An Overview of Geometry for Post-primary School Mathematics in Ireland
Work in Progress

David R. Wilkins

January 28, 2020

1 An Account of the Geometry Syllabus for Post-primary School Mathematics

1.1 Points and Lines

PPG Notation 1 *We denote points by roman capital letters A, B, C, etc., and lines by lower-case roman letters l, m, n, etc.*

PPG Axiom 1 (Two Points Axiom)
There is exactly one line through any two given points.

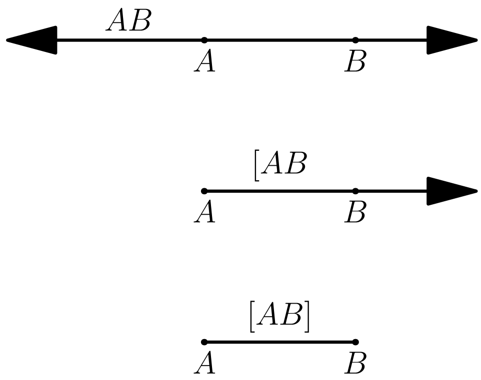
(We denote the line through points A and B by AB.)

Remark The *Two Points Axiom* states a property of the geometry of the flat Euclidean plane that is not satisfied in spherical geometry. Consider a perfect sphere centred on a point in three-dimensional (Euclidean) space. Each plane passing through the centre of the sphere intersects the sphere in a circle. The circles obtained in this fashion are known as **great circles**. The great circles are the analogue, in spherical geometry, of the straight lines of plane Euclidean geometry. Two points on the sphere are said to be **antipodal** if the line joining those two points in three-dimensional space passes through the centre of the sphere. Given two points of the sphere that are not antipodal, there exists a unique great circle on the sphere that passes through the two points. However if two points on the sphere are antipodal, then there are an infinite number of great circles that pass through both of those points.

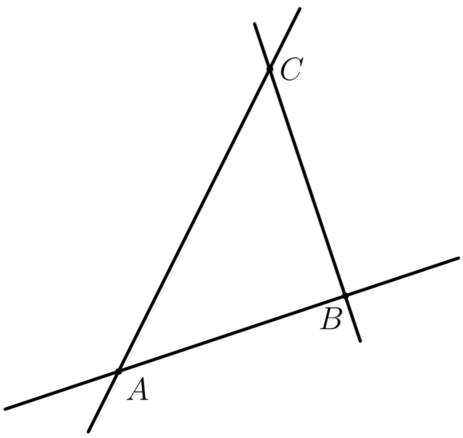
We may choose a pair of antipodal points on the sphere, and consider them to represent the *North Pole* and the *South Pole* of the sphere. The straight line joining these “poles” then passes through the centre of the sphere. The plane through the centre of the sphere perpendicular to the line joining the North and South Poles then cuts out a great circle on the sphere: this great circle represents the *equator* of the sphere. Each point on the equator then determines a great circle, namely the unique great circle on the sphere that passes through the North Pole, the South Pole and the given point on the equator of the sphere.

PPG Definition 1 The **line segment** $[AB]$ is the part of the line AB between A and B (including the endpoints). The point A divides the line AB into two pieces, called **rays**. The point A lies between all points of one ray and all points of the other. We denote the ray that starts at A and passes through B by $[AB$.

Rays are sometimes referred to as **half-lines**.



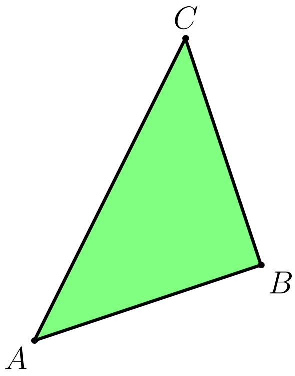
Three points usually determine three different lines.



PPG Definition 2 If three or more points lie on a single line, we say that they are **collinear**.

PPG Definition 3 Let A, B and C be points that are not collinear. The **triangle** $\triangle ABC$ is the piece of the plane enclosed by the three line segments $[AB]$, $[BC]$ and $[CA]$. The segments are called its **sides**, and the points are called its **vertices**.

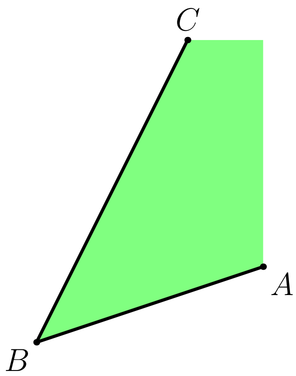
(A single point A, B or C at a corner of a triangle $\triangle ABC$ is referred to as a **vertex** of that triangle. The plural of **vertex** is **vertices**.)



1.2 Interiors of Angles and Triangles

Definition Let A, B and C be non-collinear points in a given plane. We define the *interior* of the *angle* $\angle ABC$ to consist of all points of the plane that do not lie on either of the lines AB or BC but lie on the same side of AB as the point C and on the same side of BC as the point A.

Depicting points of the plane in the usual fashion, the interior of an angle $\angle ABC$ formed by three collinear points is represented by the region shaded in green in the diagram below.



1.3 The Ruler Axiom

We denote the set of all **real numbers** by $\mathbb{R}$.

PPG Definition 4 We denote the **distance** between the points A and B by $|AB|$. We define the **length** of the segment $[AB]$ to be $|AB|$.

PPG Axiom 2 (Ruler Axiom)
The distance between points has the following properties:—

1. *the distance $|AB|$ is never negative;*
2. $|AB| = |BA|$*;*
3. *if C lies on AB, between A and B, then $|AB| = |AC| + |CB|$;*
4. *(marking off a distance) given any ray from A, and given any real number $k \geq 0$, there is a unique point B on the ray whose distance from A is k.*

PPG Definition 5 The **midpoint** of the segment $[AB]$ is the point M of the segment with

$$|AM| = |MB| = \frac{|AB|}{2}.$$

1.4 Angles

PPG Definition 6 A subset of the plane is **convex** if it contains the whole segment that connects any two of its points.

The document *Geometry for Post-primary School Mathematics*, though not purporting to give a formal definition of angles, characterizes an **angle** as being specified by the following data:

1. A unique point A of the plane, referred to as the **vertex** of the angle;
2. two rays $[AB$ and $[AC$, both starting at the vertex, and referred to as the **arms** of the angle;
3. a piece of the plane, referred to as the **inside** of the triangle.

Angles are classified as *null angles*, *ordinary angles*, *straight angles*, *reflex angles* and *full angles*. Unless otherwise specified, angles can be presumed to be ordinary angles.

PPG Definition 7 An angle is a **null angle** if its arms coincide with one another and its inside is the empty set.

PPG Definition 8 An angle is an **ordinary angle** if its arms are not on one line, and its inside is a convex set.

PPG Definition 9 An angle is a **straight angle** if its arms are the two halves of one line, and its inside is one of the sides of that line.

PPG Definition 10 An angle is a **reflex angle** if its arms are not on one line and its inside is not a convex set.

PPG Definition 11 An angle is a **full angle** if its arms coincide with one abother and its inside is the rest of the plane.

PPG Definition 12 Given three noncollinear points A, B and C, the ordinary angle with arms $[AB$ and $[AC$ by $\angle BAC$ (and also by $\angle CAB$. The notation $\angle BAC$ is also used to refer to straight angles, where A, B and C are collinear and A lies between B and C.

When the notation $\angle BAC$ to refer to a straight angle with legs $[AB$ and $[AC$, the interior of the straight angle will consist of one of the two sides of the line formed by $[AB$ and $[AC$. Context may then determine *which* side is indicated.

Greek letters α, β, γ etc. may be used to refer to angles.

1.5 The Protractor Axiom

PPG Notation 2 *We denote the number of* ***degrees*** *in an angle $\angle BAC$ or α by the symbol $|\angle BAC|$, or $|\angle\alpha|$, as the case may be.*

PPG Axiom 3 (Protractor Axiom)
The number of degrees in an angle (also known as its degree-measure) is always a number between $0°$ and $360°$. The number of degrees of an ordinary angle is less than $180°$. It has these properties:—

1. *a straight angle has $180°$;*

2. *given a ray $[AB$, and a number d between 0 and 180, there is exactly one ray from A on each side of the line AB that makes an (ordinary) angle having d degrees with the ray $[AB$;*

3. *if D is a point inside an angle $\angle BAC$ then*

$$|\angle BAC| = |\angle BAD| + |\angle DAC|.$$

Null angles are assigned $0°$, full angles $360°$, and reflex angles have more than $180°$. To be more exact, if A, B, and C are noncollinear points, then the reflex angle "outside" the angle $\angle BAC$ is assigned an angle-measure of $360° - |\angle BAC|$.

PPG Definition 13 The ray $[AD$ is the **bisector** of the angle $\angle BAC$ if

$$|\angle BAD| = |\angle DAC| = \frac{|\angle BAC|}{2}.$$

PPG Definition 14 A **right angle** is an angle of exactly $90°$.

PPG Definition 15 An angle is **acute** if it has less than $90°$, and **obtuse** if it has more than $90°$.

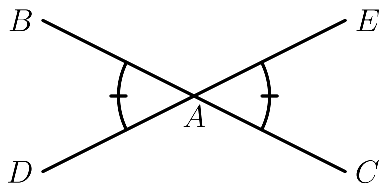
PPG Definition 16 If $\angle BAC$ is a straight angle, and D is off the line BC, then $\angle BAD$ and $\angle DAC$ are called **supplementary angles**. They add to $180°$.

PPG Definition 17 When two lines AB and AC cross at a point A, they are **perpendicular** if $\angle BAC$ is a right angle.

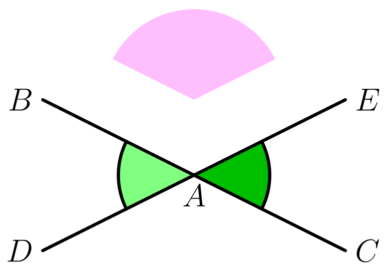
1.6 Vertically-opposite Angles

PPG Definition 18 Let A lie between B and C on the line BC, and also between D and E on the line DE. Then $\angle BAD$ and $\angle CAE$ are called **vertically-opposite** angles.

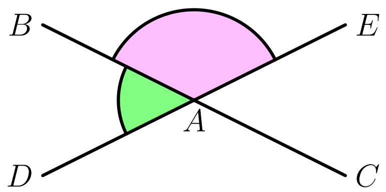
PPG Theorem 1 (Vertically-opposite Angles)
Vertically opposite angles are equal in measure.



Visual Proof Consider the effect of adding the top (pink) angle to the (green) angles to the left and to the right.



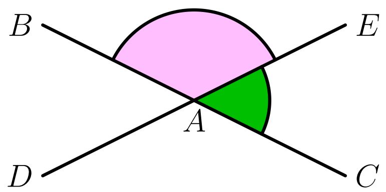
The left angle and the top angle are supplementary angles whose sum is equal to 180°.



In symbols,

$$|\angle BAD| + |\angle BAE| = 180^\circ.$$

Similarly the right angle and the top angle are supplementary angles whose sum is equal to 180°.



In symbols,

$$|\angle CAE| + |\angle BAE| = 180^\circ.$$

Thus

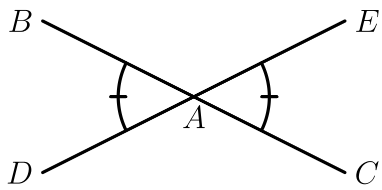
$$|\angle BAD| + |\angle BAE| = |\angle CAE| + |\angle BAE|.$$

Subtracting the angle BAE from both sides, we find that

$$|\angle BAD| = |\angle CAE|,$$

as required. ∎

Symbolic Proof Let the line BC intersect the line DE at the point A, as shown in the diagram below.



Then $\angle BAD$ and $\angle BAE$ are supplementary angles whose sum is a straight angle, and therefore

$$|\angle BAD| + |\angle BAE| = 180^\circ.$$

Similarly $\angle CAE$ and $\angle BAE$ are supplementary angles whose sum is a straight angle, and therefore

$$|\angle CAE| + |\angle BAE| = 180^\circ.$$

It follows that

$$|\angle BAD| + |\angle BAE| = |\angle CAE| + |\angle BAE|.$$

Subtracting $|\angle BAE|$ from both sides, we find that

$$|\angle BAD| = |\angle CAE|,$$

as required. ∎

Remark PPG Theorem 1 corresponds to Proposition 15 in Book I of Euclid's *Elements*. It is proved in the same fashion. Euclid makes use of the result of Proposition 13 of that book, which ensures that supplementary angles add up to two right angles. Euclid also makes use of *Common Notion 3*, which asserts that *if equals be subtracted from equals, the remainders are equal.*

1.7 Congruent Triangles

PPG Definition 19 Let A, B, C be a triple of non-collinear points, and let A', B' and C' be another triple of non-collinear points. We say that the triangles $\angle ABC$ and $\angle A'B'C'$ are **congruent** if all the sides and angles of one are equal to the corresponding sides and angles of the other, i.e., $|AB| = |A'B'|$, $|BC| = |B'C'|$, $|CA| = |C'A'|$, $|\angle ABC| = |\angle A'B'C'|$, $|\angle BCA| = |\angle B'C'A'|$, and $|\angle CAB| = |\angle C'A'B'|$.

PPG Notation 3 *The names of angles in a triangle can be abbreviated, in contexts where no ambiguity would result, by labelling them by the names of the vertices. Thus, in a triangle $\triangle ABC$, the angle $\angle CAB$ may be denoted simply by $\angle A$ (provided of course that there is no* other *triangle present that has A as one of its vertices).*

PPG Axiom 4 (Congruence Rules) *Let $\triangle ABC$ and $\triangle A'B'C'$ be triangles. If*

*(1) (**SAS Congruence Rule**) $|AB| = |AB|$, $|AC| = |A'C'|$ and $|\angle A| = |\angle A'|$,*

or

*(2) (**ASA Congruence Rule**) $|BC| = |B'C'|$, $|\angle B| = |\angle B'|$ and $|\angle C| = |\angle C'|$,*

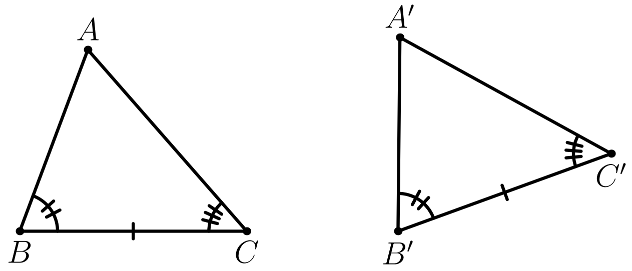
or

*(3) (**SSS Congruence Rule**) $|AB| = |A'B'|$, $|BC| = |B'C'|$ and $|CA| = |C'A'|$,*

then the triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent.

1.8 The ASA Congruence Rule

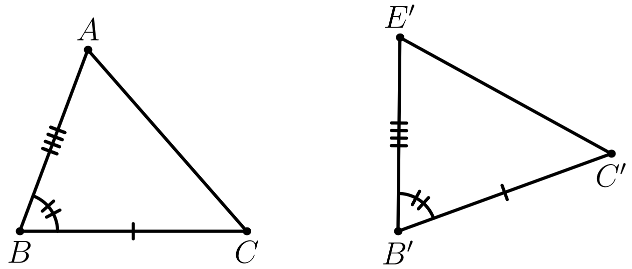
Let $\triangle ABC$ and $\triangle A'B'C'$ be triangles. Suppose that $|BC| = |B'C'|$, $|\angle B| = |\angle B'|$ and $|\angle C| = |\angle C'|$.



Now, in this situation, the ASA Congruence Rule (PPG Axiom 4, (2)) requires that the triangles $\triangle ABC$ and $\triangle A'B'C'$ be congruent. We shall show that, in fact, it is not necessary to include the ASA Congruence Rule as a separate requirement in the statement of PPG Axiom 4, because, as we shall explain, it is possible to justify the rule through appropriate applications of the *Two Points Axiom* (PPG Axiom 1), the *Ruler Axiom* (PPG Axiom 2), the *Protractor Axiom* (PPG Axiom 3) and the following axiom:—

> (Hilbert's SAS Axiom.) *Let $\triangle ABC$ and $\triangle A'B'C'$ be triangles. If $|AB| = |AB|$, $|AC| = |A'C'|$ and $|\angle A| = |\angle A'|$, then $|\angle B| = |\angle B'|$ and $|\angle C| = |\angle C'|$.*

To show this, we first note that the *Ruler Axiom* (PPG Axiom 2) ensures the existence of a point E' on the ray $[B'A'$ determined so that $|AB| = |E'B'|$. It then follows from the *Protractor Axiom* (PPG Axiom 3) that $|\angle E'B'C'| = |\angle A'B'C'|$ because the points A' and E', both lie on the same ray starting at the point B'. It follows that, in the triangles $\triangle ABC$ and $\triangle E'B'C'$, the sides and angles satisfy $|AB| = |E'B'|$, $|BC| = |B'C'|$, and $|\angle ABC| = |\angle E'B'C'|$.



Hilbert's SAS Axiom, stated above, therefore ensures that $|\angle BCA| = |\angle B'C'E'|$.

But then

$$|\angle B'C'E'| = |\angle BCA| = |\angle B'C'A'|.$$

Moreover the points A' and E' lie on the same side of the line $B'C'$. It therefore follows from the *Protractor Axiom* (PPG Axiom 3) that the ray $[C'E'$ coincides with the ray $[C'A'$. Thus the point E' lies on the line $C'A'$.

Now the point E' was determined so as to lie on the line $B'A'$. We conclude therefore that the lines $C'A'$ and $B'A'$ intersect at the point E'.

But the lines $C'A'$ and $B'A'$ intersect at the point A, and moreover the Two Points Axiom (PPG Axiom 1) ensures that two distinct lines cannot intersect at more than one point. Therefore the point E' must coincide with the point A'. It follows from this that $|A'B'| = |AB|$. It then follows from *Hilbert's SAS Axiom*, stated above, that $|\angle BAC| = |\angle B'A'C'|$.

We have now shown that if, in two triangles $\triangle ABC$ and $\triangle A'B'C'$, $|BC| = |B'C'|$, $|\angle B| = |\angle B'|$ and $|\angle C| = |\angle C'|$, then $|AB| = |A'B'|$ and $|\angle A| = |\angle A'|$. Applying this result with B, B', C and C' replaced by C, C', B and B' respectively, we can also conclude that $|AC| = |A'C'|$. We can conclude therefore that if, in two triangles $\triangle ABC$ and $\triangle A'B'C'$, $|BC| = |B'C'|$, $|\angle B| = |\angle B'|$ and $|\angle C| = |\angle C'|$, then the triangle $\triangle ABC$ and $\triangle A'B'C'$ are congruent. This demonstrates that the *ASA Congruence Rule* does indeed follow as a consequence of the the *Two Points Axiom* (PPG Axiom 1), the *Ruler Axiom* (PPG Axiom 2), the *Protractor Axiom* (PPG Axiom 3) and *Hilbert's SAS Axiom*.

Now *Hilbert's SAS Axiom* can be combined with the ASA Congruence rule to show that if, in triangles $\triangle ABC$ and $\triangle A'B'C'$, $|AB| = |AB|$, $|AC| = |A'C'|$ and $|\angle A| = |\angle A'|$, then $|BC| = |B'C'|$, and therefore the triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent. Indeed *Hilbert's SAS Axiom* ensures that, in these circumstances $|\angle B| = |\angle B'|$. The ASA Congruence rule then ensures that the triangles are congruent. We conclude therefore that both the SAS Congruence Rule and the ASA Congruence Rule are consequences of the *Two Points Axiom* (PPG Axiom 1), the *Ruler Axiom* (PPG Axiom 2), the *Protractor Axiom* (PPG Axiom 3) and *Hilbert's SAS Axiom* stated above.

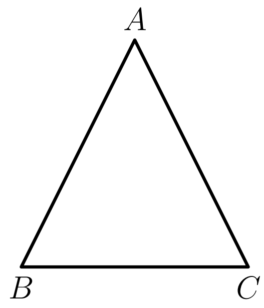
1.9 Right-Angled Triangles

PPG Definition 20 A triangle is called **right-angled** if one of its angles is a right angle. The side opposite the right angle is called the **hypotenuse**.

1.10 Isosceles Triangles

Remark If one angle of a triangle is a right angle then the other two angles are acute angles. This is a consequence of the result that the sum of the three angles of a triangle in the Euclidean plane is equal to 180° (see Theorem 4).

PPG Definition 21 A triangle is called **isosceles** if two sides are equal. It is **equilateral** if three sides are equal. It is **scalene** if no two sides are equal.



PPG Theorem 2 (Isosceles Triangles)

(1) In an isosceles triangle the angles opposite the equal sides are equal.

(2) Conversely, if two angles are equal, then the triangle is isosceles.

Proof Suppose that, in a triangle $\triangle ABC$, the edges with endpoint A satisfy $|AB| = |AC|$. It then follows on applying the SAS Congruence Rule that the triangles $\triangle ABC$ and $\triangle ACB$ are congruent, and therefore $\angle B = \angle C$. This proves (1).

Next suppose that, in a triangle $\triangle ABC$, the angles $\angle B$ and $\angle C$ at the endpoints of the side $[BC]$ satisfy $|\angle B| = |\angle C|$. The ASA Congruence Rule then ensures that the triangles $\triangle ABC$ and $\triangle ACB$ are congruent, and therefore $|AB| = |AC|$. This proves (2), thus completing the proof of the theorem. ∎

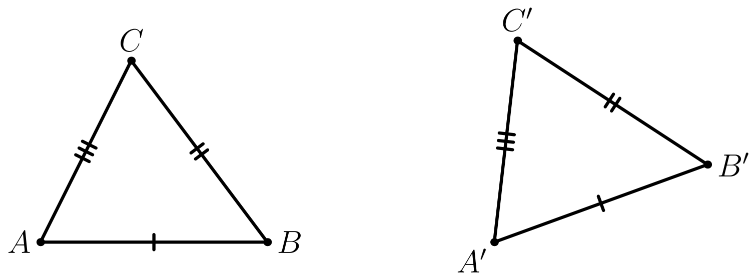
Remark PPG Theorem 2 corresponds to Propositions 5 and 6 in Book I of Euclid's *Elements*. The proof of Proposition 5 given by Euclid is far more elaborate, and became known as the *Pons Asinorum* ("Bridge of Asses"). The Greek mathematician Pappus of Alexandria (4th century C.E.) gave a short proof of (1), essentially the same in conception as that given above. This proof was included in Proclus's *commentaries on Euclid* (see Heath, *The Thirteen Books of Euclid's Elements*, vol. I, p.254). Others, certainly until well into the 19th century considered proofs of the Isosceles Triangle Theorem along these lines to be unsatisfactory, on the basis that the triangles $\triangle ABC$ and $\triangle ACB$ are in fact the *same triangle* (but with the vertices lettered according to different schemes).

1.11 The SSS Congruence Rule

Let $\triangle ABC$ and $\triangle A'B'C'$ be triangles. Suppose that corresponding sides of these triangles are equal, so that $|AB| = |A'B'|$, $|BC| = |B'C'|$ and $|CA| = |C'A'|$, We show how the the *Two Points Axiom* (PPG Axiom 1), the *Ruler Axiom* (PPG Axiom 2), the *Protractor Axiom* (PPG Axiom 3) the *SAS Congruence Rule* and the Isosceles Triangle Theorem (PPG Theorem 2) can be used to prove that the triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent.

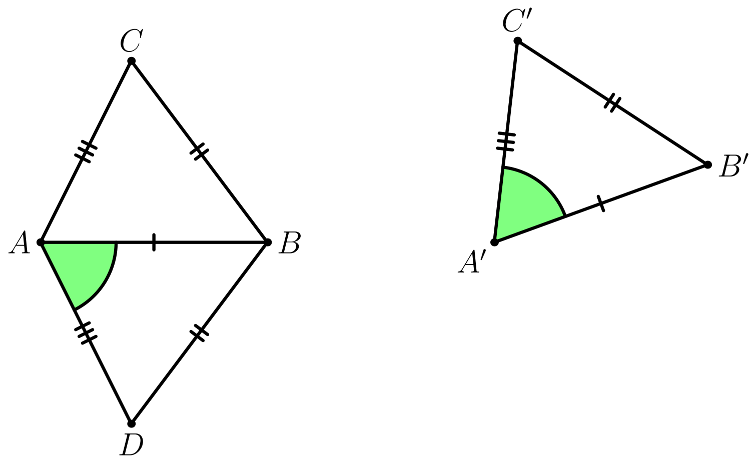
The argument uses a proof strategy attributed to Philo of Byzantium (3rd century B.C.E.).

Let $\triangle ABC$ and $\triangle A'B'C'$ be triangles. Suppose that $|AB| = |A'B'|$, $|BC| = |B'C'|$ and $|CA| = |C'A'|$.

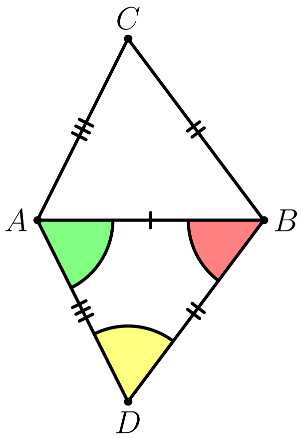


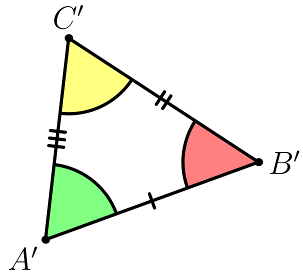
It follows from the *Protractor Axiom* that there exists a ray starting at the point A, on the opposite side of the line AB to the point C, that makes an angle with the line segment $[AB]$ equal to $\angle B'A'C'$. It then follows from the *Ruler Axiom* that there exists a point D on this line segment for which $|AD| = |A'C'|$. Then $|\angle BAD| = |\angle B'A'C'|$ and

$$|AD| = |A'C'| = |AC|.$$



Now the sides of the triangle $\triangle A'B'C'$ meeting at the point A' are equal in length to corresponding sides of the triangle $\triangle ABD$ meeting at the point A, and $|\angle B'A'C'| = |\angle BAD|$. It follows from the SAS Congruence Rule that the triangles $\triangle A'B'C'$ are $\triangle ABD$ congruent, and therefore $|BD| = |A'C'|$, $|\angle ABD| = |\angle A'B'C'|$ and $|\angle ADB| = |\angle A'D'B'|$.





Join the points C and D. The proof of the SSS Congruence Rule now breaks down into five cases as follows:

(1) the line $[CD]$ passes through the point A;

(2) the line $[CD]$ passes through the point B;

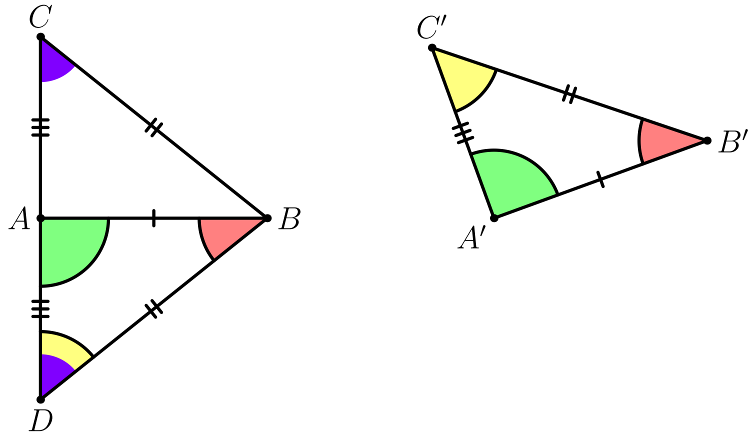
(3) the line $[CD]$ passes between A and B;

(4) the point A lies between the point B and the point where the lines AB and CD intersect;

(5) the point B lies between the point A and the point where the lines AB and CD intersect;

Case (1)

In this case the line segment CD passes through the point A, as shown in the following figure:



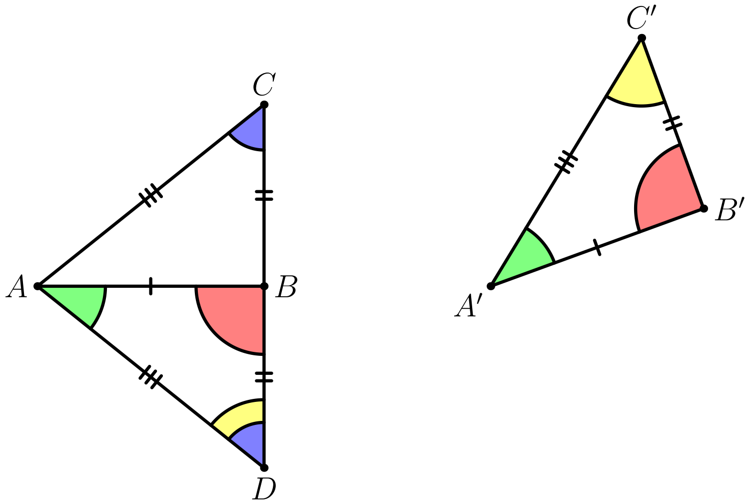
In this case $\triangle BCD$ is an isosceles triangle with $|BC| = |BD|$. Also the triangles $\triangle ABD$ and $\triangle A'B'C'$ are congruent. Therefore

$$|\angle ACB| = |\angle DCB| = |\angle CDB| = |\angle ADB| = |\angle A'C'B'|.$$

It now follows from the SAS Congruence Rule that the triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent.

Case (2)

In this case the line segment CD passes through the point B, as shown in the following figure:



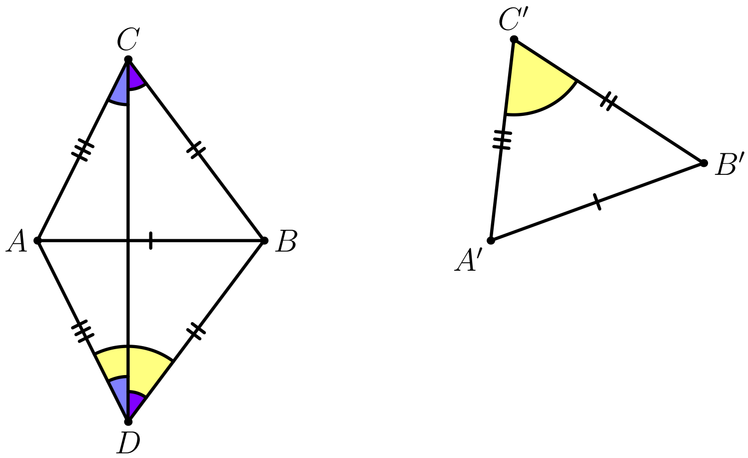
In this case $\triangle ACD$ is an isosceles triangle with $|AC| = |AD|$. Also the triangles $\triangle ABD$ and $\triangle A'B'C'$ are congruent. Therefore

$$|\angle ACB| = |\angle ACD| = |\angle ADC| = |\angle ADB| = |\angle A'C'B'|.$$

It now follows from the SAS Congruence Rule that the triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent.

Case (3)

In this case the line segment CD passes between the points A and B as shown in the following figure:



In this case $\triangle ACD$ and $\triangle BCD$ are both isosceles triangles with $|AC| = |AD|$ and $|BC| = |BD|$. It follows that

$$|\angle ACD| = |\angle ADC| \quad \text{and} \quad |\angle CDB| = |\angle DCB|.$$

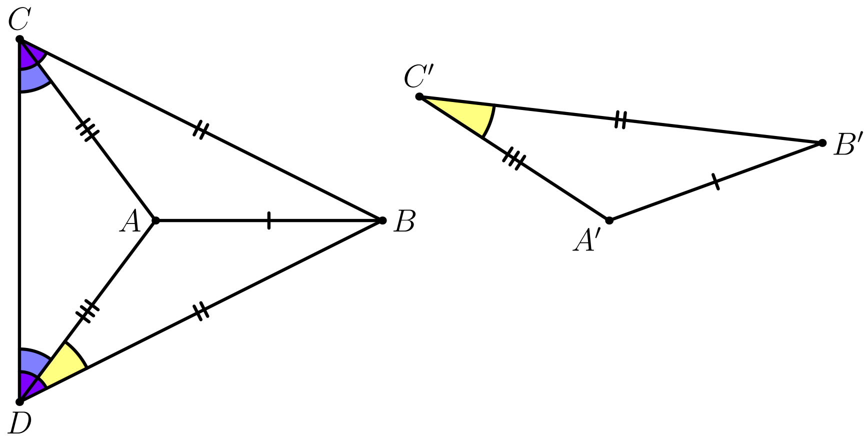
Now D lies in the interior of $\angle ACB$, and C lies in in the interior of $\angle ADB$. Therefore, adding angles, and using the congruence of the triangles $\angle ABD$ and $\angle A'B'C'$, we find that

$$\begin{aligned} |\angle ACB| &= |\angle ACD| + |\angle DCB| \\ &= |\angle ADC| + |\angle CDB| \\ &= |\angle ADB| = |\angle A'C'B'|. \end{aligned}$$

It now follows from the SAS Congruence Rule that the triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent.

Case (4)

In this case the point A lies between the point B and the point where the lines AB and CD intersect, as shown in the following figure:



In this case $\triangle ACD$ and $\triangle BCD$ are both isosceles triangles with $|AC| = |AD|$ and $|BC| = |BD|$. It follows that

$$|\angle ACD| = |\angle ADC| \quad \text{and} \quad |\angle BCD| = |\angle BDC|.$$

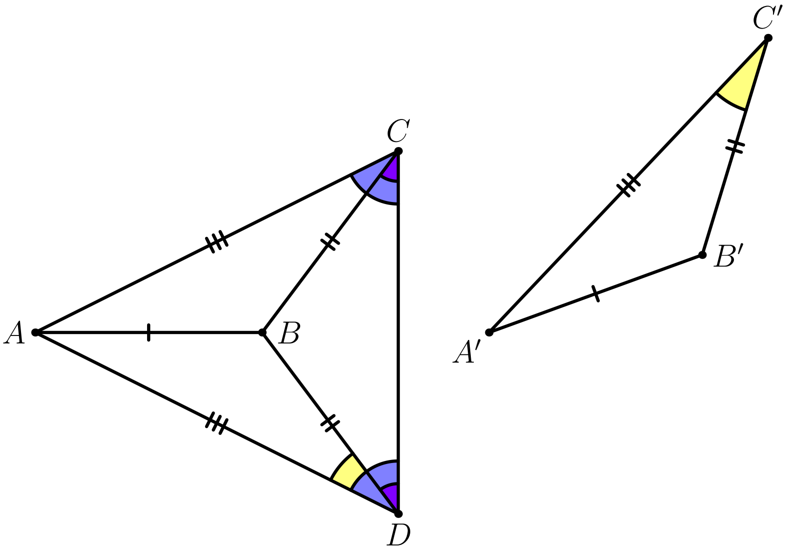
Now A lies in the interior of $\angle BCD$. Therefore, subtracting angles, and using the congruence of the triangles $\angle ABD$ and $\angle A'B'C'$, we find that

$$\begin{aligned} |\angle ACB| &= |\angle DCB| - |\angle DCA| \\ &= |\angle CDB| - |\angle CDA| \\ &= |\angle ADB| = |\angle A'C'B'|. \end{aligned}$$

It now follows from the SAS Congruence Rule that the triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent.

Case (5)

In this case the point B lies between the point A and the point where the lines AB and CD intersect, as shown in the following figure:



In this case $\triangle ACD$ and $\triangle BCD$ are both isosceles triangles with $|AC| = |AD|$ and $|BC| = |BD|$. It follows that

$$|\angle ACD| = |\angle ADC| \quad \text{and} \quad |\angle BCD| = |\angle BDC|.$$

Now B lies in the interior of $\angle ACD$. Therefore, subtracting angles, and using the congruence of the triangles $\angle ABD$ and $\angle A'B'C'$, we find that

$$\begin{aligned} |\angle ACB| &= |\angle ACD| - |\angle BCD| \\ &= |\angle ADC| - |\angle BDC| \\ &= |\angle ADB| = |\angle A'C'B'|. \end{aligned}$$

It now follows from the SAS Congruence Rule that the triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent.

This completes the proof of the SSS Congruence Rule.

1.12 Parallels and the Alternate Angles Theorem

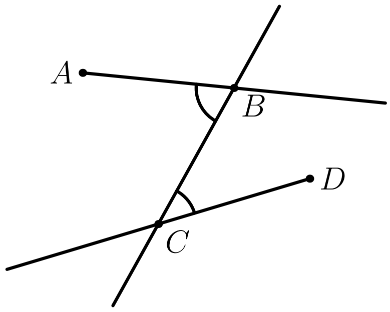
PPG Definition 22 Two lines l and m are **parallel** if they are either identical, or have no common point.

PPG Notation 4 *We write $l \parallel m$ for "l is parallel to m."*

PPG Axiom 5 (Axiom of Parallels) *Given any line l and a point P, there is exactly one line through P that is parallel to l.*

PPG Definition 23 If l and m are lines, then a line n is called a **transversal** of l and m if it meets them both.

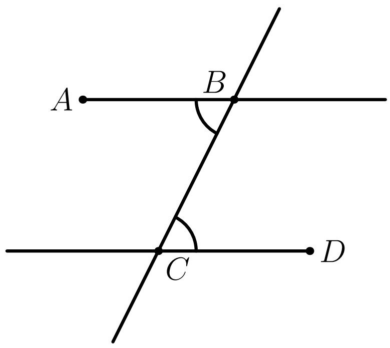
PPG Definition 24 Given two lines AB and CD and a transversal BC of them, as depicted in the figure immmediately below, the angles $\angle ABC$ and $\angle BCD$ are called **alternate angles**.



PPG Theorem 3 (Alternate Angles)
Suppose that A and D are on opposite sides of the line BC.

(1) If $|\angle ABC| = |\angle BCD|$, then the line AB is parallel to the line CD. In other words, if a transversal makes equal alternate angles on two lines, then the lines are parallel.

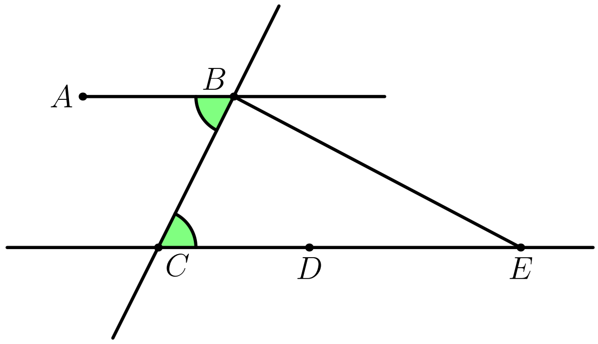
(2) Conversely, if the line AB is parallel to the line CD, then $|\angle ABC| = |\angle BCD|$. In other words, if two lines are parallel, then any transversal will make equal alternate angles with them.



Proof Let A, B, C and D be distinct points, where the points A and D lie on opposite sides of the line BC. We begin the proof of the Alternate Angles Theorem by showing that if the alternate angles $\angle ABC$ and $\angle BCD$ are equal, and if E is a point that lies on the ray $[CD$ starting from C that passes through D, then

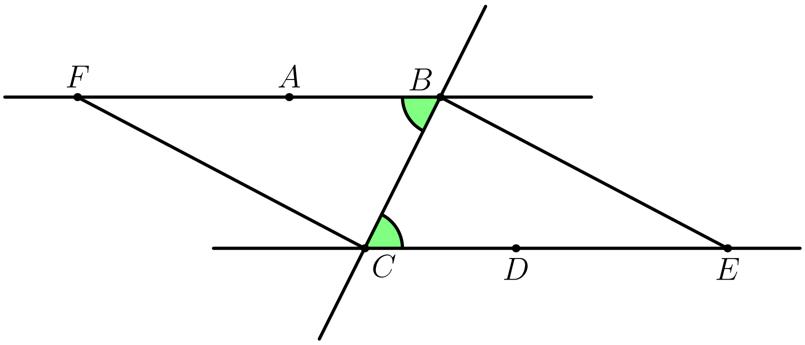
$$|\angle ABC| + |\angle CBE| \neq 180^\circ.$$

Suppose therefore that $|\angle ABC| = |\angle BCD|$ and that the point E lies on the line segment starting at C that passes through the point D.

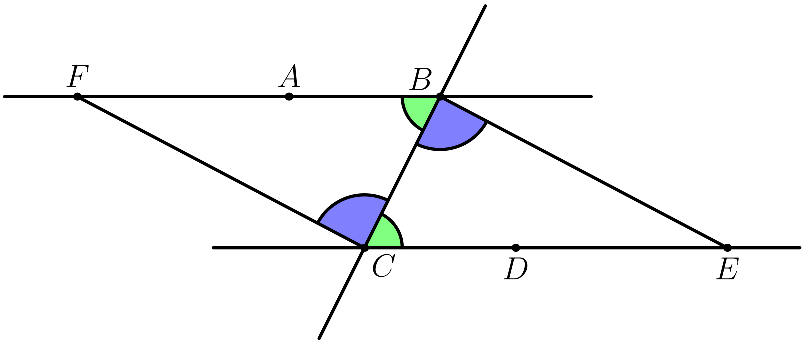


The *Ruler Axiom* (PPG Axiom 2) ensures the existence of a point F on the ray $[BA$ starting at A and passing through the point A characterized by the property that $|FB| = |CE|$.

Consider the triangles $\triangle BCE$ and $\triangle CBF$.



The sides $[BC]$ and $[CE]$ of the triangle $\triangle BCE$ are equal in length the corresponding sides $[CB]$ and $[BF]$ of the triangle $\triangle CBF$, and moreover $|\angle BCE| = |\angle CBF|$. Applying the SAS Congruence Rule (PPG Axiom 4), we conclude that the triangles $\triangle BCE$ and $\triangle CBF$ are congruent, and therefore $|\angle EBC| = |\angle FCB|$.



The results obtained show that the sum of the angles of the triangles $\triangle BCE$ and $\triangle BCF$ that adjoin the common vertex at B is equal to the sum of the angles of those triangles that adjoin the common vertex at C. Specifically

$$|\angle FBC| + |\angle CBE| = |\angle ECB| + |\angle BCF|.$$

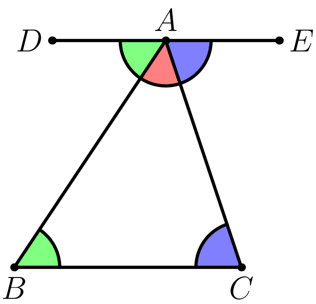
It follows that if $\angle FBC$ and $\angle CBE$ were supplementary angles whose sum were equal to 180°, then $\angle ECB$ and $\angle BCF$ would also be supplementary angles whose sum was equal to 180°, and thus there would exist two distinct lines passing through the points E and F: one of those lines would also pass through the point B, and the other would also pass through the point C. But this would contradict the *Two Points Axiom* (PPG Axiom 1). Therefore the angles $\angle FBC$ and $\angle CBE$ cannot be supplementary angles whose sum is equal to 180°, and similarly $\angle ECB$ and $\angle BCF$ cannot be supplementary angles whose sum is equal to 180°. Thus when the alternate angles $\angle ABC$ and $\angle BCD$ are equal, the line AB cannot contain any segment with endpoint A whose other endpoint lies on the line CD and on the same side of the transversal BC as the point D. Thus if $|\angle ABC| = |\angle BCD|$ then the lines AB and CD cannot intersect at any point of the plane that lies on the same side of the transversal BC as the point D. The same argument, with A, B, C, D, E and F replaced by D, C, B, A, F and E respectively shows that if $|\angle ABC| = |\angle BCD|$ then the lines AB and CD cannot intersect at any point of the plane that lies on the same side of the transversal BC as the point A. Combining these results, we conclude that if $|\angle ABC| = |\angle BCD|$ then the lines AB and CD must be parallel.

Conversely we must show that if the lines AB and CD are parallel to one another then the alternate angles $\angle ABC$ and $\angle BCD$ formed by the transversal CD are equal. To prove this, we make use of the Axiom of Parallels (PPG Axiom 5). A straightforward application of the *Protractor Axiom* (PPG Axiom 3) enables us to conclude that there exists a point G on the same side of the transversal BC as the point D for which CG for which $\angle ABC = \angle BCG$. The result already obtained then guarantees that the lines AB and CG are parallel. Now the Axiom of Parallels (PPG Axiom 5) ensures that there cannot exist more than one line through the point C that is parallel to the line AB. It follows that if CD is parallel to AB then the lines CD and CG coincide, and therefore the points C, D and G are collinear. But then $\angle BCD = \angle BCG$, and therefore $|\angle ABC| = |\angle BCD|$. This completes the proof. ∎

1.13 Basic Properties of Triangles

PPG Theorem 4 (Sum of the Interior Angles of a Triangle)
The angles in any triangle add up to $180°$.

Proof Let $\triangle ABC$ be a triangle. Let points D and E lie on the line through the vertex A of the triangle parallel to the side $[BC]$, taken so that the point A lies between D and E, the point D lies on the opposite side of AB to the point C, and the point E lies on the opposite side of the line AC to the point B.



The line DE is parallel to the line BC. It follows from the Alternate Angles Theorem (PPG Theorem 3) that

$$|\angle DAB| = |\angle ABC| = |\angle B| \quad \text{and} \quad |\angle CAE| = |\angle BCA| = |\angle C|.$$

Now the sum of the three angles $|\angle DAB|$, $|\angle BAC|$ and $|\angle CAE|$ at A is equal to a straight angle. It follows that

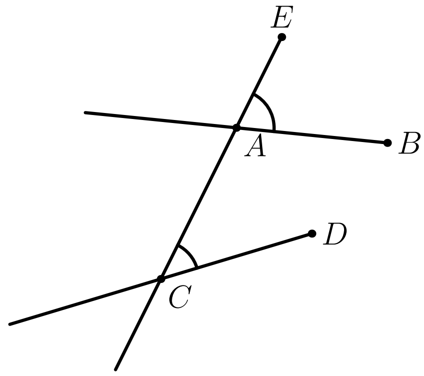
$$|\angle A| + |\angle B| + |\angle C| = |\angle BAC| + |\angle DAB| + |\angle EAC| = 180°,$$

as required. ∎

Remark Heath observes[1] that the proof strategy used here to prove that the sum of the three angles of a triangle is equal to two right angles was attributed by the ancient Greeks to the Pythagoreans. It is recorded in the commentaries of Proclus.

[1]T.L. Heath, *The Thirteen Books of Euclid's Elements*, Volume 1, p. 317, 320.

PPG Definition 25 Given two lines AB and CD and a transversal AE of them, as depicted in the figure immediately below, the angles $\angle EAB$ and $\angle ACD$ are called **corresponding angles** (with respect to the two lines and the given transversal).

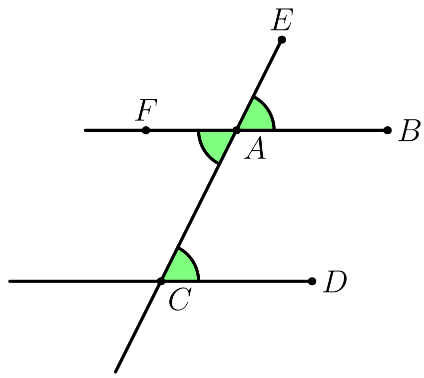


PPG Theorem 5 (Corresponding Angles)
Two lines are parallel if and only if, for any transversal, corresponding angles are equal.

Proof Suppose that the corresponding angles $\angle EAB$ and $\angle ACD$ are equal. Let F be a point on the line AB that lies on the opposite side of A to the point B. The Vertical Angles Theorem (PPG Theorem 1) then ensures that the vertical angles $\angle EAB$ and $\angle CAF$ are equal. Therefore

$$|\angle ACD| = |\angle EAB| = \angle CAF.$$

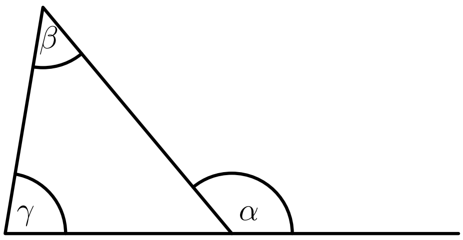
Moreover the angles $\angle ACD$ and $\angle CAF$ are alternating angles.



It follows from the Alternate Angles Theorem (PPG Theorem 3) that the lines AB and CD are parallel to one another.

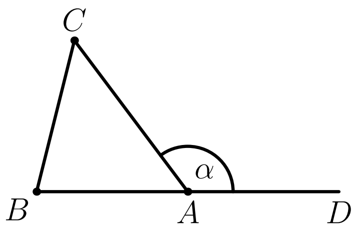
Conversely suppose that the lines AB and CD are parallel to one another. It then follows from the Alternate Angles Theorem (PPG Theorem 3) $|\angle ACD| = |\angle CAF|$. The Vertical Angles Theorem (PPG Theorem 1) ensures that $|\angle EAB| = |\angle CAF|$. Thus the corresponding angles $\angle ACD$ and $\angle EAB$ are both equal to $\angle CAF$. They are therefore equal to one another. This completes the proof. ∎

PPG Definition 26 In the figure immediately below, the angle α is called an **exterior angle** of the triangle, and the angles β and γ are called (corresponding) **interior opposite angles**.[2]



PPG Theorem 6 (Exterior Angle)
Each exterior angle of a triangle is equal to the sum of the interior opposite angles.

Proof (using PPG Theorem 4) Let $\triangle ABC$ be a triangle. We denote the angles $\angle CAB$, $\angle ABC$ and $\angle BCA$ by $\angle A$, $\angle B$ and $\angle C$ respectively. Let the line segment $\angle BA$ be produced beyond A to a point D, so that the point A lies between B and D. We denote by α the exterior angle $\angle DAC$ at the point A.



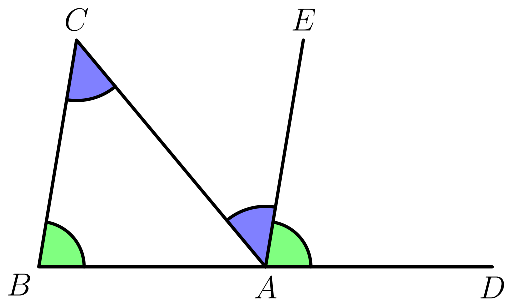
Now the angles $\angle A$ and α are supplementary angles, and therefore $\angle A + \alpha = 180°$. Moreover it follows from PPG Theorem 4 that the sum of the three angles of the triangle $\triangle ABC$ is also equal to 180°. Therefore

$$|\angle B| + |\angle C| + |\angle A| = 180° = \alpha + |\angle A|,$$

and consequently $|\angle B| + |\angle C| = \alpha$, as required. ∎

[2] The phrase **interior remote angles** is sometimes used instead of **interior opposite angles**.

Euclid's Proof (using Theorems 3 and 5) Let $\triangle ABC$ be a triangle. Produce the side $\triangle BA$ of the triangle to a point D, so that the point A lies between the points B and D. Also take a point E distinct from A on the line through the point A parallel to BC, so that the point E lies on the same side of the line BA as the point C.



The points B and E then lie on opposite sides of the line AC. It follows from the Alternate Angles Theorem (PPG Theorem 3) that $|\angle C| = |\angle BCA| = |\angle CAE|$. Also it follows from the Corresponding Angles Theorem (PPG Theorem 5) that $|\angle B| = |\angle ABC| = |\angle DAE|$. Now the point E lies in the interior of the angle $\angle DAC$. It follows that

$$|\angle DAC| = |\angle DAE| + |\angle CAE| = |\angle B| + |\angle C|,$$

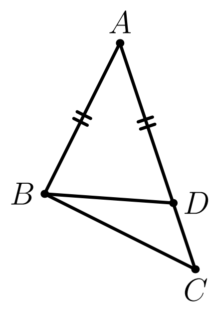
as required. ∎

PPG Theorem 7

(1) In a triangle $\triangle ABC$, suppose that $|AC| > |AB|$. Then $|\angle ABC| > |\angle ACB|$. In other words, the angle opposite the greater of the two sides is greater than the angle opposite the lesser side.

(2) Conversely, if $|\angle ABC| > |\angle ACB|$, then $|AC| > |AB|$. In other words, the side opposite the greater of the two angles is greater than the side opposite the lesser angle.

Proof We first prove (1). Let $\triangle ABC$ be a triangle. Suppose that $|AC| > |AB|$. There then exists a point D on the side $[AC]$ of the triangle for which $|AD| = |AB|$. This point D lies between A and C, as shown in the following figure:



The Isosceles Triangle Theorem (PPG Theorem 2) ensures that $|\angle ABD| = |\angle ADB|$. Now it follows from the Exterior Angle Theorem (PPG Theorem 6), applied to the triangle $\triangle DCB$, that

$$|\angle ADB| = |\angle DCB| + |\angle DBC| > |\angle DCB|.$$

Moreover $\angle DCB = \angle ACB$, because the point D lies between A and C. Thus $|\angle ADB| > |\angle ACB|$.

Now $|\angle ABD| < |\angle ABC|$, because the point B lies between A and C and therefore lies in the interior of the angle $\angle ABC$. Putting these inequalities and equalities together, we find that

$$|\angle ACB| < |\angle ADB| = |\angle ABD| < |\angle ABC|.$$

This completes the proof of (1).

We now prove (2) using the result (1) just obtained. Let $\triangle ABC$ be a triangle. Suppose that $|AB|C > |AC|B$.

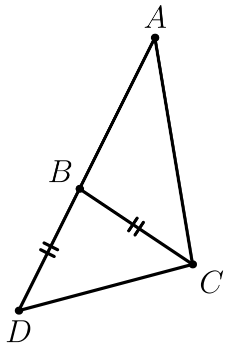
If it were the case that $|AC| = |AB|$ then the Isosceles Triangle Theorem (PPG Theorem 2) would ensure that $|AB|C = |AC|B$, contrary to assumption. Therefore it cannot be the case that $|AC| = |AB|$.

If it were the case that $|AC| < |AB|$ then the result (1) just proved would ensure that $|AB|C < |AC|B$, contrary to assumption. Therefore it cannot be the case that $|AC| < |AB|$.

Given that the possibilities that $|AC| = |AB|$ and $|AC| < |AB|$ have been ruled out, it must be the case that $|AC| > |AB|$. This completes the proof of (2), and thus completes the proof of the theorem. ∎

Remark The statement of claim (1) of PPG Theorem 7, and the proof strategy, do not differ significantly from the statement and proof of Proposition 18 in Book I of Euclid's *Elements of Geometry*. The statement of claim (2) of PPG Theorem 7, and the proof strategy, do not differ significantly from the statement and proof of Proposition 19 in Book I of Euclid's *Elements of Geometry*.

PPG Theorem 8 (Triangle Inequality)
Two sides of a triangle are together greater than the third.



Proof Let $\triangle ABC$ be a triangle. We shall show that $|AC| < |AB| + |BC|$.

Produce the side $[AB]$ of this triangle beyond B to a point D so that the point B lies between A and D and $|BD| = |BC|$. Then

$$|AD| = |AB| + |BD| = |AB| + |BC|.$$

The Isosceles Triangle Theorem (PPG Theorem 2) ensures that $|\angle BDC| = |\angle BCD|$. Now $\angle BDC = \angle ADC$, because the point B lies between D and A. Also $|\angle BCD| < |\angle ACD|$, because the point B lies between A and D and therefore lies in the interior of the angle $\angle ACD$. It follows that $|\angle ADC| < |\angle ACD|$. It then follows from PPG Theorem 7 that $[AC] < [AD]$, and thus

$$|AC| < |AB| + |BC|,$$

as required. ∎

Remark The statement of PPG Theorem 8, and the proof strategy, do not differ significantly from the statement and proof of Proposition 20 in Book I of Euclid's *Elements of Geometry*.

1.14 Perpendicular Lines

PPG Proposition 1 *Two lines perpendicular to the same line are parallel to one another.*

Proof This is a special case of the Alternate Angles Theorem (PPG Theorem 3). ∎

PPG Proposition 2 *There is a unique line perpendicular to a given line and passing through a given point. This applies to a point on or off the line.*

PPG Definition 27 The **perpendicular bisector** of a segment $[AB]$ is the line through the midpoint of $[AB]$. perpendicular to AB.

1.15 Quadrilaterals and Parallelograms

PPG Definition 28 A closed chain of line segments laid end-to-end, not crossing anywhere, and not making a straight angle at any endpoint encloses a piece of the plane called a **polygon**. The segments are called the **sides** or edges of the polygon, and the endpoints where they meet are called the **vertices**. Sides that meet are called **adjacent sides**, and the ends of a side are called **adjacent vertices**. The angles at adjacent vertices are called **adjacent angles**. A polygon is called **convex** if it contains the whole segment connecting any two of its points.

PPG Definition 29 A **quadrilateral** is a polygon with four vertices.

Two sides of a quadrilateral that are not adjacent are called **opposite sides**. Similarly, two angles of a quadrilateral that are not adjacent are called **opposite angles**.

PPG Definition 30 A **rectangle** is a quadrilateral having right angles at all four vertices.

PPG Definition 31 A **rhombus** is a quadrilateral having all four sides equal.

PPG Definition 32 A **square** is a rectangular rhombus.

PPG Definition 33 A polygon is **equilateral** if all its sides are equal, and **regular** if all its sides and angles are equal.

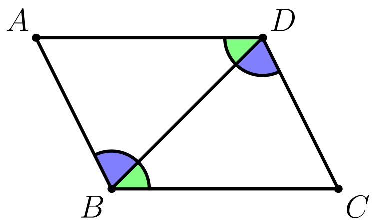
PPG Definition 34 A **parallelogram** is a quadrilateral in which both pairs of opposite sides are parallel.

PPG Proposition 3 *Each rectangle is a parallelogram.*

PPG Theorem 9

In a parallelogram, opposite sides are equal, and opposite angles are equal.

Proof Join the vertices B and D by a line segment. The line BD is a transversal of the parallel lines AD and BC. It therefore follows from the Alternate Angles Theorem (PPG Theorem 3) that $|\angle ADB| = |\angle DBC|$. The line BD is also a transversal of the parallel lines AB and DC. It therefore follows from the Alternate Angles Theorem (PPG Theorem 3) that $|\angle ABD| = |\angle BDC|$.



Now consider the triangles $\triangle ABD$ and $\triangle CDB$. The side $[BD]$ is common to both triangles. The angles $\angle ABD$ and $\angle ADB$ adjacent to the side $[BD]$ in the triangle $\triangle ABD$ are respectively equal to the angles $\angle CDB$ and $\angle CBD$ adjacent to the side $[BD]$ in the triangle $\triangle CBD$. Applying the ASA Congruence Rule (PPG Axiom 4), we conclude that the triangles $\triangle ABD$ and $\triangle CDB$ are congruent, and therefore $|AB| = |CD|$, $|AD| = |CB|$ and $|\angle BAD| = |DC|B$. We conclude therefore that the opposite sides $[AB]$ and $[DC]$ of the parallelogram $ABCD$ are equal to one another, and the opposite sides $[AD]$ and $[BC]$ of this parallelogram are also equal to one another. Also the opposite angles of the parallelogram $ABCD$ at A and C are equal to one another. Finally we note that

$$|\angle ABC| = |\angle ABD| + |\angle DBC| = |\angle CDB| + |\angle BDA| = |\angle CDA|,$$

and thus the opposite angles of the parallelogram $ABCD$ at B and D are equal to one another. This completes the proof. ∎

PPG Remark 1 Sometimes it happens that the converse of a true statement is false. For example, it is true that if a quadrilateral is a rhombus, then its diagonals are perpendicular. But it is not true that a quadrilateral whose diagonals are perpendicular is always a rhombus.

Converse 1 to PPG Theorem 9 *If the opposite angles of a convex quadrilateral are equal, then it is a parallelogram.*

Converse 2 to PPG Theorem 9 *If the opposite sides of a convex quadrilateral are equal, then it is a parallelogram.*

PPG Corollary 1 *A diagonal divides a parallelogram into two congruent triangles.*

PPG Remark 2 The converse is false: it may happen that a diagonal divides a convex quadrilateral into two congruent triangles, even though the quadrilateral is not a parallelogram.

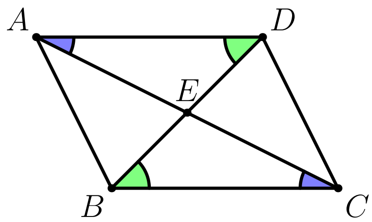
PPG Proposition 4 *A quadrilateral in which one pair of opposite sides is equal and parallel, is a parallelogram.*

PPG Proposition 5 *Each rhombus is a parallelogram.*

PPG Theorem 10
The diagonals of a parallelogram bisect one another.

Proof Each of the lines BD and AC is a transversal of the parallel lines AD and BC. It therefore follows from the Alternate Angles Theorem (PPG Theorem 3) that

$$|\angle ADB| = |\angle DBC| \quad \text{and} \quad |\angle DAC| = |\angle ACB|.$$



Now consider the triangles $\triangle EAD$ and $\triangle ECB$, where E is the point where the diagonals $[AC]$ and $[BD]$ intersect. The side $[AD]$ of the first triangle is equal to the side $[BC]$ of the second, because opposite sides of a parallelogram are equal to one another (PPG Theorem 10). The angles $\angle EDA$ and $\angle EAD$ adjacent to the side $[AD]$ of the first triangle are respectively equal to the angles $\angle EBC$ and $\angle ECB$ adjacent to the side $[BC]$ of the second triangle. Applying the ASA Congruence Rule (PPG Axiom 4), we conclude that the triangles $\triangle EAD$ and $\triangle ECB$ are congruent, and therefore $|EA| = |EC|$ and $|EB| = |ED|$. Thus the diagonals of the parallelogram $ABCD$ bisect one another, as claimed. ∎

PPG Proposition 6 *If the diagonals of a quadrilateral bisect one another, then the quadrilateral is a parallelogram.*

1.16 Ratio and Proportion

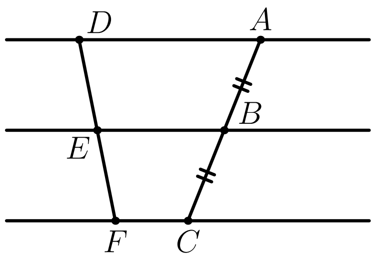
PPG Definition 35 If the three sides of one triangle are equal, respectively, to those of another, then the two triangles are said to be **similar**.

PPG Remark 3 Obviously, two right-angled triangles are similar if they have a common angle other than the right angle.

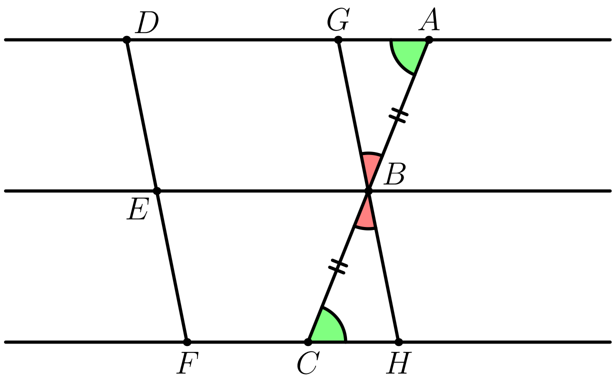
(The angles sum to 180°, so the third angle must agree as well.)

PPG Theorem 11
If three parallel lines cut off equal segments on some transversal line, then they will cut off equal segments on any other transversal.

Proof Let AC be a transversal cutting the three parallel lines at points A, B and C, where B lies between B and C, and let DF be another transversal cutting the three parallel lines at points D, E and F, where E lies between D and F. Suppose also that $|AB| = |BC|$. We must prove that $|DE| = |EF|$.



Let the line through the point B parallel to the transversal DF meet the parallel lines AD and CF at points G and H respectively.



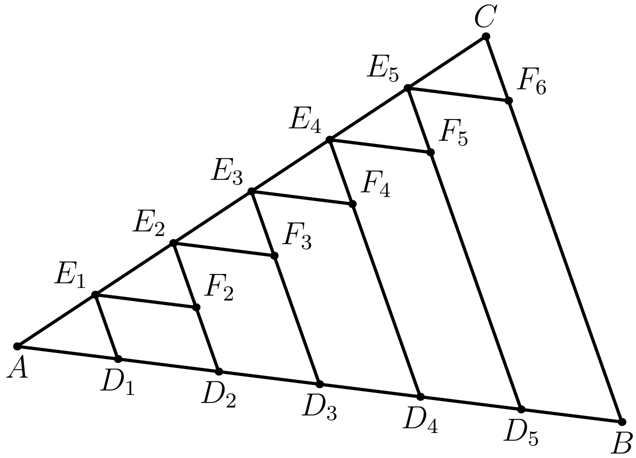
It follows from the Vertical Angles Theorem (PPG Theorem 1) that $|\angle GBA| = |\angle HBC|$. It follows from the Alternate Angles Theorem (PPG Theorem 3) that $|\angle GAB| = |\angle HCB|$. Also it is given that $|AB| = |CB|$. It follows from the ASA Congruence Rule (PPG Axiom 4), that the triangles $\triangle GAB$ and $\triangle HCB$ are congruent, and therefore $|GB| = |HB|$.

Now the quadrilaterals $DGBE$ and $FHBE$ are parallelograms, and opposite sides of parallelograms are equal (PPG Theorem 9). It follows that

$$|DE| = |GB| = |BH| = |EF|,$$

as claimed. ∎

Lemma A-1 *Let $D_0, D_1, \ldots, D_n$ be points that occur in that order along a side $[AB]$ of a triangle $\triangle ABC$, where $D_0 = A$ and $D_n = B$. For each integer i between 1 and n let the line through the point D_i that is parallel to the line BC meet the side $[AC]$ of the triangle $\triangle ABC$ at the point E_i, and, for each integer i between 2 and n, let the line through the point E_{i-1} that is parallel to the side AB of the triangle $\triangle ABC$ meet the line D_iE_i at the point F_i. Then the points $E_0, E_1, \ldots, E_n$ occur, in that order along the line segment $[AC]$, so that E_i lies between E_{i-1} and E_{i+1} for all integers i between 1 and $n-1$. Furthermore $E_0 = A$ and $E_n = C$, and, for each integer i between 2 and n, the point F_i lies between D_i and E_i*



Proof *Pasch's Axiom* ensures that when a line does not pass through any vertex of a triangle but does meet at least one side of the triangle, then it must meet some other side of that triangle. It follows that, when a line does not pass through any vertex of a triangle, meets one side of the triangle, and is parallel to a second side of the triangle, then it must also meet the third side of the triangle. The lemma is proved on making repeated applications of this principle.

Lines that are parallel to the same line are parallel to one another. (This is a consequence of PPG Theorem 5. See also Euclid's *Elements*, Book I, Proposition 30.) For each integer i between 1 and $n-1$, the line D_iE_i is parallel to the line BC. Also $D_n = B$ and $E_n = C$. It follows that, for each integer i between 1 and $n-1$, the line D_iE_i is parallel to the line $D_{i+1}E_{i+1}$.

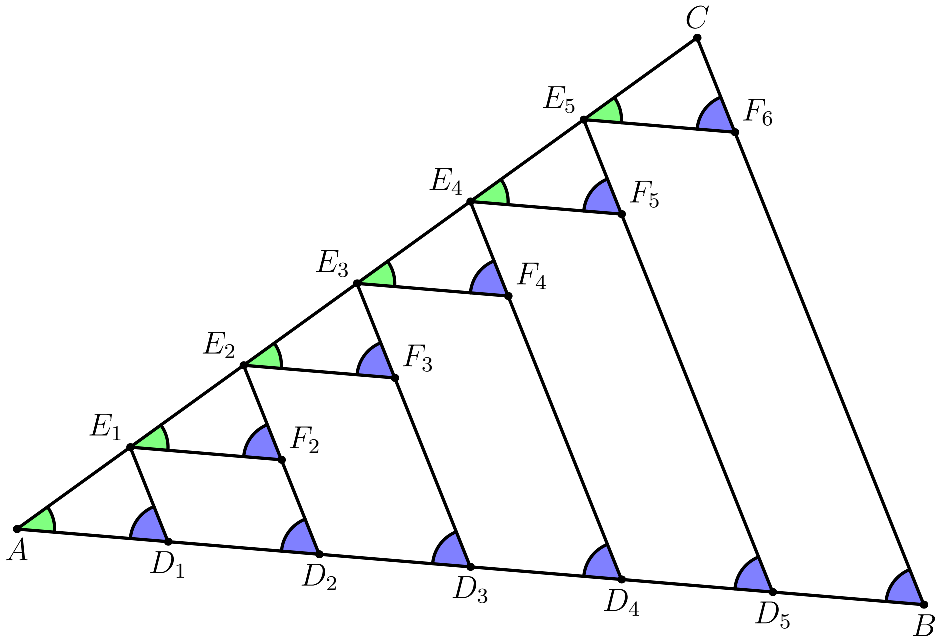
Let i be an integer between 1 and $n-1$. Then the line D_iE_i is parallel to the side $[D_{i+1}E_{i+1}]$ of the triangle $\triangle AD_{i+1}E_{i+1}$ and meets the side $[AD_{i+1}]$ at D_{i+1}. This line must therefore also meet the side $[AE_{i+1}]$ of this triangle (*Pasch's Axiom*), Therefore, for all integers i between 1 and $n-1$ the point E_i must lie between A and E_{i+1} on the line segment $[AC]$. Applying this result with i replaced by $i - 1$ enables us to conclude also that, for all integers i between 2 and n, the point E_{i-1} lies between the points A and E_i. Therefore, for all integers i between 1 and $n-1$, the point E_i must lie between E_{i-1} and E_{i+1}. Thus the points $E_0, E_1, \ldots, E_n$ all belong to the line segment $[AC]$, and moreover they occur in the specified order along that line segment. Moreover $E_0 = A$ and $E_n = C$.

For each integer i between 2 and n, the point F_i is defined to be the unique point at which the line through the point E_{i-1} parallel to the line AB intersects the line D_iE_i. Now the line $E_{i-1}F_i$ meets the side $[AE_i]$ of the triangle $\triangle AD_iE_i$ at the point E_{i-1} but does not meet the side $[AD_i]$ of this triangle. It follows from *Pasch's Axiom* that the line $E_{i-1}F_i$ must therefore meet the side $[D_iE_i]$ of the triangle $\triangle AD_iE_i$, and therefore the point F_i must lie between D_i and E_i. This completes the proof. ∎

PPG Theorem 12

Let $\triangle ABC$ be a triangle. If a line l is parallel to BC and cuts $[AB]$ in the ratio $s : t$, then it also cuts $[AC]$ is the same ratio.

Proof We first prove the result in the commensurable case in which $s/t = k/(n-k)$, where k and n are integers satisfying $0 < k < n$. Let $D_0, D_1, \ldots, D_n$ be evenly-spaced points along the side $[AB]$ of the triangle $\triangle ABC$, where $D_0 = A$ and $D_n = B$. For each integer i between 1 and n let the line through the point D_i that is parallel to the line BC meet the line segment $[AC]$ of the triangle $\triangle ABC$ at the point E_i, and, for each integer i between 2 and n, let the line through the point E_{i-1} that is parallel to the side AB of the triangle $\triangle ABC$ meet the line D_iE_i at the point F_i. Then the points $E_0, E_1, \ldots, E_n$ occur in that order along the line segment $[AC]$. Furthermore $E_0 = A$ and $E_n = C$, and, for each integer i between 2 and n, the point F_i lies between D_i and E_i. Let $F_1 = D_1$. (Lemma A-1).



The commensurable case with $n = 6$

Let i be an integer between 2 and n. Then the quadrilateral $D_{i-1}D_iF_iE_{i-1}$ is a parallelogram. Opposite sides of any parallelogram are equal to one another (PPG Theorem 9). Therefore

$$|E_{i-1}F_i| = |D_{i-1}D_i| = |AD_1|.$$

Moreover it follows from repeated applications of the Corresponding Angles Theorem (PPG Theorem 5) that

$$|\angle E_iE_{i-1}F_i| = |\angle CAB| = |\angle E_1AD_1|$$

and

$$|\angle E_iF_iE_{i-1}| = |\angle E_iD_iD_{i-1}| = |\angle CBA| = |\angle E_1D_1A|.$$

Given these results, the ASA Congruence Rule (PPG Axiom 4) ensures that the triangles $\triangle E_iE_{i-1}F_i$ are $\triangle E_1AD_1$ are congruent. It follows that $|E_{i-1}E_i| = |AE_1|$.

We conclude from this that the points $E_0, E_1, \ldots, E_n$ are evenly-spaced points on the line segment $[AB]$, and therefore

$$|AE_i| = \frac{i}{n}|AC|$$

for $i = 0, 1, 2, \ldots, n$.

Now the unique line parallel to BC that cuts $[AB]$ in the ratio $k : n-k$ is the line D_kE_k. This line cuts $[AB]$ in the ratio $k : n-k$ at the point D_k, and also cuts $[AC]$ in the ratio $k : n-k$ at the point E_k. The result in the commensurable case follows immediately.

We now use the result in the commensurable case in order to prove PPG Theorem 12 in the general case. Let u and v be real numbers satisfying the inequalities $0 < u < 1$ and $0 < v < 1$, let G be the unique point on the line segment $[AB]$ for which $|AG| = u|AB|$, and let H be the unique point on the line segment $[AC]$ for which $|AH| = v|AB|$. Note that the point G divides the line segment $[AB]$ in the ratio $s : t$ if and only if

$$u = \frac{s}{s+t}.$$

Thus the proof of PPG Theorem 12 will be completed once we prove that the line GH will not be parallel to BC unless $u = v$.

Suppose that $u \neq v$. Then there exists a rational number q that lies strictly between u and v. Then either $u < q < v$ or else $u > q > v$. Then, because q is a rational number satisfying $0 < q < 1$, there exist positive integers k and n for which

$$q = \frac{k}{n}.$$

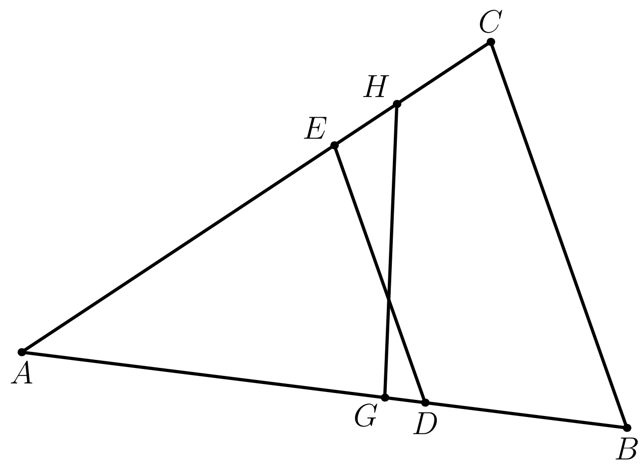
Let D denote the unique point on the line segment $[AB]$ that divides the line segment in the ratio $k : n - k$. Then $|AD| = q|AB|$. Then let E be the unique point on the line segment $[AC]$ at which that line segment meets the line parallel to BC that passes through D. It follows from the result proved in the commensurable case that the point E divides the line segment $[AC]$ in the ratio $k : n - k$, and therefore $|AE| = q|AC|$.

Suppose in particular that $u < v$ and that q is a rational number satisfying the inequalities $u < q < v$. Then

$$|AG| = u|AB| < q|AB| = |AD|$$

and

$$|AH| = v|AC| > q|AC| = |AE|.$$



the case when $u < q < v$

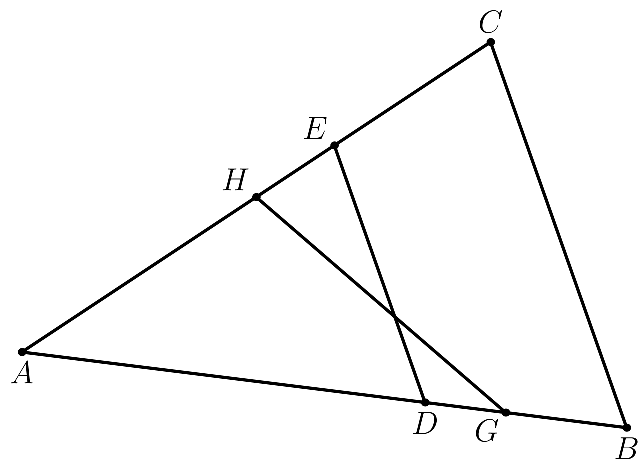
Thus the point G lies between A and D, but the point H does not lie between A and E. It follows that the line GH meets the side $[AD]$ of the triangle $\triangle ADE$ at G, but does not meet the side $[AE]$ of this triangle. The line GH must therefore meet the remaining side $[DE]$ of this triangle (*Pasch's Axiom*). Moreover DE is parallel to the line BC. It follows that if $u < v$ then the lines BC and GH are not parallel.

Next suppose that $u > v$ and that q is a rational number satisfying the inequalities $u > q > v$. Then

$$|AG| = u|AB| > q|AB| = |AD|$$

and

$$|AH| = v|AC| < q|AC| = |AE|.$$



the case when $u > q > v$

Thus the point H lies between A and E, but the point G does not lie between A and D. It follows that the line GH meets the side $[AE]$ of the triangle $\triangle ADE$ at H, but does not meet the side $[AD]$ of this triangle. The line GH must therefore meet the remaining side $[DE]$ of this triangle (*Pasch's Axiom*). Moreover DE is parallel to the line BC. It follows that if $u > v$ then the lines BC and GH are not parallel.

We have now shown that if a line GH cuts the sides $[AB]$ and $[AC]$ of the triangle $\triangle ABC$ at points G and H respectively, where $[AG] = u[AB]$ and $[AH] = v[AC]$, and if $u \neq v$, then the line GH is not parallel to BC. We can conclude immediately from this that if a line l is parallel to $[BC]$ then it must cut the other two sides $[AB]$ and $[AC]$ of that triangle in the same ratio. This completes the proof of PPG Theorem 12 in the general case. ∎

PPG Proposition 7 *Let $\triangle ABC$ and $\triangle A'B'C'$ be triangles. Suppose that*

$$|\angle A| = |\angle A'| \quad \textit{and} \quad \frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|}.$$

Let the triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar.

PPG Theorem 13

If two triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar, then their sides are proportional, in order:

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|}.$$

Proof The *Ruler Axiom* (PPG Axiom 2) ensures the existence of points B'' and C'' determined by the following conditions:

- $|AB''| = |A'B'|$ and $|AC''| = |A'C'|$;
- the point B'' lies on the ray $[AB$ starting at A that passes through the point B;
- the point C'' lies on the ray $[AC$ starting at A that passes through the point C.

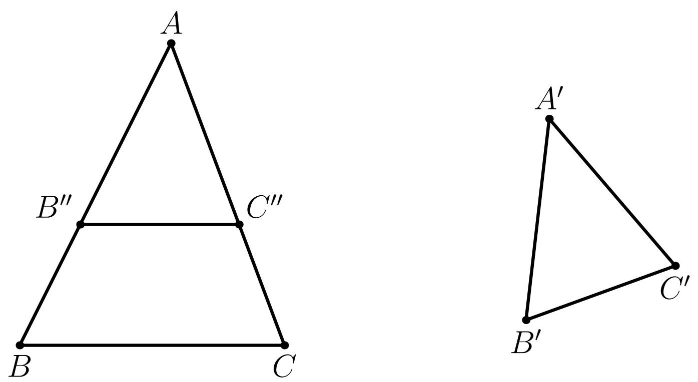
The angle $\angle B''AC''$ then coincides with the angle $\angle BAC$. But the triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar. It follows that

$$|\angle AB''C''| = |\angle ABC| = |\angle A'B'C'|.$$

Now the points B'' and C'' have been determined to ensure that $|AB''| = |A'B'|$ and $|AC''| = |A'C'|$. Applying the SAS Congruence Rule (PPG Axiom 4), it follows that the triangles $\triangle AB''C''$ and $\triangle A'B'C'$ are congruent, and therefore $|\angle AB''C''| = |\angle A'B'C'|$. It follows from the similarity of the triangles $\triangle ABC$ and $\triangle A'B'C'$ that

$$|\angle AB''C''| = |\angle ABC|.$$

Moreover the points A, B and B'' are collinear, and the points C and C'' lie on the same side of the line AB. Therefore the angles $|\angle AB''C''|$ and $|\angle ABC|$ are corresponding angles. It then follows from the Corresponding Angles Theorem (PPG Theorem 5) that the lines BC and $B''C''$ are parallel.



Let
$$r = \frac{|A'B'|}{|AB|}.$$
Then $r > 0$, and $|AB''| = r|AB|$. If $r = 1$ then the points B and B'' coincide, and therefore the line segments $[B''C'']$ and $[BC]$ will also coincide. In this case the triangles $\triangle ABC$ and $\triangle A'B'C'$ are not merely similar but congruent, and it follows immediately that corresponding sides of these triangles are equal in length, and are therefore proportional. If $r < 1$ then the line $B''C''$ cuts the side $[AB]$ of the triangle $\triangle ABC$ at a point lying between A and B. Because the lines $B''C''$ and BC are parallel, Theorem 12 ensures that the line $B''C''$ cuts the sides $[AB]$ and $[AC]$ of the triangle $\triangle ABC$ in the same ratio, and therefore the point C'' lies between A and C, and $|AC''| = r|AC|$. In the case where $r > 1$, Theorem 12 ensures that the line BC cuts the sides $[AB'']$ and $[AC'']$ of the triangle $\triangle AB''C''$ in the same ratio, and therefore $|AC| = r^{-1}|AC''|$. Thus, in all cases
$$\frac{|A'B'|}{|AB|} = \frac{|AB''|}{|AB|} = r = \frac{|AC''|}{|AC|} = \frac{|C'A'|}{|CA|}.$$
Replacing A, B, C, A', B', C' by B, C, A, B', C' and A' respectively, we find that
$$\frac{|B'C'|}{|BC|} = \frac{|A'B'|}{|AB|}.$$
Thus corresponding sides of the triangles $\triangle ABC$ and $\triangle A'B'C'$ are proportional, as required. ∎

Remark In order to prove Theorem 13, it is of course permissible to assume, without loss of generality, that $|A'B'| \leq |AB|$. But this in itself provides no warrant for assuming that $|A'C'| \leq |AC|$.

1.17 Theorem Concordance

Post-Primary	Euclid's *Elements*
Axiom 4	Book I, Propositions 4, 8, 26
Theorem 1	Book I, Proposition 15
Theorem 2	Book I, Propositions 5 and 6
Theorem 3	Book I, Propositions 27 and 29
Theorem 4	Book I, Propositions 32
Theorem 5	Book I, Propositions 28 and 29
Theorem 6	Book I, Proposition 32
Theorem 7	Book I, Propositions 18 and 19
Theorem 8	Book I, Proposition 20
Theorem 9	Book I, Proposition 34
Theorem 10	—
Theorem 11	—
Theorem 12	Book VI, Proposition 2
Theorem 13	Book VI, Proposition 4
Theorem 14	Book I, Proposition 47 and Book VI, Proposition 31
Theorem 15	Book I, Proposition 48
Theorem 16	—
Theorem 17	Book I, Proposition 34
Theorem 18	—
Theorem 19	Book III, Proposition 20
Theorem 20	Book III, Proposition 16 and 18
Theorem 21	Book III, Propositions 1 and 3

In relation to the proof of Theorem 6, see also Heath's commentary on Proposition 32 of Book I of Euclid's *Elements*[3]. According to Proclus, this proof was handed down by Eudemus, who attributed to the Pythagoreans.

In relation to Theorem 12, see also Heath's account of a proof of the result of Proposition 2 of Book VI of Euclid's *Elements*, presented by De Morgan, which Heath discusses in the second volume of his translation and commentary on Euclid's *Elements*[4]. This proof is formulated the context of the ancient Greek theory of ratio and proportion, generally attributed to Eudoxus, which is the subject matter of Book V of Euclid's *Elements*.

[3] T.L. Heath, *The thirteen books of Euclid's Elements*, Vol. 1, p. 320.

[4] T.L. Heath, *The thirteen books of Euclid's Elements*, Vol. 2, p. 124.