Module MAU34201: Algebraic Topology I
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Section 6: Discontinuous Group Actions and Orbit Spaces

D. R. Wilkins
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6 Discontinuous Group Actions and Orbit Spaces

6.1 Path-Lifting and the Fundamental Group

Let \( \tilde{X} \) and \( X \) be topological spaces, let \( \rho: \tilde{X} \to X \) be a covering map from \( \tilde{X} \) to \( X \), and let \( \alpha: [0, 1] \to X \) and \( \beta: [0, 1] \to X \) be paths in the base space \( X \) which both start at some point \( b_0 \) of \( X \) and finish at some point \( b_1 \) of \( X \), so that
\[
\alpha(0) = \beta(0) = b_0 \quad \text{and} \quad \alpha(1) = \beta(1) = b_1.
\]

Let \( \tilde{b}_0 \) be some point of the covering space \( \tilde{X} \) that projects down to \( b_0 \), so that \( \rho(\tilde{b}_0) = b_0 \). It follows from the Path-Lifting Theorem (Theorem 4.13) that there exist paths \( \tilde{\alpha}: [0, 1] \to \tilde{X} \) and \( \tilde{\beta}: [0, 1] \to \tilde{X} \) in the covering space \( \tilde{X} \) that both start at \( \tilde{b}_0 \) and are lifts of the paths \( \alpha \) and \( \beta \) respectively. Thus
\[
\tilde{\alpha}(0) = \tilde{\beta}(0) = \tilde{b}_0,
\]
\[
\rho(\tilde{\alpha}(t)) = \alpha(t) \quad \text{and} \quad \rho(\tilde{\beta}(t)) = \beta(t) \quad \text{for all} \quad t \in [0, 1].
\]

These lifts \( \tilde{\alpha} \) and \( \tilde{\beta} \) of the paths \( \alpha \) and \( \beta \) are uniquely determined by their starting point \( \tilde{b}_0 \) (see Proposition 4.11).

Now, though the lifts \( \tilde{\alpha} \) and \( \tilde{\beta} \) of the paths \( \alpha \) and \( \beta \) have been chosen such that they start at the same point \( \tilde{b}_0 \) of the covering space \( \tilde{X} \), they need not in general end at the same point of \( \tilde{X} \). However we shall prove that if \( \alpha \simeq \beta \) rel \( \{0, 1\} \), then the lifts \( \tilde{\alpha} \) and \( \tilde{\beta} \) of \( \alpha \) and \( \beta \) respectively that both start at some point \( \tilde{b}_0 \) of \( \tilde{X} \) will both finish at some point \( \tilde{b}_1 \) of \( \tilde{X} \), so that \( \tilde{\alpha}(1) = \tilde{\beta}(1) = \tilde{b}_1 \). This result is established in Proposition 6.1 below.

**Proposition 6.1** Let \( \tilde{X} \) and \( X \) be topological spaces, and let \( \rho: \tilde{X} \to X \) be a covering map from \( \tilde{X} \) to \( X \). Also let \( \alpha: [0, 1] \to X \) and \( \beta: [0, 1] \to X \) be paths in \( X \), where \( \alpha(0) = \beta(0) \) and \( \alpha(1) = \beta(1) \), and let \( \tilde{\alpha}: [0, 1] \to \tilde{X} \) and \( \tilde{\beta}: [0, 1] \to \tilde{X} \) be paths in \( \tilde{X} \) such that \( \rho \circ \tilde{\alpha} = \alpha \) and \( \rho \circ \tilde{\beta} = \beta \). Suppose that \( \tilde{\alpha}(0) = \tilde{\beta}(0) \) and that \( \alpha \simeq \beta \) rel \( \{0, 1\} \). Then \( \tilde{\alpha}(1) = \tilde{\beta}(1) \) and \( \tilde{\alpha} \simeq \tilde{\beta} \) rel \( \{0, 1\} \).

**Proof** Let \( b_0 \) and \( b_1 \) be the points of \( X \) given by
\[
b_0 = \alpha(0) = \beta(0), \quad b_1 = \alpha(1) = \beta(1).
\]
Now \( \alpha \simeq \beta \) rel \( \{0, 1\} \), and therefore there exists a homotopy \( F: [0, 1] \times [0, 1] \to X \) such that
\[
F(t, 0) = \alpha(t) \quad \text{and} \quad F(t, 1) = \beta(t) \quad \text{for all} \quad t \in [0, 1],
\]
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and
\[ F(0, \tau) = b_0 \quad \text{and} \quad F(1, \tau) = b_1 \quad \text{for all} \ \tau \in [0, 1]. \]

It then follows from the Homotopy-Lifting Theorem (Theorem 4.14) that there exists a continuous map \( G: [0, 1] \times [0, 1] \to \tilde{X} \) such that \( \rho \circ G = F \) and \( G(0, 0) = \tilde{\alpha}(0) \). Then \( \rho(G(0, \tau)) = b_0 \) and \( \rho(G(1, \tau)) = b_1 \) for all \( \tau \in [0, 1] \).

A straightforward application of Proposition 4.11 shows that any continuous lift of a constant path must itself be a constant path. Therefore \( G(0, \tau) = \tilde{b}_0 \) and \( G(1, \tau) = \tilde{b}_1 \) for all \( \tau \in [0, 1] \), where
\[ \tilde{b}_0 = G(0, 0) = \tilde{\alpha}(0), \quad \tilde{b}_1 = G(1, 0). \]

However
\[ G(0, 0) = G(0, 1) = \tilde{b}_0 = \tilde{\alpha}(0) = \tilde{\beta}(0). \]

Also
\[ \rho(G(t, 0)) = F(t, 0) = \alpha(t) = \rho(\tilde{\alpha}(t)) \]
and
\[ \rho(G(t, 1)) = F(t, 1) = \beta(t) = \rho(\tilde{\beta}(t)) \]
for all \( t \in [0, 1] \). It follows that the map that sends \( t \in [0, 1] \) to \( G(t, 0) \) is a lift of the path \( \alpha \) that starts at \( \tilde{b}_0 \), and the map that sends \( t \in [0, 1] \) to \( G(t, 1) \) is a lift of the path \( \beta \) that also starts at \( \tilde{b}_0 \).

However the lifts \( \tilde{\alpha} \) and \( \tilde{\beta} \) of the paths \( \alpha \) and \( \beta \) are uniquely determined by their starting points (see Proposition 4.11). It follows that \( G(t, 0) = \tilde{\alpha}(t) \) and \( G(t, 1) = \tilde{\beta}(t) \) for all \( t \in [0, 1] \). In particular,
\[ \tilde{\alpha}(1) = G(1, 0) = \tilde{b}_1 = G(1, 1) = \tilde{\beta}(1). \]

Moreover the map \( G: [0, 1] \times [0, 1] \to \tilde{X} \) is a homotopy between the paths \( \tilde{\alpha} \) and \( \tilde{\beta} \) which satisfies \( G(0, \tau) = \tilde{b}_0 \) and \( G(1, \tau) = \tilde{b}_1 \) for all \( \tau \in [0, 1] \). It follows that \( \tilde{\alpha} \simeq \tilde{\beta} \) rel \( \{0, 1\} \), as required.

Let \( \tilde{X} \) and \( X \) be topological spaces, and let \( \rho: \tilde{X} \to X \) be a covering map from \( \tilde{X} \) to \( X \). Also let \( \tilde{b}_0 \) be a point of the covering space \( \tilde{X} \), and let \( b_0 = \rho(\tilde{b}_0) \). Then the covering map \( \rho \) induces a group homomorphism
\[ \rho_{\#}: \pi_1(\tilde{X}, \tilde{b}_0) \to \pi_1(X, b_0) \]
from the fundamental group \( \pi_1(\tilde{X}, \tilde{b}_0) \) of the covering space with basepoint \( \tilde{b}_0 \) to the fundamental group \( \pi_1(X, b_0) \) of the base space with basepoint \( b_0 \).

This induced homomorphism \( \rho_{\#} \) is defined so that \( \rho_{\#}[\gamma] = [\rho \circ \tilde{\gamma}] \) for all loops \( \tilde{\gamma} \) in the covering space \( \tilde{X} \) based at the point \( \tilde{b}_0 \) (see Proposition 5.2).
Proposition 6.2 Let \( \tilde{X} \) and \( X \) be topological spaces, and let \( \rho: \tilde{X} \to X \) be a covering map from \( \tilde{X} \) to \( X \). Also let \( b_0 \) be a point of the covering space \( \tilde{X} \), and let \( b_0 = \rho(b_0) \). Then the homomorphism

\[
\rho_#: \pi_1(\tilde{X}, \tilde{b}_0) \to \pi_1(X, b_0)
\]

of fundamental groups induced by the covering map \( \rho \) is injective.

**Proof** Let \( \sigma_0 \) and \( \sigma_1 \) be loops in \( \tilde{X} \) based at the point \( \tilde{b}_0 \), representing elements \([\sigma_0]\) and \([\sigma_1]\) of \( \pi_1(\tilde{X}, \tilde{b}_0) \). Suppose that \( \rho_#([\sigma_0]) = \rho_#([\sigma_1]) \). Then \( \rho \circ \sigma_0 \simeq \rho \circ \sigma_1 \) rel \( \{0,1\} \). Also \( \sigma_0(0) = \tilde{b}_0 = \sigma_1(0) \). It therefore follows (on applying Proposition 6.1) that \( \sigma_0 \simeq \sigma_1 \) rel \( \{0,1\} \), and thus \([\sigma_0] = [\sigma_1] \).

We conclude therefore that the homomorphism \( \rho_# : \pi_1(\tilde{X}, \tilde{b}_0) \to \pi_1(X, b_0) \) is injective.

Proposition 6.3 Let \( \tilde{X} \) and \( X \) be topological spaces, and let \( \rho: \tilde{X} \to X \) be a covering map from \( \tilde{X} \) to \( X \). Also let \( b_0 \) be a point of the covering space \( \tilde{X} \), let \( b_0 = \rho(b_0) \), and let \( \gamma \) be a loop in \( X \) based at \( b_0 \). Then \([\gamma] \in \rho_#(\pi_1(\tilde{X}, \tilde{b}_0)) \) if and only if there exists a loop \( \tilde{\gamma} \) in \( \tilde{X} \), based at the point \( \tilde{b}_0 \), such that \( \rho \circ \tilde{\gamma} = \gamma \).

**Proof** If \( \gamma = \rho \circ \tilde{\gamma} \) for some loop \( \tilde{\gamma} \) in \( \tilde{X} \) based at \( \tilde{b}_0 \) then \([\gamma] = \rho_#([\tilde{\gamma}]) \), and therefore \([\gamma] \in \rho_#(\pi_1(\tilde{X}, \tilde{b}_0)) \).

Conversely suppose that \([\gamma] \in \rho_#(\pi_1(\tilde{X}, \tilde{b}_0)) \). We must show that there exists some loop \( \tilde{\gamma} \) in \( \tilde{X} \) based at \( \tilde{b}_0 \) such that \( \gamma = \rho \circ \tilde{\gamma} \). Now there exists a loop \( \sigma \) in \( \tilde{X} \) based at the point \( \tilde{b}_0 \) such that \([\gamma] = \rho_#([\sigma]) \) in \( \pi_1(X, b_0) \). Then \( \gamma \simeq \rho \circ \sigma \) rel \( \{0,1\} \). It follows from the Path-Lifting Theorem for covering maps (Theorem 4.13) that there exists a unique path \( \tilde{\gamma}: [0, 1] \to \tilde{X} \) in \( \tilde{X} \) for which \( \tilde{\gamma}(0) = \tilde{b}_0 \) and \( \rho \circ \tilde{\gamma} = \gamma \). It then follows from Proposition 6.1 that \( \tilde{\gamma}(1) = \sigma(1) \) and \( \tilde{\gamma} \simeq \sigma \) rel \( \{0,1\} \). But \( \sigma(1) = \tilde{b}_0 \). Therefore the path \( \tilde{\gamma} \) is the required loop in \( \tilde{X} \) based the point \( \tilde{b}_0 \) which satisfies \( \rho \circ \tilde{\gamma} = \gamma \).

Corollary 6.4 Let \( \tilde{X} \) and \( X \) be topological spaces, and let \( \rho: \tilde{X} \to X \) be a covering map from \( \tilde{X} \) to \( X \). Also let \( q_0 \) and \( q_1 \) be points of \( X \) satisfying \( \rho(q_0) = \rho(q_1) \), and let \( \eta: [0, 1] \to \tilde{X} \) be a path in \( \tilde{X} \) from \( q_0 \) to \( q_1 \). Suppose that \([\rho \circ \eta] \in \rho_#(\pi_1(\tilde{X}, q_0)) \). Then the path \( \eta \) is a loop in \( \tilde{X} \), and thus \( q_0 = q_1 \).

**Proof** It follows from Proposition 6.3 that there exists a loop \( \sigma \) based at \( q_0 \) satisfying \( \rho \circ \sigma = \rho \circ \eta \). Then \( \eta(0) = \sigma(0) \). Now Proposition 4.11 ensures that the lift to \( \tilde{X} \) of any path in \( X \) is uniquely determined by its starting point. It follows that \( \eta = \sigma \). But then the path \( \eta \) must be a loop in \( \tilde{X} \), and therefore \( q_0 = q_1 \), as required.
**Theorem 6.5** Let \( \tilde{X} \) and \( X \) be topological spaces and let \( \rho: \tilde{X} \to X \) be a covering map from \( \tilde{X} \) to \( X \). Suppose that \( \tilde{X} \) is path-connected and that \( X \) is simply connected. Then the covering map \( \rho: \tilde{X} \to X \) is a homeomorphism.

**Proof** We show that the map \( \rho: \tilde{X} \to X \) is a bijection. This map is surjective (because covering maps are by definition surjective). We must show that it is injective. Let \( q_0 \) and \( q_1 \) be points of \( \tilde{X} \) with the property that \( \rho(q_0) = \rho(q_1) \). Then there exists a path \( \eta: [0, 1] \to \tilde{X} \) with \( \eta(0) = q_0 \) and \( \eta(1) = q_1 \), because the covering space \( \tilde{X} \) is path-connected. Then \( \rho \circ \eta \) is a loop in \( X \) based at the point \( b_0 \), where \( b_0 = \rho(q_0) \). However \( \pi_1(X, b_0) \) is the trivial group, because \( X \) is simply connected. It follows from Corollary 6.4 that the path \( \eta \) is a loop in \( \tilde{X} \) based at \( q_0 \), and therefore \( q_0 = q_1 \). This shows that the covering map \( \rho: \tilde{X} \to X \) is injective.

Accordingly the map \( \rho: \tilde{X} \to X \) is a bijection. But any bijective covering map is a homeomorphism (Corollary 4.8). The result follows. \( \blacksquare \)

### 6.2 Discontinuous Group Actions

**Definition** Let \( G \) be a group, and let \( X \) be a set. The group \( G \) is said to act on the set \( X \) (on the left) if each element \( g \) of \( G \) determines a corresponding function \( \theta_g: X \to X \) from the set \( X \) to itself, where

(i) \( \theta_{gh} = \theta_g \circ \theta_h \) for all \( g, h \in G \);

(ii) the function \( \theta_e \) determined by the identity element \( e \) of \( G \) is the identity function of \( X \).

Let \( G \) be a group acting on a set \( X \). Given any element \( p \) of \( X \), the orbit \([p]_G \) of \( p \) (under the group action) is defined to be the subset \( \{\theta_g(p): g \in G\} \) of \( X \), and the stabilizer of \( p \) is defined to be the subgroup \( \{g \in G : \theta_g(p) = p\} \) of the group \( G \). Thus the orbit of an element \( p \) of \( X \) is the set consisting of all points of \( X \) to which \( p \) gets mapped under the action of elements of the group \( G \). The stabilizer of \( p \) is the subgroup of \( G \) consisting of all elements of this group that fix the point \( p \). The group \( G \) is said to act freely on \( X \) if \( \theta_g(p) \neq p \) for all \( p \in X \) and \( g \in G \) satisfying \( g \neq e \). Thus the group \( G \) acts freely on \( X \) if and only if the stabilizer of every element of \( X \) is the trivial subgroup of \( G \).

Let \( e \) be the identity element of \( G \). Then \( p = \theta_e(p) \) for all \( p \in X \), and therefore \( p \in [p]_G \) for all \( p \in X \), where \([p]_G = \{\theta_g(p) : g \in G\}\).

Let \( p \) and \( q \) be elements of \( X \) for which \([p]_G \cap [q]_G \) is non-empty, and let \( r \in [p]_G \cap [q]_G \). Then there exist elements \( h \) and \( k \) of \( G \) such that \( r = \theta_h(p) = \theta_k(q) \). Then \( \theta_g(r) = \theta_{gh}(p) = \theta_{gk}(q) \), \( \theta_g(p) = \theta_{gh^{-1}}(r) \) and \( \theta_g(q) = \theta_{gk^{-1}}(r) \)
for all $g \in G$. Therefore $[p]_G = [r]_G = [g]_G$. It follows from this that the group action partitions the set $X$ into orbits, so that each element of $X$ determines an orbit which is the unique orbit for the action of $G$ on $X$ to which it belongs. We denote by $X/G$ the set of orbits for the action of $G$ on $X$.

Now suppose that the group $G$ acts on a topological space $X$. Then there is a surjective function $\rho: X \to X/G$, where $\rho(p) = [p]_G$ for all $p \in X$. This surjective function induces a quotient topology on the set of orbits: a subset $W$ of $X/G$ is open in this quotient topology if and only if $\rho^{-1}(W)$ is an open set in $X$ (see Lemma 2.13). We define the orbit space $X/G$ for the action of $G$ on $X$ to be the topological space whose underlying set is the set of orbits for the action of $G$ on $X$, the topology on $X/G$ being the quotient topology induced by the function $\rho: X \to X/G$. This function $\rho: X \to X/G$ is then an identification map: we shall refer to it as the quotient map from $X$ to $X/G$.

We shall be concerned here with situations in which a group action on a topological space gives rise to a covering map. The relevant group actions are those where the group acts freely and properly discontinuously on the topological space.

**Definition** Let $G$ be a group with identity element $e$, and let $X$ be a topological space. The group $G$ is said to act freely and properly discontinuously on $X$ if each element $g$ of $G$ determines a corresponding continuous map $\theta_g: X \to X$, where the following conditions are satisfied:

(i) $\theta_{gh} = \theta_g \circ \theta_h$ for all $g, h \in G$;

(ii) the continuous map $\theta_e$ determined by the identity element $e$ of $G$ is the identity map of $X$;

(iii) given any point $p$ of $X$, there exists an open set $V$ in $X$ such that $p \in V$ and $\theta_g(V) \cap V = \emptyset$ for all $g \in G$ satisfying $g \neq e$.

Let $G$ be a group which acts freely and properly discontinuously on a topological space $X$. Given any element $g$ of $G$, the corresponding continuous function $\theta_g: X \to X$ determined by $g$ is a homeomorphism. Indeed it follows from conditions (i) and (ii) in the above definition that $\theta_{g^{-1}} \circ \theta_g$ and $\theta_g \circ \theta_{g^{-1}}$ are both equal to the identity map of $X$, and therefore $\theta_g: X \to X$ is a homeomorphism with inverse $\theta_{g^{-1}}: X \to X$.

**Remark** The terminology ‘freely and properly discontinuously’ is traditional, but is hardly ideal. The adverb ‘freely’ refers to the requirement
that \( \theta_g(p) \neq p \) for all \( p \in X \) and for all \( g \in G \) satisfying \( g \neq e \). The adverb ‘discontinuously’ refers to the fact that, given any point \( x \) of \( X \), the elements of the orbit \( \{ \theta_g(p) : g \in G \} \) of \( p \) are separated; it does not signify that the functions defining the action are in any way discontinuous or badly-behaved. The adverb ‘properly’ refers to the fact that, given any compact subset \( K \) of \( X \), the number of elements \( g \) of the group \( G \) for which \( K \cap \theta_g(K) \neq \emptyset \) is finite. Moreover the definitions of proper discontinuous actions in textbooks and in sources of reference are not always in agreement: some say that an action of a group \( G \) on a topological space \( X \) (where each group element determines a corresponding homeomorphism of the topological space) is properly discontinuous if, given any \( p \in X \), there exists an open set \( V \) in \( X \) such that the number of elements \( g \) of the group for which \( g(V) \cap V \neq \emptyset \) is finite; others say that the action is properly discontinuous if it satisfies the conditions given in the definition above for a group acting freely and properly discontinuously on the set. William Fulton, in his textbook *Algebraic topology: a first course* (Springer, 1995), introduced the term ‘evenly’ in place of ‘freely and properly discontinuously’, but this change in terminology does not appear to have been generally adopted.

### 6.3 Orbit Spaces

**Example** The cyclic group \( C_2 \) of order 2 consists of a set \( \{ e, a \} \) with two elements \( e \) and \( a \), together with a group multiplication operation defined so that \( e^2 = a^2 = e \) and \( ea = ae = a \). The identity element of \( C_2 \) is thus \( e \).

Let us represent the \( n \)-dimensional sphere \( S^n \) as the unit sphere in \( \mathbb{R}^{n+1} \) centred on the origin. Let \( \theta_e : S^n \to S^n \) be the identity map of \( S^n \) and let \( \theta_a : S^n \to S^n \) be the antipodal map of \( S^n \), defined such that \( \theta_a(p) = -p \) for all \( p \in S^n \). Then the group \( C_2 \) acts on \( S^n \) (on the left) so that elements \( e \) and \( a \) of \( S^n \) correspond under this action to the homeomorphisms \( \theta_e \) and \( \theta_a \) respectively. Points \( p \) and \( q \) are said to be antipodal to one another if and only if \( q = -p \). Each orbit for the action of \( C_2 \) on \( S^n \) thus consists of a pair of antipodal points on \( S^n \).

Let \( n \) be a point on the \( n \)-dimensional sphere \( S^n \), and let

\[
V = \{ p \in S^n : p \cdot n > 0 \}.
\]

Then \( V \) is open in \( S^n \) and \( n \in V \). Also

\[
\theta_a(V) = \{ p \in S^n : p \cdot n < 0 \},
\]

and therefore \( V \cap \theta_a(V) = \emptyset \). Consequently the group \( C_2 \) acts freely and properly discontinuously on \( S^n \).
Distinct points of $S^n$ belong to the same orbit under the action of $C_2$ on $S^n$ if and only if the line in $\mathbb{R}^{n+1}$ passing through those points also passes through the origin. It follows that lines in $\mathbb{R}^{n+1}$ that pass through the origin are in one-to-one correspondence with orbits for the action of $C_2$ on $S^n$. The orbit space $S^n/C_2$ thus represents the set of lines through the origin in $\mathbb{R}^{n+1}$. We define $n$-dimensional real projective space $\mathbb{R}P^n$ to be the topological space whose elements are the lines in $\mathbb{R}^{n+1}$ passing through the origin, with the topology obtained on identifying $\mathbb{R}P^n$ with the orbit space $S^n/C_2$. The quotient map $\rho: S^n \to \mathbb{R}P^n$ then sends each point $p$ of $S^n$ to the orbit consisting of the two points $p$ and $-p$. Thus each pair of antipodal points on the $n$-dimensional sphere $S^n$ determines a single point of $n$-dimensional real projective space $\mathbb{R}P^n$.

**Proposition 6.6** Let $G$ be a group acting freely and properly discontinuously on a topological space $X$, let $X/G$ denote the resulting orbit space, and let $\rho: X \to X/G$ be the quotient map that sends each element of $X$ to its orbit under the action of the group $G$. Let $\varphi: X \to Y$ be a continuous surjective map from $X$ to a topological space $Y$. Suppose that elements $p$ and $q$ of $X$ satisfy $\varphi(p) = \varphi(q)$ if and only if $\rho(p) = \rho(q)$. Suppose also $\varphi(V)$ is open in $Y$ for every open set $V$ in $X$. Then the surjective continuous map $\varphi: X \to Y$ induces a homeomorphism $\psi: X/G \to Y$ between the topological spaces $X/G$ and $Y$, where $\psi(\rho(p)) = \varphi(p)$ for all $p \in X$.

**Proof** The function $\psi: X/G \to Y$ is continuous because $\varphi: X \to Y$ is continuous and $\rho: X \to Y$ is a quotient map (see Lemma 2.14). Moreover it is surjective because $\varphi: X \to Y$ is surjective, and it is injective because elements $p$ and $q$ satisfy $\varphi(p) = \varphi(q)$ if and only if $\rho(p) = \rho(q)$. It follows that $\psi: X/G \to Y$ is a bijection.

Let $W$ be an open set in $X/G$. It follows from the definition of the quotient topology that $\rho^{-1}(W)$ is open in $X$. The map $\varphi$ maps open sets to open sets. Therefore $\varphi(\rho^{-1}(W))$ is open in $Y$. But $\varphi(\rho^{-1}(W)) = \psi(W)$. Thus $\psi(W)$ is open in $Y$ for every open set $W$ in $X/G$, and therefore the inverse of the map $\psi$ is continuous. Thus the continuous bijection $\psi: X/G \to Y$ is a homeomorphism, as required. 

**Corollary 6.7** Let the group $\mathbb{Z}$ act on the real line $\mathbb{R}$ by translation, where the action sends each integer $n$ to the translation function $\theta_n: \mathbb{R} \to \mathbb{R}$ that is defined so that $\theta_n(t) = t + n$ for all real numbers $t$. Let $\mathbb{R}/\mathbb{Z}$ denote the orbit space for this action, and let $\rho: \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the quotient map that sends each real number to its orbit under the action of the group $\mathbb{Z}$. Let $S^1$ denote the unit circle centred on the origin in $\mathbb{R}^2$, let $\kappa: \mathbb{R} \to S^1$ be defined such that $\kappa(t) = (\cos 2\pi t, \sin 2\pi t)$

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for all real numbers \( t \), and let \( \psi: \mathbb{R}/\mathbb{Z} \to S^1 \) be the map defined such that \( \psi(\rho(t)) = \kappa(t) \) for all real numbers \( t \). Then \( \psi: \mathbb{R}/\mathbb{Z} \to S^1 \) is a homeomorphism.

**Proof** The map \( \kappa: \mathbb{R} \to S^1 \) maps open sets to open sets. The result therefore follows directly on applying Proposition 6.6.

**Proposition 6.8** Let \( G \) be a group acting freely and properly discontinuously on a topological space \( X \), let \( X/G \) denote the resulting orbit space, and let \( \rho: X \to X/G \) be the quotient map that sends each element of \( X \) to its orbit under the action of the group \( G \). Let \( \varphi: X \to Y \) be a continuous surjective map from \( X \) to a Hausdorff topological space \( Y \). Suppose that elements \( p \) and \( q \) of \( X \) satisfy \( \varphi(p) = \varphi(q) \) if and only if \( \rho(p) = \rho(q) \). Suppose also that there exists a compact subset \( K \) of \( X \) that intersects every orbit for the action of \( G \) on \( X \). Then the surjective continuous map \( \varphi: X \to Y \) induces a homeomorphism \( \psi: X/G \to Y \) between the topological spaces \( X/G \) and \( Y \), where \( \psi(\rho(p)) = \varphi(p) \) for all \( p \in X \).

**Proof** The function \( \psi: X/G \to Y \) is continuous because \( \varphi: X \to Y \) is continuous and \( \rho: X \to X/G \) is a quotient map (see Lemma 2.14). Moreover it is surjective because \( \varphi: X \to Y \) is surjective, and it is injective because elements \( p \) and \( q \) satisfy \( \varphi(p) = \varphi(q) \) if and only if \( \rho(p) = \rho(q) \). It follows that \( \psi: X/G \to Y \) is a bijection.

The orbit space \( X/G \) is compact, because it is the image \( \rho(K) \) of the compact set \( K \) under the continuous map \( \rho: X \to X/G \). (see Lemma 1.29). Thus \( \psi: X/G \to Y \) is a continuous bijection from a compact topological space to a Hausdorff space. This map is therefore a homeomorphism (see Theorem 1.35).

**Example** Let the group \( \mathbb{Z} \) of integers under addition act on the real line \( \mathbb{R} \) by translation so that, under this action, an integer \( n \) corresponds to the homeomorphism \( \theta_n: \mathbb{R} \to \mathbb{R} \) defined such that \( \theta_n(t) = t + n \) for all real numbers \( t \). Let \( \rho: \mathbb{R} \to \mathbb{R}/\mathbb{Z} \) be the quotient map onto the orbit space, and let \( \kappa: \mathbb{R} \to S^1 \) be defined such that

\[
\kappa(t) = (\cos 2\pi t, \sin 2\pi t)
\]

for all real numbers \( t \), and let \( \psi: \mathbb{R}/\mathbb{Z} \to S^1 \) be the map defined such that \( \psi(\rho(t)) = \kappa(t) \) for all real numbers \( t \).

Now \( S^1 \) is a Hausdorff space, as it is a subset of the metric space \( \mathbb{R}^2 \). Also the map \( \kappa: \mathbb{R} \to S^1 \) is surjective. Real numbers \( t_1 \) and \( t_2 \) satisfy \( \kappa(t_1) = \kappa(t_2) \) if and only if \( t_1 = t_2 + n \) for some integer \( n \). It follows that \( \kappa(t_1) = \kappa(t_2) \).
if and only if \( \rho(t_1) = \rho(t_2) \). The compact subset \([0, 1]\) of \( \mathbb{R} \) intersects every orbit for the action of \( \mathbb{Z} \) on \( \mathbb{R} \). It therefore follows from Proposition 6.8 that \( \psi: \mathbb{R}/\mathbb{Z} \to S^1 \) is a homeomorphism. (This result was also shown to follow from the fact that \( \kappa: \mathbb{R} \to S^1 \) maps open sets to open sets: see Corollary 6.7.)

**Proposition 6.9** Let \( G \) be a group acting freely and properly discontinuously on a topological space \( X \). Then the quotient map \( \rho: X \to X/G \) from \( X \) to the corresponding orbit space \( X/G \) is a covering map.

**Proof** The quotient map \( \rho: X \to X/G \) is surjective. Let \( V \) be an open set in \( X \). Then \( \rho^{-1}(\rho(V)) \) is the union \( \bigcup_{g \in G} \theta_g(V) \) of the open sets \( \theta_g(V) \) as \( g \) ranges over the group \( G \), because \( \rho^{-1}(\rho(V)) \) is the subset of \( X \) consisting of all elements of \( X \) that belong to the orbit of some element of \( V \). Moreover each set \( \theta_g(V) \) is an open set in \( X \), because each map \( \theta_g \) is a homeomorphism mapping the set \( X \) onto itself. Also any union of open sets in a topological space is an open set. We conclude therefore that if \( V \) is an open set in \( X \) then \( \rho(V) \) is an open set in \( X/G \).

Let \( p \) be a point of \( X \). Then there exists an open set \( V \) in \( X \) such that \( p \in V \) and \( \theta_g(V) \cap V = \emptyset \) for all \( g \in G \) satisfying \( g \neq e \). Now \( \rho^{-1}(\rho(V)) = \bigcup_{g \in G} \theta_g(V) \). We claim that the sets \( \theta_g(V) \) are pairwise disjoint. Let \( g \) and \( h \) be elements of \( G \). Suppose that \( \theta_g(V) \cap \theta_h(V) \neq \emptyset \). Then \( \theta_{h^{-1}}(\theta_g(V) \cap \theta_h(V)) \neq \emptyset \). But \( \theta_{h^{-1}}: X \to X \) is a bijection. Consequently

\[
\theta_{h^{-1}}(\theta_g(V) \cap \theta_h(V)) = \theta_{h^{-1}}(\theta_g(V)) \cap \theta_{h^{-1}}(\theta_h(V)) = \theta_{h^{-1}}(V) \cap V,
\]

and therefore \( \theta_{h^{-1}}(V) \cap V \neq \emptyset \). It follows that \( h^{-1}g = e \), where \( e \) denotes the identity element of \( G \), and therefore \( g = h \). It follows from this that if \( g \) and \( h \) are elements of the group \( G \), and if \( g \neq h \), then \( \theta_g(V) \cap \theta_h(V) = \emptyset \). We conclude therefore that the preimage \( \rho^{-1}(\rho(V)) \) of \( \rho(V) \) is indeed the disjoint union of the sets \( \theta_g(V) \) as \( g \) ranges over the group \( G \). Moreover each of these sets \( \theta_g(V) \) is an open set in \( X \).

Now \( V \cap [p]_G = \{p\} \) for all \( p \in V \), because \( [p]_G = \{\theta_g(p): g \in G\} \) and \( V \cap \theta_g(V) = \emptyset \) whenever \( g \) is an element of the group \( G \) distinct from the identity element of that group. It follows that if \( p \) and \( q \) are elements of \( V \), and if \( \rho(p) = \rho(q) \) then \( [p]_G = [q]_G \) and therefore \( p = q \). Consequently the restriction \( \rho|V: V \to X/G \) of the quotient map \( \rho \) to \( V \) is injective, and therefore \( \rho \) maps \( V \) bijectively onto \( \rho(V) \). But \( \rho \) maps open sets onto open sets, and any continuous bijection that maps open sets onto open sets is a homeomorphism. We conclude therefore that the restriction of \( \rho: X \to X/G \) to the open set \( V \) maps \( V \) homeomorphically onto \( \rho(V) \). Moreover, given any element \( g \) of \( G \), the quotient map \( \rho \) satisfies \( \rho = \rho \circ \theta_g^{-1} \), and the
homeomorphism $\varphi^{-1}$ maps $\varphi(V)$ homeomorphically onto $V$. It follows that the quotient map $\rho$ maps $\varphi(V)$ homeomorphically onto $\rho(V)$ for all $g \in V$.

We conclude therefore that $\rho(V)$ is an evenly covered open set in $X/G$ whose preimage $\rho^{-1}(\rho(V))$ is the disjoint union of the open sets $\varphi(V)$ as $g$ ranges over the group $G$. Consequently the quotient map $\rho: X \to X/G$ is a covering map, as required.

6.4 Fundamental Groups of Orbit Spaces

**Theorem 6.10** Let $G$ be a group acting freely and properly discontinuously on a path-connected topological space $X$, let $\rho: X \to X/G$ be the quotient map from $X$ to the orbit space $X/G$, let $b_0$ be a point of $X$, and let $c_0 = \rho(b_0) = [b_0]_G$. Then there exists a surjective homomorphism $\lambda: \pi_1(X/G, c_0) \to G$ characterized by the property that $\tilde{\gamma}(1) = \theta_{\lambda([\gamma])}(b_0)$ for any loop $\gamma$ in $X/G$ based at $c_0$, where $\tilde{\gamma}$ denotes the unique path in $X$ for which $\tilde{\gamma}(0) = b_0$ and $\rho \circ \tilde{\gamma} = \gamma$. The kernel of this homomorphism is the subgroup $\rho_{\#}(\pi_1(X, b_0))$ of $\pi_1(X/G, c_0)$.

**Proof** Let $\gamma: [0, 1] \to X/G$ be a loop in the orbit space with $\gamma(0) = \gamma(1) = c_0$. It follows from the Path-Lifting Theorem for covering maps (Theorem 4.13) that there exists a unique path $\tilde{\gamma}: [0, 1] \to X$ for which $\tilde{\gamma}(0) = b_0$ and $\rho \circ \tilde{\gamma} = \gamma$. Now $\tilde{\gamma}(0)$ and $\tilde{\gamma}(1)$ must belong to the same orbit under the action of the group $G$ on the topological space $X$, because

$$\rho(\tilde{\gamma}(0)) = \gamma(0) = \gamma(1) = \rho(\tilde{\gamma}(1)).$$

Therefore there exists some element $g$ of $G$ such that $\tilde{\gamma}(1) = \varphi(g)(b_0)$. This element $g$ is uniquely determined, because the group $G$ acts freely on $X$. Moreover the value of $g$ is determined by the based homotopy class $[\gamma]$ of $\gamma$ in $\pi_1(X/G, c_0)$. Indeed it follows from Proposition 6.1 that if $\sigma$ is a loop in $X/G$ based at $c_0$, if $\tilde{\sigma}$ is the lift of $\sigma$ starting at $b_0$ (so that $\rho \circ \tilde{\sigma} = \sigma$ and $\tilde{\sigma}(0) = b_0$), and if $[\gamma] = [\sigma]$ in $\pi_1(X/G, c_0)$ (so that $\gamma \simeq \sigma$ rel $\{0, 1\}$), then $\tilde{\gamma}(1) = \tilde{\sigma}(1)$. We conclude therefore that there exists a well-defined function

$$\lambda: \pi_1(X/G, c_0) \to G,$$

which is characterized by the property that $\tilde{\gamma}(1) = \theta_{\lambda([\gamma])}(b_0)$ for any loop $\gamma$ in $X/G$ based at $c_0$, where $\tilde{\gamma}$ denotes the unique path in $X$ for which $\tilde{\gamma}(0) = b_0$ and $\rho \circ \tilde{\gamma} = \gamma$.

Now let $\alpha: [0, 1] \to X/G$ and $\beta: [0, 1] \to X/G$ be loops in $X/G$ based at $c_0$, and let $\tilde{\alpha}: [0, 1] \to X$ and $\tilde{\beta}: [0, 1] \to X$ be the lifts of $\alpha$ and $\beta$ respectively starting at $b_0$, so that $\rho \circ \tilde{\alpha} = \alpha$, $\rho \circ \tilde{\beta} = \beta$ and $\tilde{\alpha}(0) = \tilde{\beta}(0) = b_0$. Then
Then the path \( \theta_{\lambda(\alpha,\beta)}(b_0) \) is also a lift of the loop \( \beta \), and is the unique lift of \( \beta \) starting at \( \tilde{\alpha}(1) \). Let \( \alpha \cdot \beta \) be the concatenation of the loops \( \alpha \) and \( \beta \), where

\[
(\alpha,\beta)(t) = \begin{cases} 
\alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\
\beta(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases}
\]

Then the unique lift of \( \alpha \cdot \beta \) to \( X \) starting at \( b_0 \) is the path \( \sigma: [0,1] \to X \), where

\[
\sigma(t) = \begin{cases} 
\tilde{\alpha}(2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\
\theta_{\lambda(\alpha\beta)}(\tilde{\beta}(2t - 1)) & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases}
\]

It follows that

\[
\theta_{\lambda(\alpha,\beta)}(b_0) = \theta_{\lambda(\alpha,\beta)}(b_0) = \sigma(1) = \theta_{\lambda(\alpha\beta)}(1) = \theta_{\lambda(\alpha)}\theta_{\lambda(\beta)}(b_0).
\]

Consequently \( \lambda([\alpha],[\beta]) = \lambda([\alpha])\lambda([\beta]) \). Thus the function

\[
\lambda: \pi_1(X/G, c_0) \to G
\]

is a homomorphism.

Let \( g \in G \). Then there exists a path \( \alpha \) in \( X \) from \( b_0 \) to \( \theta_g(b_0) \), because the space \( X \) is path-connected. Then \( \rho \circ \alpha \) is a loop in \( X/G \) based at \( c_0 \), and \( g = \lambda([\rho \circ \alpha]) \). This shows that the homomorphism \( \lambda \) is surjective.

Let \( \gamma: [0,1] \to X/G \) be a loop in \( X/G \) based at \( c_0 \). Suppose that \( \gamma \in \ker \lambda \). Then \( \tilde{\gamma}(1) = \theta_e(b_0) = b_0 \), and therefore \( \tilde{\gamma} \) is a loop in \( X \) based at \( b_0 \). Moreover \( \gamma = \rho_\#[\gamma] \). Consequently \( \gamma \in \rho_\#(\pi_1(X, b_0)) \). On the other hand, if \( \gamma \in \rho_\#(\pi_1(X, b_0)) \) then \( \gamma = \rho \circ \tilde{\gamma} \) for some loop \( \tilde{\gamma} \) in \( X \) based at \( b_0 \) (see Proposition 6.3). But then \( b_0 = \tilde{\gamma}(1) = \theta_{\lambda(\gamma)}(b_0) \), and therefore \( \lambda([\gamma]) = e \), where \( e \) is the identity element of \( G \). Thus \( \ker \lambda = \rho_\#(\pi_1(X, b_0)) \), as required.

Corollary 6.11 Let \( G \) be a group acting freely and properly discontinuously on a path-connected topological space \( X \), let \( \rho: X \to X/G \) be the quotient map from \( X \) to the orbit space \( X/G \), and let \( b_0 \) be a point of \( X \). Then \( \rho_\#(\pi_1(X, b_0)) \) is a normal subgroup of the fundamental group \( \pi_1(X/G, c_0) \) of the orbit space, and

\[
\frac{\pi_1(X/G, c_0)}{\rho_\#(\pi_1(X, b_0))} \cong G.
\]

Proof The subgroup \( \rho_\#(\pi_1(X, b_0)) \) is the kernel of the homomorphism

\[
\lambda: \pi_1(X/G, c_0) \to G
\]

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characterized by the property that \( \tilde{\gamma}(1) = \theta_{\lambda(\tilde{\gamma})}(b_0) \) for any loop \( \gamma \) in \( X/G \) based at \( c_0 \), where \( \tilde{\gamma} \) denotes the unique path in \( X \) for which \( \tilde{\gamma}(0) = b_0 \) and \( \rho \circ \tilde{\gamma} = \gamma \). The image of \( \pi_1(X, b_0) \) under the homomorphism \( \rho_\# \) of fundamental groups induced by the quotient map \( \rho \) is therefore a normal subgroup of \( \pi_1(X/G, c_0) \), because the kernel of any homomorphism is a normal subgroup. The homomorphism \( \lambda \) is surjective, and the image of any group homomorphism is isomorphic to the quotient of its domain by its kernel. The result follows.

**Corollary 6.12** Let \( G \) be a group acting freely and properly discontinuously on a simply connected topological space \( X \), let \( \rho: X \to X/G \) be the quotient map from \( X \) to the orbit space \( X/G \), and let \( b_0 \) be a point of \( X \), and let \( c_0 = \rho(b_0) = [b_0]_G \). Then \( \pi_1(X/G, c_0) \cong G \).

**Proof** This is a special case of Corollary 6.11.