[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp.224–231 (1908).]

[Heath's commentary on Euclid, *Elements*, Book I, Common Notion 4.]

Common Notion 4.

Καὶ τὰ ἐφαρμόζοντα ἐπ' ἄλληλα ἴσα ἀλλήλοις ἐστίν.

Things which coincide with one another are equal to one another.

The word ἐφαρμόζειν, as a geometrical term, has a different meaning according as it is used in the active or in the passive. In the passive, ἐφαρμόζεσθαι, it means "to be *applied* to" without any implication that the applied figure will exactly fit, or coincide with, the figure to which it is applied; on the other hand the active ἐφαρμόζειν is used intransitively and means "to fit exactly," "to coincide with." In Euclid and Archimedes ἐφαρμόζειν is constructed with ἐπί and the accusative, in Pappus with the dative.

On *Common Notion* 4 Tannery observes that it is incontestably geometrical in character, and should therefore have been excluded from the *Common Notions*; again, it is difficult to see why it is not accompanied by its converse, at all events for straight lines (and, it might be added, angles also), which Euclid makes use of in I. 4. As it is, says Tannery, we have here a definition of geometrical equality more or less sufficient, but not a real axiom.

It is true that Proclus seems to recognize this *Common Notion* and the next as proper axioms in the passage (p. 196, 15–21) where he says that we should not cut down the axioms to the minimum, as Heron does in giving only three axioms; but the statement seems to rest, not on authority, but upon an assumption that Euclid would state explicitly at the beginning all axioms subsequently used and not reducible to others unquestionably included. Now in I. 4 this *Common Notion* is not quoted; it is simply inferred that "the base BC will coincide with EF, *and will be equal* to it." The position is therefore the same as it is in regard to the statement in the same proposition that, "if... the base BC does not coincide with EF, *two straight lines will enclose a space*: which is impossible"; and, if we do not admit that Euclid had the axiom that "two straight lines cannot enclose a space," neither need we infer that he had *Common Notion* 4. I am therefore inclined to think that the latter is more likely than not to be an interpolation.

It seems clear that the Common Notion, as here formulated, is intended to assert that superposition is a legitimate way of proving the equality of two figures which have the necessary parts respectively equal, or, in other words, to serve as an *axiom of congruence*.

The phraseology of the propositions, e.g. I. 4 and I. 8, in which Euclid employs the method indicated, leaves no room for doubt that he regarded one figure as actually *moved* and *placed upon* the other. Thus in I. 4 he says, "The triangle ABC being applied (ἐφαρμοζομένου) to the triangle DEF, and the point A being *placed* (τιθεμένου) upon the point D, and the straight line AB on DE, the point B will also coincide with E, because AB is equal to DE"; and in I. 8, "If the sides BA, AC do not coincide with ED, DF, but *fall beside them* (take a different position, παραλλάξουσιν), then" etc. At the same time, it is clear that Euclid disliked the method and avoided it wherever he could, e.g., in I. 26, where he proves the equality of two triangles which have two angles equal to two angles and one side of the one equal to the corresponding side of the other. It looks as though he found the method handed down by tradition (we can hardly suppose that, if Thales proved that the diameter of a circle divides it into two equal parts, he would do so by any other method than that of superposition), and followed it, in the few cases where he does so, only because he had not been able to see his way to a satisfactory substitute. But seeing how much of the *Elements* depends on I. 4, directly or indirectly, the method can hardly be regarded as being, in Euclid, of only subordinate importance; on the contrary, it is fundamental. Nor, as a matter of fact, do we find in the ancient geometers any expression of doubt as to the legitimacy of the method. Archimedes uses it to prove that any spheroidal figure cut by a plane through the centre is divided into two equal parts in respect of both its surface and its volume; he also postulates in *Equilibrium of Planes* I. that "when equal and similar plane figures coincide if applied to one another, their centres of gravity coincide also."

Killing (*Einführung in die Grundlagen der Geometrie*, II. pp. 4, 5) contrasts the attitude of the Greek geometers with that of the philosophers, who, he says, appear to have agreed in banishing motion from geometry altogether. In support of this he refers to the view frequently expressed by Aristotle that mathematics has to do with *immovable* objects (ἀκίνητα), and that only where astronomy is admitted as part of mathematical science is motion mentioned as a subject for mathematics. Cf. *Metaph.* 989 b 32 "For mathematical objects are among things which exist apart from motion, except such as relate to astronomy"; *Metaph.* 1064 a 30 "Physics deals with things which have in themselves the principle of motion; mathematics is a theoretical science and one concerned with things which are *stationary* (μένοντα) but not separable" (sc. from matter); in *Physics* II. 2, 193 b 34 he speaks of the subjects of mathematics as "in thought separable from motion."

But I doubt whether in Aristotle's use of the words "immovable," "without motion" etc. as applied to the subjects of mathematics there is any implication such as Killing supposes. We arrive at mathematical concepts by abstraction from material objects; and just as we, in thought, eliminate the matter, so according to Aristotle we eliminate the attributes of matter as such, e.g. qualitative change and *motion.* It does not appear to me that the use of "immovable" in the passages referred to means more than this. I do not think that Aristotle would have regarded it as illegitimate to *move* a geometrical figure from one position to another; and I infer this from a passage in *De caelo* III. I where he is criticising "those who make up every body that has an origin by putting together *planes*, and resolve it again into *planes*." The reference must be to the *Timaeus* (54 B sqq.) where Plato evolves the four elements in this way. He begins with a right-angled triangle in which the hypotenuse is double of the smaller side; six of these put together in the proper way produce one equilateral triangle. Making solid angles with (a) three, (b) four, and (c) five of these equilateral triangles respectively, and taking the requisite number of these solid angles, namely four of (a), six of (b) and twelve of (c) respectively, and putting them together so as to form regular solids, he obtains (α) a tetrahedron, (β) an octahedron, (γ) an icosahedron respectively. For the fourth element (earth), four isosceles right-angled triangles are first put together so as to form a square, and then six of these squares are put together to form a cube. Now, says Aristotle (299 b 23), "it is absurd that planes should only admit of being put together so as to touch in a *line*; for just as a line and a line are put together in both ways, lengthwise and breadthwise, so must a plane and a plane. A line can be combined with a line in the sense of being a line *superposed*, and not *added*"; the inference being that a *plane* can be superposed on a *plane.* Now this is precisely the sort of motion in question here; and Aristotle, so far from denying its permissibility, seems to blame Plato for not using it. Cf. also *Physics* V. 4, 228 b 25, where Aristotle speaks of "the spiral or other magnitude in which any part will not coincide with any other part," and where superposition is obviously contemplated.

Motion without deformation.

It is well known that Helmholtz maintained that geometry requires us to assume the actual existence of rigid bodies and their free mobility in space, whence he inferred that geometry is dependent on mechanics.

Veronese exposed the fallacy in this (*Fondamenti di geometrica*, pp. XXXV–XXXVI, 239–240 note, 615–7), his argument being as follows. Since geometry is concerned with empty space, which is immovable, it would be at least strange if it was necessary to have recourse to the real motion of bodies for a definition, and for the proof of the properties, of immovable space. We must distinguish the intuitive principle of motion in itself from that of motion *without deformation.* Every point of a figure which moves is transferred to another point in space. "Without deformation" means that the mutual relations between the points of the figure do not change, but the relations between them and other figures do change (for if they did not, the figure could not move). Now consider what we mean by saying that, when the figure A has moved from the position A_1 to the position A_2, the relations between the points of A in the position A_2 are unaltered from what they were in the position A_1, are the same in fact as if A had not moved but remained at A_1. We can only say that, judging of the figure (or the body with its physical qualities eliminated) by the impressions it produces in us during its movement, the impressions produced in us in the two different positions (which are in time distinct) *are equal*. In fact, we are making use of the notion of *equality* between two distinct figures. Thus if we say that two bodies are equal when they can be superposed by means of *movement without deformation*, we are committing a *petitio principii*. The notion of the equality of spaces is really prior to that of rigid bodies or of motion without deformation. Helmholtz supported his view by reference to the process of measurement in which the measure must be, at least approximately, a rigid body, but the existence of a rigid body as a standard to measure by, and the question how we discover two equal spaces to be equal, are matters of no concern to the geometer. The method of superposition, depending on motion without deformation, is only of use as a *practical* test; it has nothing to do with the *theory* of geometry.

Compare an acute observation of Schopenhauer (*Die Welt als Wille*, 2 ed. 1844, II. p. 130) which was a criticism in advance of Helmholtz' theory: "I am surprise that, instead of the eleventh axiom [the Parallel-Postulate], the eighth is not rather attacked: 'Figures which coincide (sich decken) are equal to one another.' For *coincidence* (das Sichdecken) is either mere tautology, or something entirely empirical, which belongs, not to pure intuition (Anschauung), but to external sensuous experience. It presupposes in fact the mobility of figures; but that which is movable in space is matter and nothing else. Thus this appeal to coincidence means leaving pure space, the sole element of geometry, in order to pass over to the material and empirical."

Mr Bertrand Russell observes (*Encyclopaedia Britannica*, Suppl. Vol. 4, 1902, Art. "Geometry, non-Euclidean") that the apparent use of motion here is deceptive; what in geometry is called a motion is merely the transference of our attention from one figure to another. Actual superposition, which is nominally employed by Euclid, is not required; all that is required is the transference of our attention from the original figureto a new one defined by the position of some of its elements and by certain properties which it shares with the original figure.

If the method of superposition is given up as a means of defining theoretically the equality of two figures, some other definition of equality is necessary.

But such a definition can be evolved out of *empirical* or *practical* observation of the result of superposing two material representations of figures. This is done by Veronese (*Elementi di geometria*, 1904) and Ingrami (*Elementi di geometria*, 1904). Ingrami says, namely (p. 66):

"If a sheet of paper be folded double, and a triangle be drawn upon it and then cut out, we obtain two triangles *superposed* which we in practice call *equal*. If points A, B, C, D ... be marked on one of the triangles, then, when we place this triangle upon the other (so as to coincide with it), we see that *each* of the particular points taken on the first is superposed on one particular point of the second in such a way that the segments AB, AC, AD, BC, BD, CD, ... are respectively superposed on as many segments in the second triangle and are therefore equal to them respectively. In this way we justify the following

"**Definition of equality.**

"Any two figures whatever will be called *equal* when to the points of one the points of the other can be made to correspond *univocally* [i.e. every *one* point in one to *one distinct* point in the other and *vice versa*] in such a way that the segments which join the points, two and two, in one figure are respectively equal to the segments which join, two and two, the corresponding points in the other."

Ingrami has of course previously postulated as known the signification of the phrase *equal* (*rectilineal*) *segments*, of which we get a *practical* notion when we can place one upon the other or can place a third movable segment successively on both.

New systems of Congruence-Postulates.

In the fourth Article of *Questioni riguardanti le matematiche elementari*, I., pp. 93–122, a review is given of three different systems: (1) that of Pasch in *Vorlesungen über neuere Geometrie*, 1882, p. 101 sqq., (2) that of Veronese according to the *Fondamenti di geometria*, 1891, and the *Elementi* taken together, (3) that of Hilbert (see *Grundlagen der Geometrie*, 1903, pp. 7–15).

These systems differ in the particular conceptions taken by the three authors as primary. (1) Pasch considers as primary the notion of *congruence* or *equality* between *any figures which are made up of a finite number of points only.* The definitions of congruent *segments* and of congruent *angles* have to be *deduced* in the way shown on pp. 102–103 of the Article referred to, after which Eucl. I. 4 follows immediately, and Eucl. I. 26 (I) and I. 8 by a method recalling that in Eucl. I. 7, 8.

(2) Veronese takes as primary the conception of congruence between *segments* (rectilineal). The transition to congruent *angles*, and thence to *triangles* is made by means of the following postulate:

"Let AB, AC and $A'B'$, $A'C'$ be two pairs of straight lines intersecting at A, A', and let there be determined upon them the congruent segments AB, $A'B'$ and the congruent segments AC, $A'C'$;
then, if BC, $B'C'$ are congruent, the two *pairs of straight lines* are congruent."

(3) Hilbert takes as primary the notions of congruence between *both segments and angles.*

It is observed in the Article referred to that, from the theoretical standpoint, Veronese's system is an advance upon that of Pasch, since the idea of congruence between *segments* is more simple than that of congruence between *any figures*; but, didactically, the development of the theory is more complicated when we start from Veronese's system than when we start from that of Pasch.

The system of Hilbert offers advantages over both the others from the point of view of the teaching of geometry, and I shall therefore give a short account of his system only, following the Article above quoted.

Hilbert's system

The following are substantially the Postulates laid down.

(1) *If one segment is congruent with another, the second is also congruent with the first.*

(2) *If an angle is congruent with another angle, the second angle is also congruent with the first.*

(3) *Two segments congruent with a third are congruent with one another.*

(4) *Two angles congruent with a third are congruent with one another.*

(5) *Any segment* AB *is congruent with itself, independently of its sense.*
This we may express symbolically thus:

$$AB \equiv AB \equiv BA.$$

(6) *Any angle* ab *is congruent with itself, independently of its sense.*
This we may express symbolically thus:

$$(ab) \equiv (ab) \equiv (ba).$$

(7) On any straight line r′, starting from any one of its points A′, and on each side of it respectively, there exists one and only one segment congruent with a segment AB belonging to the straight line r.

(8) *Given a ray* a, *issuing from a point* O, *in any plane which contains it and on each of the two sides of it, there exists one and only one ray* b *issuing from* O *such that the angle* (ab) *is congruent with a given angle* (a′b′).

(9) *If* AB, BC *are two consecutive segments of the same straight line* r *(segments, that is, having an extremity and no other point common), and* A′B′, B′C′ *two consecutive segments on another straight line* r′, *and if* AB ≡ A′B′, BC ≡ B′C′, *then*

$$AC \equiv A'C'.$$

(10) *If* (ab), (bc) *are two consecutive angles in the same plane* π *(angles, that is, having the vertex and one side common), and* (a′b′), (b′c′) *two consecutive angles in another plane* π', *and if*[1] (ab) ≡ (a′b′), (bc) ≡ (b′c′) *then*

$$(ac) \equiv (a'c').$$

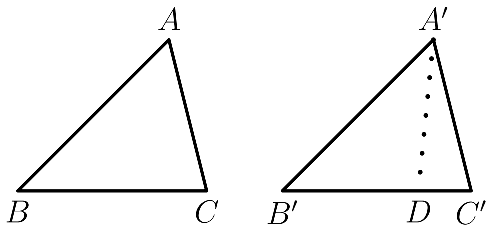
(11) *If two triangles have two sides and the included angles respectively congruent, they have also their third sides congruent as well as the angles opposite to the congruent sides respectively.*

As a matter of fact, Hilbert's postulate corresponding to (11) does not assert the equality of the third sides in each, but only the equality of the two remaining angles in one triangle to the two remaining angles in the other respectively. He proves the equality of the third sides (thereby completing the theorem of Eucl. I. 4) by *reductio ad absurdum* thus. Let ABC, $A'B'C'$ be the two triangles which have the sides AB, AC respectively congruent with the sides $A'B'$, $A'C'$ and the included angle at A congruent with the included angle at A'.

Then, by Hilbert's own postulate, the angles ABC, $A'B'C'$ are congruent, as also the angles ACB, $A'C'B'$.

If BC is not congruent with $B'C'$, let D be taken on $B'C'$ such that BC, $B'D$ are congruent and join $A'D$.

[1] [Note added by DRW: the relevant formulae following are printed as $(bc) = (b'c')$ and $(ac) = (a'c')$, i.e., with = in place of the congruence sign ≡ implied by the context.]



Then the two triangles ABC, $A'B'D$ have two sides and the included angles congruent respectively; therefore, by the same postulate, the angles BAC, $B'A'D$ are congruent.

But the angles BAC, $B'A'C'$ are congruent; therefore by (4) above, the angles $B'A'C'$, $B'A'D$ are congruent: which is impossible, since it contradicts (8) above.

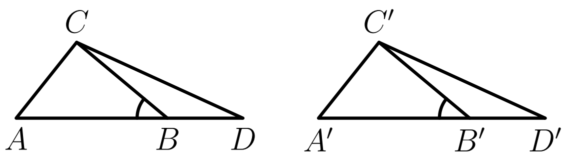
Hence BC, $B'C'$ cannot but be congruent.

Euclid I. 4 is thus proved; but it seems to be as well to include all of that theorem in the postulate, as is done in (11) above, since the two parts of its are equally suggested by empirical observation of the result of one superposition.

A proof similar to that just given immediately establishes Eucl. I. 26 (1), and Hilbert next proves that

If two angles ABC, $A'B'C'$ are congruent with one another, their supplementary angles CBD, $C'B'D'$ are also congruent with one another.

We choose A, D on one of the straight lines forming the first angle, and A', D' on one of those forming the second angle, and again C, C' on the other straight lines forming the angles, so that $A'B'$ is congruent with AB, $C'B'$ with CB, and $D'B'$ with DB.



The triangles ABC, $A'B'C'$ are congruent, by (11) above; and AC is congruent with $A'C'$, and the angle CAB with the angle $C'A'B'$.

Thus AD, $A'D'$ being congruent, by (9), the triangles CAD, $C'A'D'$ are also congruent, by (11);
whence CD is congruent with $C'D'$, and the angle ADC with the angle $A'D'C'$.

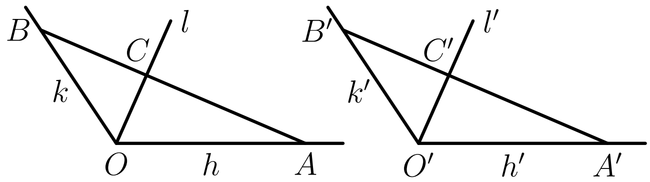
Lastly, by (11), the triangles CDB, $C'D'B'$ are congruent, and the angles CBD, $C'B'D'$ are thus congruent.

Hilbert's next proposition is that

Given that the angle (h, k) *in the plane* α *is congruent with the angle* $(\mathrm{h}', \mathrm{k}')$ *in the plane* α'*, and that* l *is a half-ray in the plane* α *starting from the vertex of the angle* (h, k) *and lying within that angle, there always exists a half-ray* l' *in the second plane* α'*, starting from the vertex of the angle* $(\mathrm{h}', \mathrm{k}')$ *and lying within that angle, such that*

$$(\mathrm{h}, \mathrm{l}) \equiv (\mathrm{h}', \mathrm{l}'), \quad \textit{and} \quad (\mathrm{k}, \mathrm{l}) \equiv (\mathrm{k}', \mathrm{l}').$$

If O, O' are the vertices, we choose points A, B on h, k, and points A', B' on h', k' respectively, such that OA, $O'A'$ are congruent and also OB, $O'B'$.



The triangles OAB, $O'A'B'$ are then congruent; and, if l meets AB in C, we can determine C' on $A'B'$ such that $A'C'$ is congruent with AC.

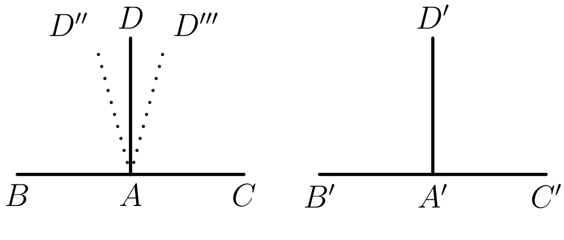
Then l' drawn from O' through C' is the half-ray required.

The congruence of the angles (h, l), (h', l') follows from (11) directly, and that of (k, l) and (k', l') follows in the same way after we have inferred by means of (9) that, AB, AC being respectively congruent with $A'B'$, $A'C'$, the difference BC is congruent with the difference $B'C'$.

It is by means of the two propositions just given that Hilbert proves that

All right angles are congruent with one another.

Let the angle BAD be congruent with its adjacent angle CAD, and likewise the angle $B'A'D'$ congruent with its adjacent angle $C'A'D'$. All four angles are then right angles.



If the angle $B'A'D'$ is not congruent with the angle BAD, let the angle with AB for one side and congruent with the angle $B'A'D'$ be the angle BAD'', so that AD'' falls either within the angle BAD or within the angle DAC. Suppose the former.

By the last proposition but one (about adjacent angles), the angles $B'A'D'$, BAD'' being congruent, the angles $C'A'D'$, CAD'' are congruent.

Hence, by the hypothesis and postulate (4) above, the angles BAD'', CAD'' are also congruent.

And, since the angles BAD, CAD are congruent, we can find within the angle CAD a half-ray CAD''' such that the angles BAD'', CAD''' are congruent, and likewise the angles DAD'', DAD''' (by the last proposition).

But the angles BAD'' and CAD'' were congruent (see above); and it follows, by (4), that the angles CAD'', CAD''' are congruent, which is impossible, since it contradicts postulate (8).

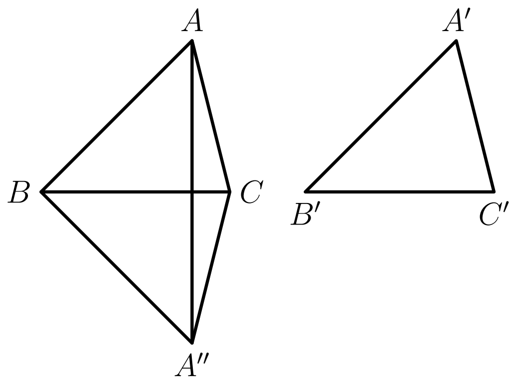
Therefore etc.

Euclid I. 5 follows directly by applying the postulate (11) above to ABC, ACB as distinct triangles.

Postulates (9), (10) above give in substance the proposition that "the sums or differences of segments, or of angles, respectively equal, are equal."

Lastly, Hilbert proves Eucl. I. 8 by means of the theorem of Eucl. I. 5 and the proposition just stated as applied to angles.

ABC, $A'B'C$ being the given triangles with three sides respectively congruent, we suppose an angle CBA'' to be determined, on the side of BC opposite to A, congruent with the angle $A'B'C'$, and we make BA'' equal to $A'B'$.



The proof is obvious, being equivalent to the alternative proof often given in our text-books for Eucl. I. 8.

[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp.248–250 (1908).]

[Heath's commentary on Euclid, *Elements*, Book I, Proposition 4.]

1–3. It is a fact that Euclid's enunciations not infrequently leave something to be desired in point of clearness and precision. Here he speaks of the triangles having "the angle equal to the angle, namely the angle contained by the equal straight lines" (τὴν γωνίαν τῇ γωνίᾳ ἴσαν ἔχῃ τὴν ὑπό τῶν ἴσων εὐθειῶν περιεχομένην), only one of the two angles being described in the latter expression (in the accusative), and a similar expression in the dative being left to be understood of the other angle. It is curious too that, after mentioning two "*sides*," he speaks of the angles contained by the equal "*straight lines*," not "*sides*." It may be that he wished to adhere scrupulously, at the outset, to the phraseology of the definitions, where the angle is the inclination to one another of the two *lines* or *straight lines*. Similarly in the enunciation of I. 5 he speaks of producing the equal "straight lines" as if to keep strictly to the wording of Postulate 2.

2. **respectively.** I agree with Mr H. M. Taylor (*Euclid*, p. ix) that it is best to abandon the traditional translation of "each to each," which would naturally seem to imply that all the four magnitudes are equal rather than (as the Greek ἑκατέρα ἑκατέρᾳ does) that one is equal to one and the other to the other.

3. **the base.** Here we have the word *base* used for the first time in the *Elements*. Proclus explains it (p.236, 12–15) as meaning (1), when no side of a triangle has been mentioned before, the side "which is on a level with the sight" (τὴν πρὸς τῇ ὄψει κειμένην), and (2), when two sides have already been mentioned, the third side. Proclus thus avoids the mistake made by some modern editors who explain the term exclusively with reference to the case where two sides have been mentioned before. That this is an error is proved (1) by the occurrence of the term in the enunciations of I. 37 etc. about triangles on the same base and equal bases, (2) by the application of the same term to the bases of parallelograms in I. 35 etc. The truth is that the use of the term must have been suggested by the practice of drawing the particular side horizontally, as it were, and the rest of the figure above it. The *base* of a figure was therefore spoken of, primarily, in the same sense as the base of anything else, e.g., of a pedestal or column; but when, as in I. 5, two triangles were compared occupying other than the normal positions which gave rise to the name, and when two sides had been previously mentioned, the base was, as Proclus says, necessarily the third side.

6. **subtend.** ὑποτείνειν ὑπό, "to stretch under," with accusative.

9. **the angle BAC.** The full Greek expression would be ἡ ὑπὸ τῶν ΒΑ, ΑΓ περιεχομένη γωνία, "the angle contained by the (straight lines) BA, AC." But it was common practice of Greek geometers, e.g. of Archimedes and Apollonius (and Euclid too in Books X.–XIII., to use the abbreviation αἱ ΒΑΓ for αἱ ΒΑ, ΑΓ, "the (straight lines) BA, AC." Thus, on περιεχομένη being dropped, the expression would become first ἡ ὑπὸ τῶν ΒΑΓ γωνία, then ἡ ὑπὸ ΒΑΓ γωνία, and finally ἡ ὑπὸ ΒΑΓ, without γωνία, as we regularly find it in Euclid.

17. **if the triangle be applied to...**, 23. **coincide.** The difference between the technical use of the passive ἐφαρμόζεσθαι "to be *applied* (to)," and of the active ἐφαρμόζειν "to *coincide* (with)" has been noticed above (note on *Common Notion* 4, pp. 224–5).

32. [**For if, when B coincides...** 36. **coincide with EF**]. Heiberg (*Paralipomena zu Euklid* in *Hermes*, XXXVIII., 1903, p. 56) has pointed out, as a conclusive reason for regarding these words as an early interpolation, that the text of an-Nairīzī (*Codex Leidensis* 399, I, ed. Besthorn-Heiberg, p. 55) does not give the words in this place but after the conclusion Q.E.D., which shows that they constitute a *scholium* only. They were doubtless added by some commentator who thought it necessary to explain the immediate inference that, since B coincides with E and C with F, the straight line BC coincides with the straight line EF, an inference which readily follows from the definition of a straight line and Post. 1; and no doubt the Postulate that "Two straight lines cannot enclose a space" (afterwards placed among the *Common Notions*) was interpolated at the same time.

44. **Therefore etc.** Where (as here) Euclid's *conclusion* merely repeats the enunciation word for word, I shall avoid the repetition and write "Therefore etc." simply.

In the note on *Common Notion* 4 I have already mentioned that Euclid obviously used the method of superposition with reluctance, and I have given, after Veronese for the most part, the reason for holding that that method is not admissible as a *theoretical* means of proving equality, although it may be of use as a *practical* test, and may thus furnish an empirical basis on which to found a postulate. Mr. Bertrand Russell observes (*Principles of Mathematics* I. p. 405) that Euclid would have done better to assume I. 4 as an axiom, as is practically done by Hilbert (*Grundlagen der Geometrie*, p. 9). It may be that Euclid himself was as well aware of the objections to the method as are his modern critics; but at all events those objections were stated, with almost equal clearness, as early as the middle of the 16th century. Peletarius (Jacques Peletier) has a long note on this proposition (*In Euclidis Elementa geometrica demonstrationum libri sex*, 1557), in which he observes that, if superposition of lines and figures could be assumed as a method of proof, the whole of geometry would be full of such proofs, that it could equally well have been used in I. 2, 3 (thus in I. 2 we could simply have supposed the line taken up and *placed* at the point), and that in short it is obvious how far removed the method is from the dignity of geometry. The theorem, he adds, is obvious in itself and does not require proof; although it is introduced as a theorem, it would seem that Euclid intended it rather as a *definition* than a theorem, "for I cannot think that two angles are equal unless I have a conception of what equality of angles is." Why then did Euclid include the proposition among theorems, instead of placing it among the axioms? Peletarius makes the best excuse he can, but concludes thus:

"Huius itaque propositionis veritatem non aliunde quam a communi iudicio petemus; cogitabimusque figuras figuris superponere, Mechanicum quippiam esse: intelligere verò, id demum esse Mathematicum."

Expressed in terms of the modern systems of Congruence-Axioms referred to in the note on *Common Notion* 4, what Euclid really assumes amounts to the following:

(1) On the line DE, there is a point E, on either side of D, such that AB is equal to DE.

(2) On either side of the ray DE there is a ray DF such that the angle EDF is equal to the angle BAC.

It now follows that on DF there is a point F such that DF is equal to AC.

And lastly (3), we require an axiom from which to infer that the two remaining angles of the triangles are respectively equal and that the bases are equal.

I have shown above (pp. 229–230) that Hilbert has an axiom stating the equality of the remaining angles simply, but proves the equality of the bases.

Another alternative is that of Pasch (*Vorlesungen über neuere Geometrie*, p.109) who has the following "Grundsatz":

If two figures AB and FGH are given (FGH not being contained in a straight length), and AB, FG are congruent, and if a plane surface be laid through A and B, we can specify in this plane surface, produced if necessary, two points C, D, neither more nor less, such that the figures ABC and ABD are congruent with the figure FGH, and the straight line CD has with the straight line AB or with AB produced one point common.

I pass to two points of detail in Euclid's proof:

(1) The inference that, since B coincides with E, and C with F, the bases of the triangles are wholly coincident rests, as expressly stated, on the impossibility of two straight lines enclosing a space, and therefore presents no difficulty.

But (2) most editors seem to have failed to observe that at the very beginning of the proof a much more serious assumption is made without any explanation whatever, namely that, if A be placed on D, and AB on DE, the point B will coincide with E, because AB is equal to DE. That is, the *converse* of *Common Notion* 4 is assumed for straight lines. Proclus merely observes, with regard to the converse of this Common Notion, that it is only true in the case of things "of the same form" (ὁμοειδῆ), which he explains as meaning straight lines, arcs of one and the same circle, and angles "contained by lines similar and similarly situated" (p. 241, 3–8).

Savile however saw the difficulty and grappled with it in his note on the Common Notion. After stating that all straight lines with two points common are congruent between them (for otherwise two straight lines would enclose a space), he argues thus. Let there be two straight lines AB, DE, and let A be placed on D, and AB on DE. Then B will coincide with E. For, if not, let B fall somewhere short of E or beyond E; and in either case it will follow that the less is equal to the greater, which is impossible.

Savile seems to assume (and so apparently does Lardner who gives the same proof) that, if the straight lines be "applied," B will fall somewhere on DE or DE produced. But the grounds for this assumption should surely be stated; and it seems to me that it is necessary to use, not Postulate 1 alone, nor Postulate 2 alone, but both, for this purpose (in other words to assume, not only that *two straight lines cannot enclose a space*, but also that *two straight lines cannot have a common segment*). For the only safe course is to place A upon D and then turn AB about D until *some* point on AB intermediate between A and B coincides with *some* point on DE. In this position AB and DE have two points common. Then Postulate 1 enables us to infer that the straight lines coincide *between* the two common points, and Postulate 2 that they coincide beyond the second common point towards B and E. Thus the straight lines coincide throughout so far as *both* extend; and Savile's argument then proves that B coincides with E.

[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp.259–261 (1908).]

[Heath's commentary on Euclid, *Elements*, Book I, Proposition 7.]

1–6. In an English translation of the enunciation of this proposition it is absolutely necessary, in order to make it intelligible, to insert some words which are not in the Greek. The reason is partly that the Greek enunciation is itself very elliptical, and partly that some words used in it conveyed more meaning than the corresponding words in English do. Particularly is this the case with οὐ συσταθήσονται ἐπί "there shall not be constructed upon," since συνίστασθαι is the regular word for constructing a *triangle* in particular. Thus a Greek would easily understand συσταθήσονται ἐπί as meaning the construction of two lines *forming a triangle on* a given straight line as base; whereas to "construct two straight lines on a straight line" is not in English sufficiently definite unless we explain that they are drawn from the *ends* of the straight line to *meet* at a point. I have had the less hesitation in putting in the words "from the extremities" because they are actually used by Euclid in the somewhat similar enunciation of I. 21.

How impossible a literal translation into English is, if it is convey the meaning of the enunciation intelligibly, will be clear from the following attempt to render literally: "On the same straight line there shall not be constructed two other straight lines equal, each to each, to the same two straight lines (terminating) at different points on the same side, having the same extremities as the original straight lines" (ἐπὶ τῆς αὐτῆς εὐθείας δύο ταῖς αὐταῖς εὐθείαις ἄλλαι δύο εὐθεῖαι ἴσαι ἑκατέρα ἑκατέρᾳ οὐ συσταθήσονται πρὸς ἄλλῳ καὶ ἄλλῳ σημείῳ ἐπὶ τὰ αὐτὰ μέρη τὰ αὐτὰ πέρατα ἔχουσαι ταῖς ἐξ ἀρχῆς εὐθείαις).

The reason why Euclid allowed himself to use, in this enunciation, language apparently so obscure is no doubt that the phraseology was traditional and therefore, vague as it was, had a conventional meaning which the contemporary geometer well understood. This is proved, I think, by the occurrence in Aristotle (*Meteorologica* III. 5, 376 a 2 sqq.) of the very same, evidently technical expressions. Aristotle is there alluding to the theorem given by Eutocius from Apollonius' *Plane Loci* to the effect that, if H, K be two fixed points and M such a variable point that the ratio of MH to MK is a given ratio (not one of equality), the locus of M is a circle. (For an account of this theorem see note on VI. 3 below.) Now Aristotle says "The lines drawn up from H, K in this ratio cannot be constructed to two different points of the semicircle A" (αἱ οὖν ἀπὸ τῶν ΗΚ ἀναγόμεναι γραμμαὶ ἐν τούτῳ τῷ λόγῳ οὐ συσταθήσονται τοῦ ἐφ᾽ ᾧ Α ἡμικυκλίου πρὸς ἄλλο καὶ ἄλλο σημεῖον).

If a paraphrase is allowed instead of a translation adhering as closely as possible to the original, Simson's is the best that could be found, since the fact that the straight lines form *triangles* on the same base is really conveyed in the Greek. Simson's enunciation is, *Upon the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise those which are terminated at the other extremity.* Th. Taylor (the translator of Proclus) attacks Simson's alteration as "indiscreet" and as detracting from the beauty and accuracy of Euclid's enunciation which are enlarged upon by Proclus in his commentary. Yet, when Taylor says, "Whatever difficulty learners may find in conceiving this proposition abstractedly is easily removed by its exposition in the figure," he really gives his case away. The fact is that Taylor, always enthusiastic over his author, was nettled by Simson's slighting remarks on Proclus' comments on the proposition. Simson had said, with reference to Proclus' explanation of the bearing of the second part of I. 5 on I. 7, that it was not "worth while to relate his trifles at full length," to which Taylor retorts "But Mr. Simson was no philosopher; and therefore the greatest part of these Commentaries must be considered by him as trifles, from the want of a philosophic genius to comprehend their meaning, and a taste superior to that of a *mere mathematician*, to discover their beauty and elegance."

20. It would be natural to insert here the step "but the angle ACD is greater than the angle BCD. [*C.N.* 5.]"

21. **much greater,** literally "greater by much" (πολλῷ μείζων). Simson and those who follow him translate: "*much more then* is the angle BDC *greater* than the angle BCD," but the Greek for this would have to be πολλῷ (or πολὺ) μᾶλλόν ἐστι ... μείζων. πολλῷ μᾶλλον, however, though used by Apollonius, is not, apparently, found in Euclid or Archimedes.

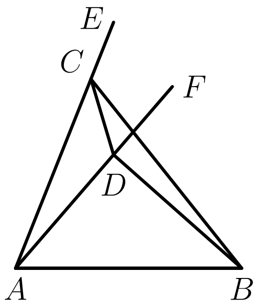
Just as in I. 6 we need a Postulate to justify theoretically the statement that CD falls within the angle ACB, so that the triangle DBC is less than the triangle ABC, so here we need Postulates which shall satisfy us as to the relative positions of CA, CB, CD on the one hand and of DC, DA, DB on the other, in order that we may be able to infer that the angle BDC is greater than the angle ADC, and the angle ACD greater than the angle BCD.

De Morgan (*op. cit.* p. 7) observes that I. 7 would be made easy to beginners if they were first familiarised, as a common notion, with "if any two magnitudes be equal, any magnitude greater than the one is greater than any magnitude less than the other." I doubt however whether a beginner would follow this easily; perhaps it would be more easily apprehended in the form "if any magnitude A is greater than a magnitude B, the magnitude A is greater than any magnitude equal to B, and (*a fortiori*) greater than any magnitude less than B."

It has been mentioned already (note on I. 5) that the second case of I. 7 given by Simson and in our text-books generally is not in the original text (the omission being in accordance with Euclid's general practice of giving only one case, and that the most difficult, and leaving the others to be worked out by the reader for himself). The second case is given by Proclus as the answer to a possible *objection* to Euclid's proposition, which should assert that the proposition is not proved to be universally true, since the proof given does not cover all possible cases. Here the objector is supposed to contend that what Euclid declares to be impossible may still be possible if one pair of lines lie wholly within the other pair of lines; and the second part of I. 5 enables the objection to be refuted.

If possible, let AD, DB be entirely within the triangle formed by AC, CB with AB, and let AC be equal to AD and BC to BD.

Join CD, and produce AC, AD to E and F.



Then, since AC is equal to AD,
the triangle ACD is isosceles,
and the angles ECD, FDC under the base are equal.

But the angle ECD is greater than the angle BCD,
therefore the angle FDC is also greater than the angle BCD.

Therefore the angle BDC is greater by far than the angle BCD.

Again, since DB is equal to CB,
the angles at the base of the triangle BDC are equal, [I. 5]
that is, the angle BDC is equal to the angle BCD.

Therefore the same angle BDC is both greater than and equal to the angle BCD: which is impossible.

The case in which D falls on AC or BC does not require proof.

I have already referred (note on I. 1) to the mistake made by those editors who regard I. 7 as being of no use except to prove I. 8. What I. 7 proves is that if, in addition to the base of a triangle, the length of the side terminating at each extremity of the base is given, only one triangle satisfying these conditions can be constructed on one and the same side of the given base. Hence not only does I. 7 enable us to prove I. 8, but it supplements I. 1 and I. 22 by showing that the constructions of those propositions give one triangle only on one and the same side of the base. But for I. 7 this could not be proved except by anticipating III. 10, of which therefore I. 7 is the equivalent for Book I. purposes. Dodgson (*Euclid and his modern Rivals*, pp. 194–5) puts it another way. "It [I. 7] shows that, of all plane figures that can be made by hingeing rods together, the *three*-sided ones (and these only) are *rigid* (which is another way of stating the fact that there cannot be *two* such figures on the same base). This is analogous to the fact, in relation to solids contained by plane surfaces hinged together, that *any* such solid is rigid, there being no maximum number of sides. And there is a close analogy between I. 7, 8 and III. 23, 24. these analogies give to geometry much of its beauty, and I think that they ought not to be lost sight of." It will therefore be apparent how ill-advised are those editors who eliminate I. 7 altogether and rely on Philo's proof for I. 8.

Proclus, it may be added, gives (pp. 268, 19–269, 10) another explanation of the retention of I. 7, notwithstanding that it was apparently only required for I. 8. It was said that astronomers used it to prove that three successive eclipses could not occur at equal intervals of time, i.e. that the third could not follow the second at the same interval as the second followed the first; and it was argued that Euclid had an eye to this astronomical application of the proposition. But, as we have seen, there are other grounds for retaining the proposition which are quite sufficient of themselves.

[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp.262–264 (1908).]

[Heath's commentary on Euclid, *Elements*, Book I, Proposition 8.]

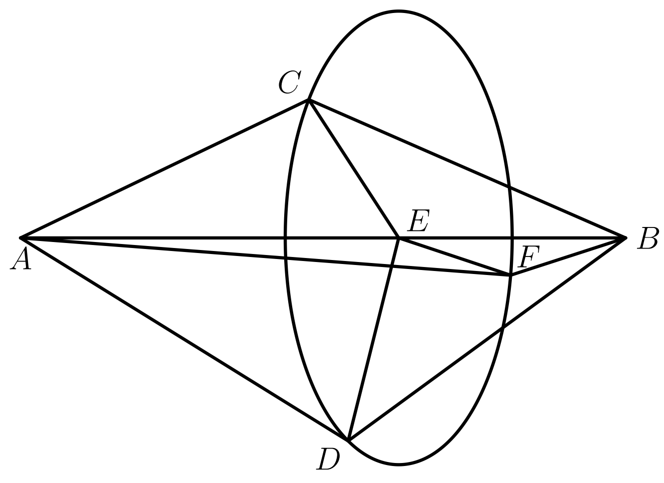
19. **BA, AC.** The text has here "BA, CA."

21. **fall beside them.** The Greek has the future, παραλλάξουσι. παραλλάττω means "to pass by without touching," "to miss" or "to deviate."

As pointed out above (p. 257) I. 8 is a *partial* converse of I. 4.

It is to be observed that in I. 8 Euclid is satisfied with proving the equality of the vertical angles and does not, as in I. 4, add that the triangles are equal, and the remaining angles are equal respectively. The reason is no doubt (as pointed out by Proclus and by Savile after him) that, when once the vertical angles are proved equal, the rest follows from I. 4, and there is no object in proving again what has been proved already.

Aristotle has an allusion to the theorem of this proposition in *Meteorologica* III. 3, 373 a 5–16. He is speaking of the rainbow and observes that, if equal rays be reflected from one and the same point to one and the same point, the points at which reflection takes place are on the circumference of a circle. "For let the broken lines ACB, AFB, ADB be all reflected from the point A to the point B (in such a way that) AC, AF, AD are all equal to one another, and the lines (terminating) at B i.e. CB, FB, DB, are likewise all equal; and let AEB be joined. It follows that *the triangles are equal*; for they are upon the equal (base) AEB."



Heiberg (*Mathematisches zu Aristoteles*, p. 18) thinks that the form of the conclusion quoted is an indication that in the corresponding proposition to Eucl. I. 8, as it lay before Aristotle, it was maintained that the *triangles* were equal, and not only the angles, and "we see here therefore, in a clear example, how the stones of the ancient fabric were recut for the rigid structure of his *Elements*." I do not, however, think that this inference from Aristotle's language as to the form of the pre-Euclidean proposition is safe. Thus if we, nowadays, were arguing from the data in the passage of Aristotle, we should doubtless infer directly that the triangles are equal in all respects, quoting I. 8 alone. Besides, Aristotle's language is rather careless, as the next sentences of the same passage show. "Let perpendiculars," he says, "be drawn to AEB from the angles CE from C, FE from F and DE from D. These, then, are equal; for they are all in equal triangles, and in one plane; for all of them are perpendicular to AEB, and they meet at one point E. Therefore the (line) drawn (through C, F, D) will be a circle, and its centre (will be) E." Aristotle should obviously have proved that the three perpendiculars *will* meet at one point E on AEB before he spoke of drawing the perpendiculars CE, FE, DE. This of course follows from their being "in equal triangles" (by means of Eucl. I. 26); and then, from the fact that the perpendiculars meet at one point on AB, it can be inferred that all three are in one plane.

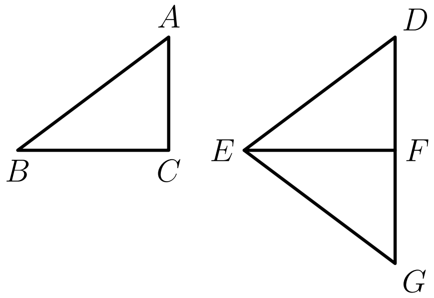
Philo's proof of I. 8.

This alternative proof avoids the use of I. 7, and it is elegant; but it is inconvenient in one respect; since three cases have to be distinguished. Proclus gives the proof in the following order (pp. 266, 15–268, 14).

Let ABC, DEF be two triangles having the sides AB, AC equal to the sides DE, DF respectively, and the base BC equal to the base EF.

Let the triangle ABC be applied to the triangle DEF, so that B is placed on E and BC on EF, but so that A falls on the opposite side of EF from D, taking the position G. Then C will coincide with F, since BC is equal to EF.

Now FG will either be in a straight line with DF, or make an angle with it, and in the latter case the angle will either be *interior* (κατὰ τὸ ἐντός) to the figure or *exterior* (κατὰ τὸ ἐκτός).

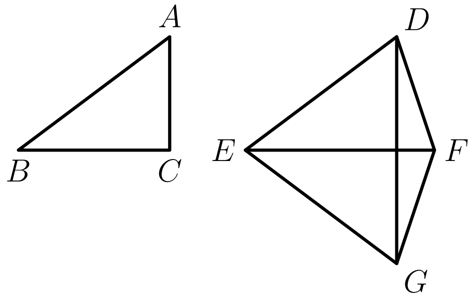


I. Let FG be in a straight line with DF.

Then, since DE is equal to EG, and DFG is a straight line, DEG is an isosceles triangle, and the angle at D is equal to the angle at G. [I. 5].

II. Let DF, FG form an angle *interior* to the figure.

Let DG be joined.



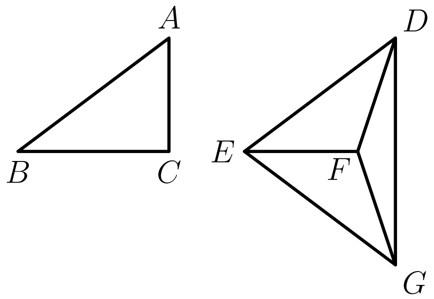
Then, since DE, EG are equal, the angle EDG is equal to the angle EGD.

Again, since DF is equal to FG, the angle FDG is equal to the angle FGD.

Therefore, by addition,
the whole angle EDF is equal to the whole angle EGF.

III. Let DF, FG form an angle *exterior* to the figure.

Let DG be joined.



The proof proceeds as in the last case, except that subtraction takes the place of addition, and the remaining angle EDF is equal to the remaining angle EGF.

Therefore in all three cases the angle EDF is equal to the angle EGF, that is, to the angle BAC.

It will be observed that, in accordance with the practice of the Greek geometers in not recognising as an "angle" any angle not less than two right angles, the re-entrant angle of the quadrilateral $DEGF$ is ignored and the angle DFG is said to be *outside* the figure.