MAU34802—The Theory of Linear Programming
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Section 5: Duality and Convexity

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1. Mathematical Programming Problems

1.1. A Furniture Retailing Problem

A retail business is planning to devote a number of retail outlets to the sale of armchairs and sofas.
The retail prices of armchairs and sofas are determined by fierce competition in the furniture retailing business. Armchairs sell for €700 and sofas sell for €1000.
However

- the amount of floor space (and warehouse space) available for stocking the sofas and armchairs is limited;
- the amount of capital available for purchasing the initial stock of sofas and armchairs is limited;
- market research shows that the ratio of armchairs to sofas in stores should neither be too low nor too high.

Specifically:

- there are 1000 square metres of floor space available for stocking the initial purchase of sofas and armchairs;
- each armchair takes up 1 square metre;
- each sofa takes up 2 square metres;
- the amount of capital available for purchasing the initial stock of armchairs and sofas is €351,000;
- the wholesale price of an armchair is €400;
- the wholesale price of a sofa is €600;
- market research shows that between 4 and 9 armchairs should be in stock for each 3 sofas in stock.

We suppose that the retail outlets are stocked with x armchairs and y sofas.
The armchairs (taking up 1 sq. metre each) and the sofas (taking up 2 sq. metres each) cannot altogether take up more than 1000 sq. metres of floor space. Therefore

$$x + 2y \leq 1000 \quad \text{(Floor space constraint).}$$

The cost of stocking the retail outlets with armchairs (costing €400 each) and sofas (costing €600 each) cannot exceed the available capital of €351000. Therefore

$$4x + 6y \leq 3510 \quad \text{(Capital constraint).}$$

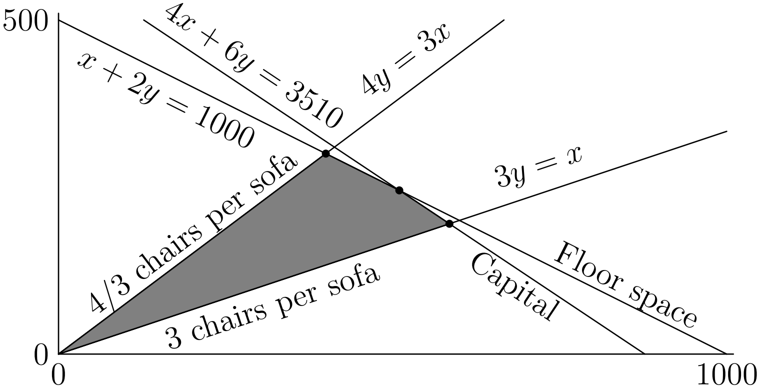
Consumer research indicates that x and y should satisfy

$$4y \leq 3x \leq 9y \quad \text{(Armchair/Sofa ratio).}$$

An ordered pair (x, y) of real numbers is said to specify a *feasible solution* to the linear programming problem if this pair of values meets all the relevant constraints.
An ordered pair (x, y) constitutes a feasible solution to the the Furniture Retailing problem if and only if all the following constraints are satisfied:

$$\begin{aligned} x - 3y &\leq 0; \\ 4y - 3x &\leq 0; \\ x + 2y &\leq 1000; \\ 4x + 6y &\leq 3510; \\ x &\geq 0; \\ y &\geq 0; \end{aligned}$$

The feasible region for the Furniture Retailing problem is depicted below:



We identify the *vertices* (or *corners*) of the feasible region for the Furniture Retailing problem. There are four of these:

- there is a vertex at $(0, 0)$;
- there is a vertex at $(400, 300)$ where the line $4y = 3x$ intersects the line $x + 2y = 1000$;
- there is a vertex at $(510, 245)$ where the line $x + 2y = 1000$ intersects the line $4x + 6y = 3510$;
- there is a vertex at $(585, 195)$ where the line $3y = x$ intersects the line $4x + 6y = 3510$.

These vertices are identified by inspection of the graph that depicts the constraints that determine the feasible region.

The furniture retail business obviously wants to confirm that the business will make a profit, and will wish to determine how many armchairs and sofas to purchase from the wholesaler to maximize expected profit.

There are fixed costs for wages, rental etc., and we assume that these are independent of the number of armchairs and sofas sold.

The *gross margin* on the sale of an armchair or sofa is the difference between the wholesale and retail prices of that item of furniture.

Armchairs cost €400 wholesale and sell for €700, and thus provide a gross margin of €300.

Sofas cost €600 wholesale and sell for €1000, and thus provide a gross margin of €400.

In a typical linear programming problem, one wishes to determine not merely *feasible* solutions to the problem. One wishes to determine an *optimal* solution that maximizes some *objective function* amongst all feasible solutions to the problem.
The objective function for the Furniture Retailing problem is the gross profit that would accrue from selling the furniture in stock. This gross profit is the difference between the cost of purchasing the furniture from the wholesaler and the return from selling that furniture.
This objective function is thus $f(x, y)$, where

$$f(x, y) = 300x + 400y.$$

We should determine the maximum value of this function on the feasible region.

Because the objective function $f(x, y) = 300x + 400y$ is linear in x and y, its maximum value on the feasable region must be achieved at one of the vertices of the region.
Clearly this function is not maximized at the origin $(0, 0)$!
Now the remaining vertices of the feasible region are at $(400, 300)$, $(510, 245)$ and $(585, 195)$, and

$$\begin{aligned} f(400, 300) &= 240,000, \\ f(510, 245) &= 251,000, \\ f(585, 195) &= 253,500. \end{aligned}$$

It follows that the objective function is maximized at $(585, 195)$.
The furniture retail business should therefore use up the available capital, stocking 3 armchairs for every sofa, despite the fact that this will not utilize the full amount of floor space available.

A linear programming problem may be presented as follows:

given real numbers c_i, $A_{i,j}$ *and* b_j *for* $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$,

find real numbers $x_1, x_2, \ldots, x_n$ *so as to*

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

subject to constraints

$x_j \geq 0$ *for* $j = 1, 2, \ldots, n$, *and*

$A_{i,1}x_1 + A_{i,2}x_2 + \cdots + A_{i,n}x_n \leq b_i$ *for* $i = 1, 2, \ldots, m$.

The furniture retailing problem may be presented in this form with $n = 2$, $m = 4$,

$$(c_1, c_2) = (300, 400),$$

$$A = \begin{pmatrix} 1 & -3 \\ -3 & 4 \\ 1 & 2 \\ 4 & 6 \end{pmatrix}, \quad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1000 \\ 3510 \end{pmatrix}.$$

Here A represents the $m \times n$ whose coefficient in the ith row and jth column is $A_{i,j}$.

Linear programming problems may be presented in matrix form. We adopt the following notational conventions with regard to transposes, row and column vectors and vector inequalities:—

- vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$ are represented as column vectors;
- we denote by M^{T} the $n \times m$ matrix that is the transpose of an $m \times n$ matrix M;
- in particular, given $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$, where b and c are represented as column vectors, we denote by b^{T} and c^{T} the corresponding row vectors obtained on transposing the column vectors representing b and c;
- given vectors u and v in $\mathbb{R}^n$ for some positive integer n, we write $u \leq v$ (and $v \geq u$) if and only if $u_j \leq v_j$ for $j = 1, 2, \ldots, n$.

Linear programming problems formulated as above may be presented in matrix notation as follows:—

Given an $m \times n$ matrix A with real coefficients,

and given column vectors $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$,

find $x \in \mathbb{R}^n$ so as to

maximize $c^{\mathrm{T}} x$

subject to constraints $Ax \leq b$ and $x \geq 0$.

1.2. A Transportation Problem concerning Dairy Produce

The *Transportation Problem* is a well-known problem and important example of a linear programming problem. Discussions of the general problem are to be found in textbooks in the following places:—

- Chapter 8 of *Linear Programming: 1 Introduction*, by George B. Danzig and Mukund N. Thapa (Springer, 1997);
- Section 18 of Chapter I of *Methods of Mathematical Economics* by Joel N. Franklin (SIAM 2002).

We discuss an example of the Transportation Problem of Linear Programming, as it might be applied to optimize transportation costs in the dairy industry.

A food business has milk-processing plants located in various towns in a small country. We shall refer to these plants as *dairies*. Raw milk is supplied by numerous farmers with farms located throughout that country, and is transported by milk tanker from the farms to the dairies. The problem is to determine the catchment areas of the dairies so as to minimize transport costs.

We suppose that there are m farms, labelled by integers from 1 to m that supply milk to n dairies, labelled by integers from 1 to n. Suppose that, in a given year, the ith farm has the capacity to produce and supply a s_i litres of milk for $i = 1, 2, \ldots, n$, and that the jth dairy needs to receive at least d_j litres of milk for $j = 1, 2, \ldots, n$ to satisfy the business obligations.

The quantity $\sum_{i=1}^{m} s_i$ then represents that *total supply* of milk, and the quantity $\sum_{j=1}^{n} d_j$ represents the *total demand* for milk.

We suppose that $x_{i,j}$ litres of milk are to be transported from the ith farm to the jth dairy, and that $c_{i,j}$ represents the cost per litre of transporting this milk.
Then the total cost of transporting milk from the farms to the dairies is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}.$$

The quantities $x_{i,j}$ of milk to be transported from the farms to the dairies should then be determined for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ so as to minimize the total cost of transporting milk.

However the ith farm can supply no more than s_i litres of milk in a given year, and that jth dairy requires at least d_j litres of milk in that year. It follows that the quantities $x_{i,j}$ of milk to be transported between farms and dairy are constrained by the requirements that

$$\sum_{j=1}^{n} x_{i,j} \leq s_i \quad \text{for } i = 1, 2, \ldots, m$$

and

$$\sum_{i=1}^{m} x_{i,j} \geq d_j \quad \text{for } j = 1, 2, \ldots, n.$$

Suppose that the requirements of supply and demand are satisfied. Then

$$\sum_{j=1}^{n} d_j \leq \sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j} \leq \sum_{i=1}^{m} s_i.$$

Thus the total supply must equal or exceed the total demand.

If it is the case that $\sum\limits_{j=1}^{n} x_{i,j} < s_i$ for at least one value of i then $\sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} x_{i,j} < \sum_{i=1}^{m} s_i$. Similarly if it is the case that $\sum\limits_{i=1}^{m} x_{i,j} > d_j$ for at least one value of j then $\sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} x_{i,j} > \sum_{j=1}^{n} d_j$.

It follows that if total supply equals total demand, so that

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j,$$

then

$$\sum_{j=1}^{n} x_{i,j} = s_i \quad \text{for } i = 1, 2, \ldots, m$$

and

$$\sum_{i=1}^{m} x_{i,j} = d_j \quad \text{for } j = 1, 2, \ldots, n.$$

The following report, published in 2006, describes a study of milk transport costs in the Irish dairy industry:

Quinlan C., Enright P., Keane M., O'Connor D. 2006. *The Milk Transport Cost Implications of Alternative Dairy Factory Location.* Agribusiness Discussion Paper No. 47. Dept of Food Business and Development. University College, Cork.

The report is available at the following URL

http://www.ucc.ie/en/media/academic/foodbusinessanddevelopment/paper47.pdf

The problem was investigated using commercial software that implements standard linear programming algorithms for the solution of forms of the Transportation Problem.

The description of the methodology used in the study begins as follows:

> A transportation model based on linear programming was developed and applied the Irish dairy industry to meet the study objectives. In such transportation models, transportation costs are treated as a direct linear function of the number of units shipped. The major assumptions are:
>
> 1. The items to be shipped are homogenous (i.e., they are the same regardless of their source or destination.
> 2. The shipping cost per unit is the same regardless of the number of units shipped.
> 3. There is only one route or mode of transportation being used between each source and each destination, Stevenson, (1993).

Sources and Destinations

In 2004 there were about 25,000 dairy farmers in the Irish Republic. Hence identifying the location and size of each individual dairy farm as sources for the transportation model was beyond available resources. An alternative approach based on rural districts was adopted. There are 156 rural districts in the state and data for dairy cow numbers by rural district from the most recent livestock census was available from the Central Statistics Office (CSO). These data were converted to milk equivalent terms using average milk yield estimates.

Typical seasonal milk supply patterns were also assumed. In this way an estimate of milk availability throughout the year by rural district was derived and this could then be further converted to milk tanker loads, depending on milk tanker size.

The following is quoted from the conclusions of that report:—

> A major report on the strategic development of the Irish dairy-processing sector proposed processing plant rationalization, 'Strategic Development Plan for the Irish Dairy Processing Sector' Prospectus, (2003). It was recommended that in the long term the number of plants processing butter, milk powder, casein and whey products in Ireland should be reduced to create four major sites for these products, with a limited number of additional sites for cheese and other products. It was estimated that savings from processing plant economies of scale would amount to €20m per annum, Prospectus (2003).

However, there is an inverse relationship between milk transport costs and plant size. Thus the optimum organisation of the industry involves a balancing of decreasing average plant costs against the increasing transport costs. In this analysis, the assumed current industry structure of 23 plants was reduced in a transportation modelling exercise firstly to 12 plants, then 9 plants and finally 6 plants and the increase in total annual milk transport costs for each alternative was calculated. Both a 'good' location and a 'poor location' 6 plant option were considered. The estimated milk transport costs for the different alternatives were; 4.60 cent per gallon for 23 plants; 4.85 cent per gallon for 12 plants; 5.04 cent per gallon for 9 plants; 5.24 cent per gallon for 6 plants ('good' location) and 5.75 cent per gallon ('poor' location) respectively.

In aggregate terms the results showed that milk transport costs would increase by €3, €5, €7 and €13 million per annum if processing plants were reduced from 23 to 12 to 9 to 6 (good location) and 6 (poor location) respectively. As the study of processing plant rationalization did not consider cheese plant rationalization in detail, it was inferred that the estimated saving from economies of scale of €20 million per annum was associated with between 6 and 12 processing sites. Excluding the 6 plant (poor location) option, the additional milk transport cost of moving to this reduced number of sites was estimated to be of the order of *5* million per annum. This represents about 25 per cent of the estimated benefits from economies of scale arising from processing plant rationalization.

The transportation model also facilitated a comparison of current milk catchment areas of processing plants with optimal catchment areas, assuming no change in number of processing plants. It was estimated that if dairies were to collect milk on an optimal basis, there would be an 11% reduction from the current (2005) milk transport costs.

In the “benchmark” model 23 plants were required to stay open at peak to accommodate milk supply and it was initially assumed that all 23 remained open throughout the year with the same catchment areas. However, due to seasonality in milk supply, it is not essential that all 23 plants remain open outside the peak.

Two options were analysed. The first involved allowing the model to determine the least cost transport pattern outside the peak i.e. a relaxation of the constraint of fixed catchment areas throughout the year, with all plants available for milk intake. Further modest reductions in milk transport costs were realisable in this case. The second option involved keeping only the bigger plants open outside the peak period. A modest increase in milk transport costs was estimated for this option due to tankers having to travel longer distances outside the peak period.

The analysis of milk transport costs in the Irish Dairy Industry is a significant topic in the Ph.D. thesis of the first author of the 2006 report from which the preceding quotation was taken:

Quinlan, Carrie, Brigid, 2013. *Optimisation of the food dairy coop supply chain.* PhD Thesis, University College Cork.

which is available at the following URL:

http://cora.ucc.ie/bitstream/handle/10468/1197/QuinlanCB_PhD2013.pdf

The Transportation Problem, with equality of total supply and total demand, can be expressed generally in the following form. Some commodity is supplied by m suppliers and is transported from those suppliers to n recipients. The ith supplier can supply at most s_i units of the commodity, and the jth recipient requires at least d_j units of the commodity. The cost of transporting a unit of the commodity from the ith supplier to the jth recipient is $c_{i,j}$.

The total transport cost is then

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}.$$

where $x_{i,j}$ denote the number of units of the commodity transported from the ith supplier to the jth recipient.

The Transportation Problem can then be presented as follows:

determine $x_{i,j}$ *for* $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$

so as minimize $\sum_{i,j} c_{i,j} x_{i,j}$

subject to the constraints

$x_{i,j} \geq 0$ *for all* i *and* j,

$\sum_{j=1}^{n} x_{i,j} \leq s_i$ *and* $\sum_{i=1}^{m} x_{i,j} \geq d_j$, *where*

$\sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j$.