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Section 5: Duality and Convexity

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2. Finite-Dimensional Vector Spaces

2.1. Real Vector Spaces

Definition

A *real vector space* consists of a set V on which there is defined an operation of vector addition, yielding an element $\mathrm{v} + \mathrm{w}$ of V for each pair v, w of elements of V, and an operation of multiplication-by-scalars that yields an element $\lambda \mathrm{v}$ of V for each $\mathrm{v} \in V$ and for each real number λ. The operation of vector addition is required to be commutative and associative. There must exist a zero element 0_V of V that satisfies $\mathrm{v} + 0_V = \mathrm{v}$ for all $\mathrm{v} \in V$, and, for each $\mathrm{v} \in V$ there must exist an element $-\mathrm{v}$ of V for which $\mathrm{v} + (-\mathrm{v}) = 0_V$. The following identities must also be satisfied for all $\mathrm{v}, \mathrm{w} \in V$ and for all real numbers λ and μ:

$$(\lambda + \mu)\mathrm{v} = \lambda \mathrm{v} + \mu \mathrm{v}, \quad \lambda(\mathrm{v} + \mathrm{w}) = \lambda \mathrm{v} + \lambda \mathrm{w},$$

$$\lambda(\mu \mathrm{v}) = (\lambda\mu)\mathrm{v}, \quad 1\mathrm{v} = \mathrm{v}.$$

Let n be a positive integer. The set $\mathbb{R}^n$ consisting of all n-tuples of real numbers is then a real vector space, with addition and multiplication-by-scalars defined such that

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots x_n + y_n)$$

and

$$\lambda(x_1, x_2, \ldots, x_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$$

for all $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in \mathbb{R}$ and for all real numbers λ.

The set $M_{m,n}(\mathbb{R})$ of all $m \times n$ matrices is a real vector space with respect to the usual operations of matrix addition and multiplication of matrices by real numbers.

2.2. Linear Dependence and Bases

Elements $u_1, u_2, \ldots, u_m$ of a real vector space V are said to be *linearly dependent* if there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$, not all zero, such that

$$\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_m u_m = 0_V.$$

If elements $u_1, u_2, \ldots, u_m$ of real vector space V are not linearly dependent, then they are said to be *linearly independent*.

Elements $u_1, u_2, \ldots, u_n$ of a real vector space V are said to *span* V if, given any element v of V, there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $v = \lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n$.

A vector space is said to be *finite-dimensional* if there exists a finite subset of V whose members span V.

Elements $u_1, u_2, \ldots, u_n$ of a finite-dimensional real vector space V are said to constitute a *basis* of V if they are linearly independent and span V.

Lemma 2.1

Elements $u_1, u_2, \ldots, u_n$ of a real vector space V constitute a basis of V if and only if, given any element v of V, there exist uniquely-determined real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$v = \lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n.$$

Proof

Suppose that $u_1, u_2, \ldots, u_n$ is a basis of V. Let v be an element V. The requirement that $u_1, u_2, \ldots, u_n$ span V ensures that there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$v = \lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n.$$

If $\mu_1, \mu_2, \ldots, \mu_n$ are real numbers for which

$$v = \mu_1 u_1 + \mu_2 u_2 + \cdots + \mu_n u_n,$$

then

$$(\mu_1 - \lambda_1)u_1 + (\mu_2 - \lambda_2)u_2 + \cdots + (\mu_n - \lambda_n)u_n = 0_V.$$

It then follows from the linear independence of $u_1, u_2, \ldots, u_n$ that $\mu_i - \lambda_i = 0$ for $i = 1, 2, \ldots, n$, and thus $\mu_i = \lambda_i$ for $i = 1, 2, \ldots, n$. This proves that the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ are uniquely-determined.

Conversely suppose that $u_1, u_2, \ldots, u_n$ is a list of elements of V with the property that, given any element v of V, there exist uniquely-determined real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$v = \lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n.$$

Then $u_1, u_2, \ldots, u_n$ span V. Moreover we can apply this criterion when $v = 0$. The uniqueness of the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ then ensures that if

$$\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n = 0_V$$

then $\lambda_i = 0$ for $i = 1, 2, \ldots, n$. Thus $u_1, u_2, \ldots, u_n$ are linearly independent. This proves that $u_1, u_2, \ldots, u_n$ is a basis of V, as required. ∎

Proposition 2.2

Let V be a finite-dimensional real vector space, let

$$\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_n$$

be elements of V that span V, and let K be a subset of $\{1, 2, \ldots, n\}$. Suppose either that $K = \emptyset$ or else that those elements u_i for which $i \in K$ are linearly independent. Then there exists a basis of V whose members belong to the list $\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_n$ which includes all the vectors u_i for which $i \in K$.

Proof

We prove the result by induction on the number of elements in the list $\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_n$ of vectors that span V. The result is clearly true when $n = 1$. Thus suppose, as the induction hypothesis, that the result is true for all lists of elements of V that span V and that have fewer than n members.

If the elements $\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_n$ are linearly independent, then they constitute the required basis. If not, then there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, not all zero, such that

$$\lambda_1 \mathrm{u}_1 + \lambda_2 \mathrm{u}_2 + \cdots + \lambda_n \mathrm{u}_n = 0_V.$$

Now there cannot exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, not all zero, such that both $\sum_{i=1}^{n} \lambda_i \mathrm{u}_i = 0_V$ and also $\lambda_i = 0$ whenever $i \neq K$. Indeed, in the case where $K = \emptyset$, this conclusion follows from the requirement that the real numbers λ_i cannot all be zero, and, in the case where $K \neq \emptyset$, the conclusion follows from the linear independence of those u_i for which $i \in K$. Therefore there must exist some integer i satisfying $1 \leq i \leq n$ for which $\lambda_i \neq 0$ and $i \notin K$.

Without loss of generality, we may suppose that $\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_n$ are ordered so that $n \notin K$ and $\lambda_n \neq 0$. Then

$$\mathrm{u}_n = -\sum_{i=1}^{n-1} \frac{\lambda_i}{\lambda_n} \mathrm{u}_i.$$

Let v be an element of V. Then there exist real numbers $\mu_1, \mu_2, \ldots, \mu_n$ such that $\mathrm{v} = \sum_{i=1}^{n} \mu_i \mathrm{u}_i$, because $\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_n$ span V. But then

$$\mathrm{v} = \sum_{i=1}^{n-1} \left(\mu_i - \frac{\mu_n \lambda_i}{\lambda_n} \right) \mathrm{u}_i.$$

We conclude that $\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_{n-1}$ span the vector space V. The induction hypothesis then ensures that there exists a basis of V consisting of members of this list that includes the linearly independent elements $\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_m$, as required. ∎

Corollary 2.3

Let V be a finite-dimensional real vector space, and let

$$\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_n$$

be elements of V that span the vector space V. Then there exists a basis of V whose elements are members of the list $\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_n$.

Proof

This result is a restatement of Proposition 2.2 in the special case where the set K in the statement of that proposition is the empty set. ■

2.3. Dual Spaces

Definition

Let V be a real vector space. A *linear functional* $\varphi\colon V \to \mathbb{R}$ on V is a linear transformation from the vector space V to the field $\mathbb{R}$ of real numbers.

Given linear functionals $\varphi\colon V \to \mathbb{R}$ and $\psi\colon V \to \mathbb{R}$ on a real vector space V, and given any real number λ, we define $\varphi + \psi$ and $\lambda\varphi$ to be the linear functionals on V defined such that
$(\varphi + \psi)(\mathrm{v}) = \varphi(\mathrm{v}) + \psi(\mathrm{v})$ and $(\lambda\varphi)(\mathrm{v}) = \lambda\,\varphi(\mathrm{v})$ for all $\mathrm{v} \in V$.
The set V^* of linear functionals on a real vector space V is itself a real vector space with respect to the algebraic operations of addition and multiplication-by-scalars defined above.

Definition

Let V be a real vector space. The *dual space* V^* of V is the vector space whose elements are the linear functionals on the vector space V.

Now suppose that the real vector space V is finite-dimensional. Let $\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_n$ be a basis of V, where $n = \dim V$. Given any $\mathrm{v} \in V$ there exist uniquely-determined real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\mathrm{v} = \sum_{j=1}^{n} \lambda_j \mathrm{u}_j$. It follows that there are well-defined functions $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ from V to the field $\mathbb{R}$ defined such that

$$\varepsilon_i \left(\sum_{j=1}^{n} \lambda_j \mathrm{u}_j \right) = \lambda_i$$

for $i = 1, 2, \ldots, n$ and for all real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$. These functions are linear transformations, and are thus linear functionals on V.

Lemma 2.4

Let V be a finite-dimensional real vector space, let

$$\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$$

be a basis of V, and let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be the linear functionals on V defined such that

$$\varepsilon_i \left(\sum_{j=1}^{n} \lambda_j \mathbf{u}_j \right) = \lambda_i$$

for $i = 1, 2, \ldots, n$ and for all real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ constitute a basis of the dual space V^* of V. Moreover $\varphi = \sum\limits_{i=1}^{n} \varphi(\mathbf{u}_\mathrm{i}) \varepsilon_i$ for all $\varphi \in V^*$.

Proof

Let $\mu_1, \mu_2, \ldots, \mu_n$ be real numbers with the property that $\sum_{i=1}^{n} \mu_i \varepsilon_i = 0_{V^*}$. Then

$$0 = \left(\sum_{i=1}^{n} \mu_i \varepsilon_i \right) (\mathsf{u}_j) = \sum_{i=1}^{n} \mu_i \varepsilon_i(\mathsf{u}_j) = \mu_j$$

for $j = 1, 2, \ldots, n$. Thus the linear functionals $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ on V are linearly independent elements of the dual space V^*.

Now let $\varphi\colon V \to \mathbb{R}$ be a linear functional on V, and let $\mu_i = \varphi(\mathrm{u}_i)$ for $i = 1, 2, \ldots, n$. Now

$$\varepsilon_i(\mathrm{u}_j) = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

It follows that

$$\begin{aligned} \left(\sum_{i=1}^{n} \mu_i \varepsilon_i\right)\left(\sum_{j=1}^{n} \lambda_j \mathrm{u}_j\right) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \mu_i \lambda_j \varepsilon_i(\mathrm{u}_j) = \sum_{j=1}^{n} \mu_j \lambda_j \\ &= \sum_{j=1}^{n} \lambda_j \varphi(\mathrm{u}_j) = \varphi\left(\sum_{j=1}^{n} \lambda_j \mathrm{u}_j\right) \end{aligned}$$

for all real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

It follows that

$$\varphi = \sum_{i=1}^{n} \mu_i \varepsilon_i = \sum_{i=1}^{n} \varphi(\mathrm{u}_i)\varepsilon_i.$$

We conclude from this that every linear functional on V can be expressed as a linear combination of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$. Thus these linear functionals span V^*. We have previously shown that they are linearly independent. It follows that they constitute a basis of V^*. Moreover we have verified that $\varphi = \sum_{i=1}^{n} \varphi(\mathrm{u}_i)\varepsilon_i$ for all $\varphi \in V^*$, as required. ∎

Definition

Let V be a finite-dimensional real vector space, let $\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_n$ be a basis of V. The corresponding *dual basis* of the dual space V^* of V consists of the linear functionals $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ on V, where

$$\varepsilon_i \left(\sum_{j=1}^{n} \lambda_j \mathrm{u}_j \right) = \lambda_i$$

for $i = 1, 2, \ldots, n$ and for all real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Corollary 2.5

Let V be a finite-dimensional real vector space, and let V^* be the dual space of V. Then $\dim V^* = \dim V$.

Proof

We have shown that any basis of V gives rise to a dual basis of V^*, where the dual basis of V has the same number of elements as the basis of V to which it corresponds. The result follows immediately from the fact that the dimension of a finite-dimensional real vector space is the number of elements in any basis of that vector space. ∎

Let V be a real-vector space, and let V^* be the dual space of V. Then V^* is itself a real vector space, and therefore has a dual space V^{**}. Now each element v of V determines a corresponding linear functional $E_{\mathsf{v}} \colon V^* \to \mathbb{R}$ on V^*, where $E_{\mathsf{v}}(\varphi) = \varphi(\mathsf{v})$ for all $\varphi \in V^*$. It follows that there exists a function $\iota \colon V \to V^{**}$ defined so that $\iota(\mathsf{v}) = E_{\mathsf{v}}$ for all $\mathsf{v} \in V$. Then $\iota(\mathsf{v})(\varphi) = \varphi(\mathsf{v})$ for all $\mathsf{v} \in V$ and $\varphi \in V^*$.

Now

$$\iota(\mathsf{v} + \mathsf{w})(\varphi) = \varphi(\mathsf{v} + \mathsf{w}) = \varphi(\mathsf{v}) + \varphi(\mathsf{w}) = (\iota(\mathsf{v}) + \iota(\mathsf{w}))(\varphi)$$

and

$$\iota(\lambda \mathsf{v})(\varphi) = \varphi(\lambda \mathsf{v}) = \lambda \varphi(\mathsf{v}) = (\lambda \iota(\mathsf{v}))(\varphi)$$

for all $\mathsf{v}, \mathsf{w} \in V$ and $\varphi \in V^*$ and for all real numbers λ. It follows that $\iota(\mathsf{v} + \mathsf{w}) = \iota(\mathsf{v}) + \iota(\mathsf{w})$ and $\iota(\lambda \mathsf{v}) = \lambda \iota(\mathsf{v})$ for all $\mathsf{v}, \mathsf{w} \in V$ and for all real numbers λ. Thus $\iota \colon V \to V^{**}$ is a linear transformation.

Proposition 2.6

Let V be a finite-dimensional real vector space, and let $\iota\colon V \to V^{**}$ be the linear transformation defined such that $\iota(\mathrm{v})(\varphi) = \varphi(\mathrm{v})$ for all $\mathrm{v} \in V$ and $\varphi \in V^*$. Then $\iota\colon V \to V^{**}$ is an isomorphism of real vector spaces.

Proof

Let $\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_n$ be a basis of V, let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be the dual basis of V^*, where

$$\varepsilon_i(\mathrm{u}_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

and let $\mathrm{v} \in V$. Then there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\mathrm{v} = \sum_{i=1}^{n} \lambda_i \mathrm{u}_i$.

Suppose that $\iota(\mathrm{v}) = 0_{V^{**}}$. Then $\varphi(\mathrm{v}) = E_{\mathrm{v}}(\varphi) = 0$ for all $\varphi \in V^*$. In particular $\lambda_i = \varepsilon_i(\mathrm{v}) = 0$ for $i = 1, 2, \ldots, n$, and therefore $\mathrm{v} = 0_V$. We conclude that $\iota\colon V \to V^{**}$ is injective.

Now let $F\colon V^* \to \mathbb{R}$ be a linear functional on V^*, let $\lambda_i = F(\varepsilon_i)$ for $i = 1, 2, \ldots, n$, let $\mathrm{v} = \sum_{i=1}^{n} \lambda_i \mathrm{u}_i$, and let $\varphi \in V^*$. Then $\varphi = \sum_{i=1}^{n} \varphi(\mathrm{u}_i)\varepsilon_i$ (see Lemma 2.4), and therefore

$$\begin{aligned}
\iota(\mathrm{v})(\varphi) &= \varphi(\mathrm{v}) = \sum_{i=1}^{n} \lambda_i \varphi(\mathrm{u}_i) = \sum_{i=1}^{n} F(\varepsilon_i)\varphi(\mathrm{u}_i) \\
&= F\left(\sum_{i=1}^{n} \varphi(\mathrm{u}_i)\varepsilon_i\right) = F(\varphi).
\end{aligned}$$

Thus $\iota(\mathrm{v}) = F$. We conclude that the linear transformation $\iota\colon V \to V^{**}$ is surjective. We have previously shown that this linear transformation is injective. There $\iota\colon V \to V^{**}$ is an isomorphism between the real vector spaces V and V^{**} as required. $\blacksquare$

The following corollary is an immediate consequence of Proposition 2.6.

Corollary 2.7

Let V be a finite-dimensional real vector space, and let V^* be the dual space of V. Then, given any linear functional $F\colon V^* \to \mathbb{R}$, there exists some $\mathrm{v} \in V$ such that $F(\varphi) = \varphi(\mathrm{v})$ for all $\varphi \in V^*$.