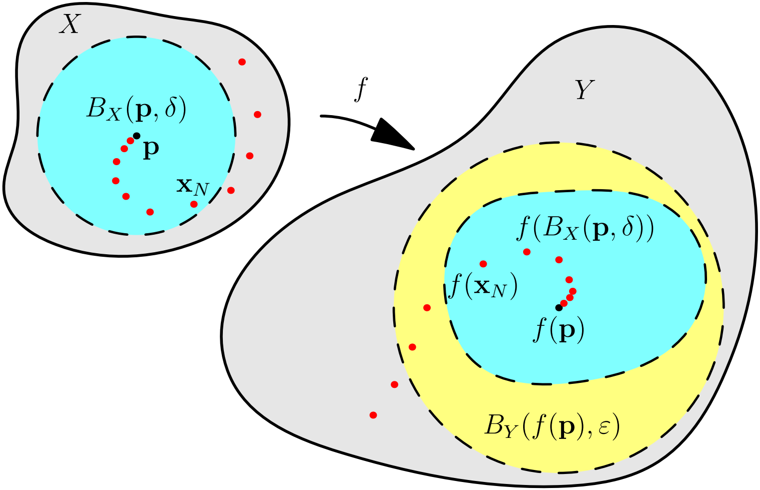
Let $\varepsilon > 0$ be given. Then there exists some $\eta > 0$ such that $|g(\mathbf{y}) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - f(\mathbf{p})| < \eta$. But then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \eta$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $|g(f(\mathbf{x})) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus $g \circ f$ is continuous at $\mathbf{p}$, as required. ∎

Lemma 1.10

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f\colon X \to Y$ be a continuous function from X to Y. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of X which converges to some point $\mathbf{p}$ of X. Then the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$.

Proof of Lemma 1.10

Let $\varepsilon > 0$ be given. Then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, since the function f is continuous at $\mathbf{p}$.


X
$B_X(\mathbf{p}, \delta)$
$\mathbf{p}$
$\mathbf{x}_N$
f
Y
$f(B_X(\mathbf{p}, \delta))$
$f(\mathbf{x}_N)$
$f(\mathbf{p})$
$B_Y(f(\mathbf{p}), \varepsilon)$

Also there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \delta$ whenever $j \geq N$, since the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Thus if $j \geq N$ then $|f(\mathbf{x}_j) - f(\mathbf{p})| < \varepsilon$. Thus the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$, as required. $\blacksquare$

Proposition 1.9

Let X, Y and Z be subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively, and let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions satisfying $f(X) \subset Y$. Suppose that f is continuous at some point $\mathbf{p}$ of X and that g is continuous at $f(\mathbf{p})$. Then the composition function $g \circ f\colon X \to Z$ is continuous at $\mathbf{p}$.

Proof of Proposition 1.9

Note that the ith component f_i of f is given by $f_i = \pi_i \circ f$, where $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$ is the continuous function which maps $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ onto its ith coordinate y_i. Now any composition of continuous functions is continuous, by Lemma 1.9. Thus if f is continuous at $\mathbf{p}$, then so are the components of f.

Conversely suppose that the components of f are continuous at $\mathbf{p} \in X$. Let $\varepsilon > 0$ be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $|f_i(\mathbf{x}) - f_i(\mathbf{p})| < \varepsilon/\sqrt{n}$ for $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_i$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - f(\mathbf{p})|^2 = \sum_{i=1}^{n} |f_i(\mathbf{x}) - f_i(\mathbf{p})|^2 < \varepsilon^2,$$

and hence $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$. Thus the function f is continuous at $\mathbf{p}$, as required. ∎

Proposition 1.12

Let X be a subset of $\mathbb{R}^n$, and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be continuous functions from X to $\mathbb{R}$. Then the functions $f + g$, $f - g$ and $f \cdot g$ are continuous. If in addition $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ then the quotient function f/g is continuous.

Proof of Proposition 1.12

First we prove that $f + g$ is continuous. Let some strictly positive real number ε be given. Then there exist strictly positive real numbers δ_1 and δ_2 such that $|f(\mathbf{x}) - f(\mathbf{p})| < \frac{1}{2}\varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_1$ and $|g(\mathbf{x}) - g(\mathbf{p})| < \frac{1}{2}\varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then

$$|(f+g)(\mathbf{x}) - (f+g)(\mathbf{p})| \leq |f(\mathbf{x}) - f(\mathbf{p})| + |g(\mathbf{x}) - g(\mathbf{p})| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

Thus the function $f + g$ is continuous at $\mathbf{p}$.

The function $-g$ is also continuous at $\mathbf{p}$, and $f - g = f + (-g)$. It follows that the function $f - g$ is continuous at $\mathbf{p}$.

Next we prove that $f \cdot g$ is continuous. Let some strictly positive real number ε be given. There exists some strictly positive real number δ_0 such that $|f(\mathbf{x}) - f(\mathbf{p})| < 1$ and $|g(\mathbf{x}) - g(\mathbf{p})| < 1$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_0$. Let M be the maximum of $|f(\mathbf{p})| + 1$ and $|g(\mathbf{p})| + 1$. Then $|f(\mathbf{x})| < M$ and $|g(\mathbf{x})| < M$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_0$. Now

$$f(\mathbf{x})g(\mathbf{x}) - f(\mathbf{p})g(\mathbf{p}) = (f(\mathbf{x}) - f(\mathbf{p}))g(\mathbf{x}) + f(\mathbf{p})(g(\mathbf{x}) - g(\mathbf{p})),$$

and thus

$$|f(\mathbf{x})g(\mathbf{x}) - f(\mathbf{p})g(\mathbf{p})| \leq M\Big(|f(\mathbf{x}) - f(\mathbf{p})| + |g(\mathbf{x}) - g(\mathbf{p})|\Big)$$

whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_0$.

There then exists some strictly positive real number δ, where $0 < \delta \leq \delta_0$, such that

$$|f(\mathbf{x}) - f(\mathbf{p})| < \frac{\varepsilon}{2M} \quad \text{and} \quad |g(\mathbf{x}) - g(\mathbf{p})| < \frac{\varepsilon}{2M}$$

whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$. But then

$$|f(\mathbf{x})g(\mathbf{x}) - f(\mathbf{p})g(\mathbf{p})| < \varepsilon$$

whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$. Thus the function $f \cdot g$ is continuous at $\mathbf{p}$.

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$. Note that $1/g = r \circ g$, where $r\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is the reciprocal function, defined by $r(t) = 1/t$. Now the reciprocal function r is continuous. Thus the function $1/g$ is a composition of continuous functions and is thus continuous. But then, using the fact that a product of continuous real-valued functions is continuous, we deduce that f/g is continuous. $\blacksquare$

Lemma 1.13

Let X be a subset of $\mathbb{R}^m$, let $f: X \to \mathbb{R}^n$ be a continuous function mapping X into $\mathbb{R}^n$, and let $|f|: X \to \mathbb{R}$ be defined such that $|f|(\mathbf{x}) = |f(\mathbf{x})|$ for all $\mathbf{x} \in X$. Then the real-valued function $|f|$ is continuous on X.

Proof of Lemma 1.13

Let $\mathbf{x}$ and $\mathbf{p}$ be elements of X. Then

$$|f(\mathbf{x})| = |(f(\mathbf{x}) - f(\mathbf{p})) + f(\mathbf{p})| \leq |f(\mathbf{x}) - f(\mathbf{p})| + |f(\mathbf{p})|$$

and

$$|f(\mathbf{p})| = |(f(\mathbf{p}) - f(\mathbf{x})) + f(\mathbf{x})| \leq |f(\mathbf{x}) - f(\mathbf{p})| + |f(\mathbf{x})|,$$

and therefore

$$\Big| |f(\mathbf{x})| - |f(\mathbf{p})| \Big| \leq |f(\mathbf{x}) - f(\mathbf{p})|.$$

The result now follows from the definition of continuity, using the above inequality. Indeed let $\mathbf{p}$ be a point of X, and let some positive real number ε be given. Then there exists a positive real number δ small enough to ensure that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then

$$\Big| |f(\mathbf{x})| - |f(\mathbf{p})| \Big| \leq |f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$$

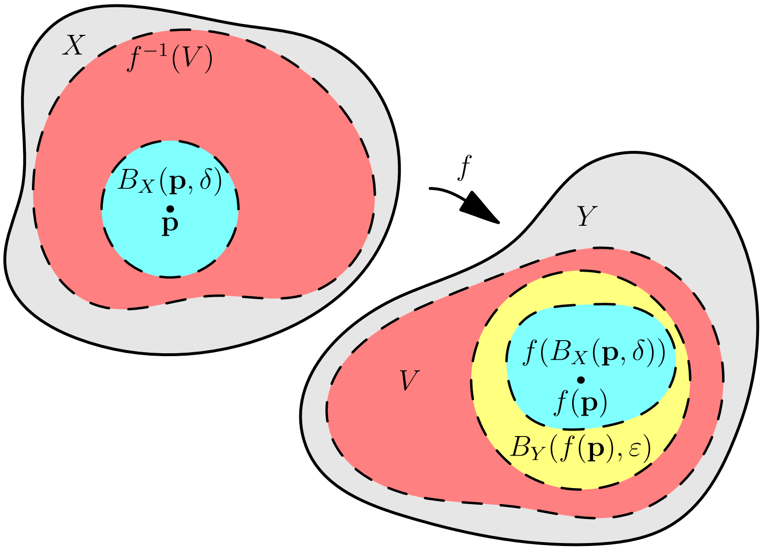
for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus the function $|f|$ is continuous, as required. ∎

Proposition 1.14

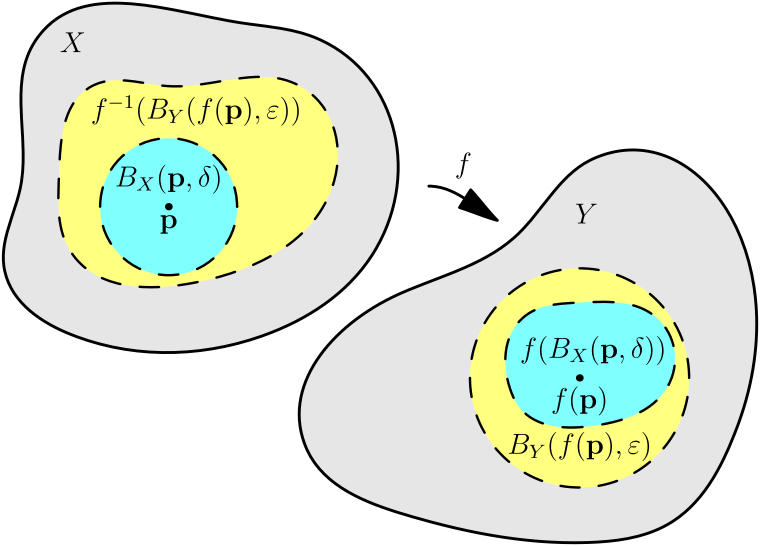
Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f\colon X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(V)$ is open in X for every open subset V of Y.

Proof of Proposition 1.14

Suppose that $f\colon X \to Y$ is continuous. Let V be an open set in Y. We must show that $f^{-1}(V)$ is open in X. Let $\mathbf{p} \in f^{-1}(V)$. Then $f(\mathbf{p}) \in V$. But V is open, hence there exists some $\varepsilon > 0$ with the property that $B_Y(f(\mathbf{p}), \varepsilon) \subset V$. But f is continuous at $\mathbf{p}$. Therefore there exists some $\delta > 0$ such that f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$ (see the remarks above). Thus $f(\mathbf{x}) \in V$ for all $\mathbf{x} \in B_X(\mathbf{p}, \delta)$, showing that $B_X(\mathbf{p}, \delta) \subset f^{-1}(V)$. This shows that $f^{-1}(V)$ is open in X for every open set V in Y.


X
$f^{-1}(V)$
$B_X(\mathbf{p}, \delta)$
$\mathbf{p}$
f
Y
V
$f(B_X(\mathbf{p}, \delta))$
$f(\mathbf{p})$
$B_Y(f(\mathbf{p}), \varepsilon)$

Conversely suppose that $f\colon X \to Y$ is a function with the property that $f^{-1}(V)$ is open in X for every open set V in Y. Let $\mathbf{p} \in X$. We must show that f is continuous at $\mathbf{p}$.


X
$f^{-1}(B_Y(f(\mathbf{p}), \varepsilon))$
$B_X(\mathbf{p}, \delta)$
$\mathbf{p}$
f
Y
$f(B_X(\mathbf{p}, \delta))$
$f(\mathbf{p})$
$B_Y(f(\mathbf{p}), \varepsilon)$

Let $\varepsilon > 0$ be given. Then $B_Y(f(\mathbf{p}), \varepsilon)$ is an open set in Y, by Lemma 1.3, hence $f^{-1}(B_Y(f(\mathbf{p}), \varepsilon))$ is an open set in X which contains $\mathbf{p}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{p}, \delta) \subset f^{-1}(B_Y(f(\mathbf{p}), \varepsilon))$. Thus, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$. We conclude that f is continuous at $\mathbf{p}$, as required. ∎

Corollary 1.15

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi\colon X \to Y$ be a continuous function from X to Y. Then $\varphi^{-1}(F)$ is closed in X for every subset F of Y that is closed in Y.

Proof of Corollary 1.15

Let F be a subset of Y that is closed in Y, and let let $V = Y \setminus F$. Then V is open in Y. It follows from Proposition 1.14 that $\varphi^{-1}(V)$ is open in X. But

$$\varphi^{-1}(V) = \varphi^{-1}(Y \setminus F) = X \setminus \varphi^{-1}(F).$$

Indeed let $\mathbf{x} \in X$. Then

$$\begin{aligned} & \mathbf{x} \in \varphi^{-1}(V) \\ \iff & \mathbf{x} \in \varphi^{-1}(Y \setminus F) \\ \iff & \varphi(\mathbf{x}) \in Y \setminus F \\ \iff & \varphi(\mathbf{x}) \notin F \\ \iff & \mathbf{x} \notin \varphi^{-1}(F) \\ \iff & \mathbf{x} \in X \setminus \varphi^{-1}(F). \end{aligned}$$

It follows that the complement $X \setminus \varphi^{-1}(F)$ of $\varphi^{-1}(F)$ in X is open in X, and therefore $\varphi^{-1}(F)$ itself is closed in X, as required. ∎

Lemma 1.16

Let X be a closed subset of n-dimensional Euclidean space $\mathbb{R}^n$. Then a subset of X is closed in X if and only if it is closed in $\mathbb{R}^n$.

Proof of Lemma 1.16

Let F be a subset of X. Then F is closed in X if and only if, given any point $\mathbf{p}$ of X for which $\mathbf{p} \notin F$, there exists some strictly positive real number δ such that there is no point of F whose distance from the point $\mathbf{p}$ is less than δ. It follows easily from this that is F is closed in $\mathbb{R}^n$ then F is closed in X.

Conversely suppose that F is closed in X, where X itself is closed in $\mathbb{R}^n$. Let $\mathbf{p}$ be a point of $\mathbb{R}^n$ that satisfies $\mathbf{p} \notin F$. Then either $\mathbf{p} \in X$ or $\mathbf{p} \notin X$.

Suppose that $\mathbf{p} \in X$. Then there exists some strictly positive real number δ such that there is no point of F whose distance from the point $\mathbf{p}$ is less than δ.

Otherwise $\mathbf{p} \notin X$. Then there exists some strictly positive real number δ such that there is no point of X whose distance from the point $\mathbf{p}$ is less than δ, because X is closed in $\mathbb{R}^n$. But $F \subset X$. It follows that there is no point of F whose distance from the point $\mathbf{p}$ is less than δ. We conclude that the set F is closed in $\mathbb{R}^n$, as required. ∎

Lemma A.3

Let X be a closed bounded set in $\mathbb{R}^m$, and let $f\colon X \to \mathbb{R}$ be a continuous real-valued function defined on X. Suppose that the set of values of the function f on X is bounded below. Then there exists a point $\mathbf{u}$ of X such that $f(\mathbf{u}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in X$.

Proof

Let

$$m = \inf\{f(\mathbf{x}) : \mathbf{x} \in X\}.$$

Then there exists an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ in X such that

$$f(\mathbf{x}_j) < m + \frac{1}{j}$$

for all positive integers j. It follows from the multidimensional Bolzano-Weierstrass Theorem (Theorem 1.2) that this sequence has a subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$ which converges to some point $\mathbf{u}$ of $\mathbb{R}^m$.

Now the point $\mathbf{u}$ belongs to X because X is closed (see Lemma 1.8). Also

$$m \leq f(\mathbf{x}_{k_j}) < m + \frac{1}{k_j}$$

for all positive integers j. It follows that $\lim_{j \to +\infty} f(\mathbf{x}_{k_j}) = m$. Consequently

$$f(\mathbf{u}) = f\left(\lim_{j \to +\infty} \mathbf{x}_{k_j}\right) = \lim_{j \to +\infty} f(\mathbf{x}_{k_j}) = m$$

(see Proposition 1.10). It follows therefore that $f(\mathbf{x}) \geq f(\mathbf{u})$ for all $\mathbf{x} \in X$, Thus the function f attains a minimum value at the point $\mathbf{u}$ of X, which is what we were required to prove. ∎

Lemma A.4

Let X be a closed bounded set in $\mathbb{R}^m$, and let $\varphi\colon X \to \mathbb{R}^n$ be a continuous function mapping X into $\mathbb{R}^n$. Then there exists a positive real number M with the property that $|\varphi(\mathbf{x})| \leq M$ for all $\mathbf{x} \in X$.

Proof

Let $g\colon X \to \mathbb{R}$ be defined such that

$$g(\mathbf{x}) = \frac{1}{1 + |\varphi(\mathbf{x})|}$$

for all $\mathbf{x} \in X$. Now the real-valued function mapping each $\mathbf{x} \in X$ to $|\varphi(\mathbf{x})|$ is continuous (see Lemma 1.13) and quotients of continuous real-valued functions are continuous where they are defined (see Lemma 1.12). It follows that the function $g\colon X \to \mathbb{R}$ is continuous. Moreover the values of this function are bounded below by zero. Consequently there exists some point $\mathbf{w}$ of X with the property that $g(\mathbf{x}) \geq g(\mathbf{w})$ for all $\mathbf{x} \in X$ (see Lemma A.3). Let $M = |\varphi(\mathbf{w})|$. Then $|\varphi(\mathbf{x})| \leq M$ for all $\mathbf{x} \in X$. The result follows. ∎

Theorem 1.17

Let X be a closed bounded set in $\mathbb{R}^m$, and let $f\colon X \to \mathbb{R}$ be a continuous real-valued function defined on X. Then there exist points $\mathbf{u}$ and $\mathbf{v}$ of X such that $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$.

Proof of Theorem 1.17

It follows from Lemma A.4 that there exists positive real number M with the property that $-M \leq f(\mathbf{x}) \leq M$ for all $\mathbf{x} \in X$. Thus the set of values of the function f is bounded above and below on X. Consequently there exist points $\mathbf{u}$ and $\mathbf{v}$ where the functions f and $-f$ respectively attain their minimum values on the set X (see Lemma A.3). The result follows. ∎