MAU23302—Euclidean and Non-Euclidean Geometry
School of Mathematics, Trinity College
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Part II, Section 1:
Möbius Transformations and Cross-Ratio

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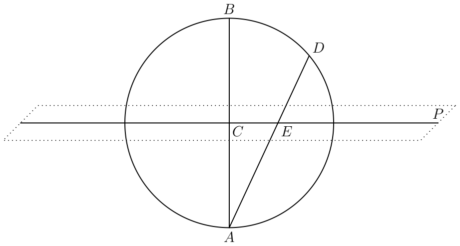
1.1. Stereographic Projection

Let a sphere in three-dimensional space be given, let C be the centre of that sphere, let AB be a diameter of that sphere with endpoints A and B, and let P be the plane through the centre of the sphere that is perpendicular to the diameter AB. Given a point D of the sphere distinct from the point A, the image of D under *stereographic projection* from the point A is defined to be the point E at which the line passing through the points A and D intersects the plane P.



Proposition 1.1

Let S^2 be the unit sphere in $\mathbb{R}^3$, consisting of those points (u, v, w) of $\mathbb{R}^3$ that satisfy the equation $u^2 + v^2 + w^2 = 1$, and let P be the plane consisting of those points (u, v, w) of $\mathbb{R}^3$ for which $w = 0$. Then, for each point (u, v, w) of S^2 distinct from the point $(0, 0, -1)$, the straight line passing through the points (u, v, w) and $(0, 0, -1)$ intersects the plane P at the point $(x, y, 0)$ at which

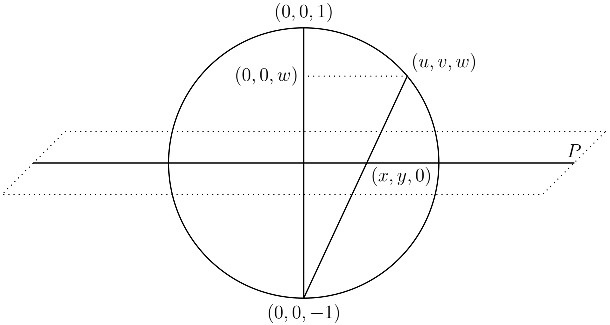
$$x = \frac{u}{w+1} \quad \text{and} \quad y = \frac{v}{w+1}.$$

Proof

Let $A = (0, 0, -1)$, $D = (u, v, w)$ and $E = (x, y, 0)$. Then the displacements of the points D and E from the point A are represented by the vectors $(u, v, w + 1)$ and $(x, y, 1)$ respectively. These vectors are parallel because the points A, D and E are collinear. Consequently

$$\frac{x}{u} = \frac{y}{v} = \frac{1}{w + 1}.$$

The result follows. ∎


$(0,0,1)$
$(0,0,w)$
(u,v,w)
P
$(x,y,0)$
$(0,0,-1)$

Definition

Let (u, v, w) be a point on the unit sphere distinct from the point $(0, 0, -1)$, where $u^2 + v^2 + w^2 = 1$, and let (x, y) be a point of the plane $\mathbb{R}^2$. We say that the point (x, y) is the *image* of the point (u, v, w) under *stereographic projection* from the point $(0, 0, -1)$ if

$$x = \frac{u}{w+1} \quad \text{and} \quad y = \frac{v}{w+1}.$$

Proposition 1.2

Each point (x, y) of $\mathbb{R}^2$ is the image, under stereographic projection from the point $(0, 0, -1)$, of the point (u, v, w) of the unit sphere for which

$$u = \frac{2x}{1 + x^2 + y^2}, \quad v = \frac{2y}{1 + x^2 + y^2} \quad \text{and} \quad w = \frac{1 - x^2 - y^2}{1 + x^2 + y^2}.$$

This point (u, v, w) is distinct from the point $(0, 0, -1)$.

Proof

Given a point (x, y) of $\mathbb{R}^2$, the straight line passing through the points $(0, 0, -1)$ and $(x, y, 0)$ is not tangent to the unit sphere, and therefore intersects the unit sphere at some point distinct from $(0, 0, -1)$. It follows that every point of $\mathbb{R}^2$ is the image, under stereographic projection from $(0, 0, -1)$, of some point of the unit sphere distinct from the point $(0, 0, -1)$.

Let (x, y) be the image, under stereographical projection from the point $(0, 0, -1)$, of a point (u, v, w), where $u^2 + v^2 + w^2 = 1$ and $w \neq -1$. Then

$$x = \frac{u}{w+1}, \quad y = \frac{v}{w+1}.$$

It follows that

$$x^2 + y^2 = \frac{u^2 + v^2}{(w+1)^2} = \frac{1 - w^2}{(w+1)^2} = \frac{1 - w}{w+1}.$$

It follows that

$$w(x^2 + y^2) + x^2 + y^2 = 1 - w,$$

and therefore

$$w = \frac{1 - x^2 - y^2}{1 + x^2 + y^2}.$$

But then

$$1 + w = 1 + \frac{1 - x^2 - y^2}{1 + x^2 + y^2} = \frac{2}{1 + x^2 + y^2},$$

and therefore

$$\begin{aligned} u &= (1 + w)x = \frac{2x}{1 + x^2 + y^2}, \\ v &= (1 + w)y = \frac{2y}{1 + x^2 + y^2}. \end{aligned}$$

Conversely if

$$u = \frac{2x}{1 + x^2 + y^2}, \quad v = \frac{2y}{1 + x^2 + y^2} \quad \text{and} \quad w = \frac{1 - x^2 - y^2}{1 + x^2 + y^2}.$$

then

$$u^2 + v^2 + w^2 = \frac{4(x^2 + y^2) + (1 - x^2 - y^2)^2}{(1 + x^2 + y^2)^2} = 1,$$

because

$$\begin{aligned} 4(x^2 + y^2) &+ (1 - x^2 - y^2)^2 \\ &= 4(x^2 + y^2) + 1 - 2(x^2 + y^2) + (x^2 + y^2)^2 \\ &= 1 + 2(x^2 + y^2) + (x^2 + y^2)^2 \\ &= (1 + x^2 + y^2)^2. \end{aligned}$$

Also $w > -1$ and

$$x = \frac{u}{w + 1} \quad \text{and} \quad y = \frac{v}{w + 1}.$$

The result follows. ∎

1. Möbius Transformations and Cross-Ratios

1.2. The Riemann Sphere

The *Riemann sphere* $\mathbb{P}^1$ may be defined as the set $\mathbb{C} \cup \{\infty\}$ obtained by augmenting the system $\mathbb{C}$ of complex numbers with an additional element, denoted by ∞, where ∞ is not itself a complex number, but is an additional element added to the set, with the additional conventions that

$$z + \infty = \infty, \quad \infty \times \infty = \infty, \quad \frac{z}{\infty} = 0 \quad \text{and} \quad \frac{\infty}{z} = \infty$$

for all complex numbers z, and

$$z \times \infty = \infty, \qquad \text{and} \quad \frac{z}{0} = \infty$$

for all non-zero complex numbers z. The symbol ∞ cannot be added to, or subtracted from, itself. Also 0 and ∞ cannot be divided by themselves.

Note that, because the sum of two elements of $\mathbb{P}^1$ is not defined for every single pair of elements of $\mathbb{P}^1$, this set cannot be regarded as constituting a group under the operation of addition. Similarly its non-zero elements cannot be regarded as constituting a group under multiplication. In particular, the Riemann sphere cannot be regarded as constituting a field.

Note that any element of the Riemann sphere can be represented in the form $\dfrac{u}{v}$, where u and v are complex numbers that are not both equal to zero. Moreover the values of this fraction are determined as follows:

- $\dfrac{u}{v} = z$ for some non-zero complex number z if and only if $u \neq 0$, $v \neq 0$ and $u = zv$;
- $\dfrac{u}{v} = 0$ if and only if $u = 0$ and $v \neq 0$;
- $\dfrac{u}{v} = \infty$ if and only if $u \neq 0$ and $v = 0$.

Lemma 1.3

Let u, v, u' and v' be complex numbers, where u and v are not both zero and also u' and v' are not both zero. Then the following are true:

(i) $\dfrac{u}{v} = \dfrac{u'}{v'}$ *if and only if* $v'u = u'v$;

(ii) $\dfrac{u}{v} = \dfrac{u'}{v'}$ *if and only if there exists some non-zero complex number* w *for which* $u' = wu$ *and* $v' = wv$;

(iii) *in cases where* $\dfrac{u}{v} = \dfrac{u'}{v'}$ *it follows that* $u = 0$ *if and only if* $u' = 0$;

(iv) *in cases where* $\dfrac{u}{v} = \dfrac{u'}{v'}$ *it follows that* $v = 0$ *if and only if* $v' = 0$.

Proof

First suppose that the complex numbers u, v, u' and v' are all non-zero. Then all four properties follow directly.

Next suppose that $u = 0$. Then $v \neq 0$ and $\dfrac{u}{v} = 0$. It follows in this case that $\dfrac{u}{v} = \dfrac{u'}{v'}$ if and only if $\dfrac{u'}{v'} = 0$, in which case $u' = 0$. Thus in cases where $\dfrac{u}{v} = \dfrac{u'}{v'}$ we find that $u = 0$ implies that $u' = 0$. Similarly $u' = 0$ if and only if $u = 0$, and thus $u = 0$ if and only if $u' = 0$. Note also that in cases where $u = 0$ and $\dfrac{u}{v} = \dfrac{u'}{v'}$, the complex numbers v and v' are both non-zero, and consequently the identities $u' = wu$ and $v' = wv$ hold simultaneously on taking $w = \dfrac{v'}{v}$.

Next suppose that $v = 0$. Then $u \neq 0$ and $\dfrac{u}{v} = \infty$. It follows in this case that $\dfrac{u}{v} = \dfrac{u'}{v'}$ if and only if $\dfrac{u'}{v'} = \infty$, in which case $v' = 0$. Thus in cases where $\dfrac{u}{v} = \dfrac{u'}{v'}$ we find that $v = 0$ implies that $v' = 0$. Similarly $v' = 0$ if and only if $v = 0$, and thus $v = 0$ if and only if $v' = 0$. Note also that in cases where $v = 0$ and $\dfrac{u}{v} = \dfrac{u'}{v'}$, the complex numbers u and u' are both non-zero, and consequently the identities $u' = wu$ and $v' = wv$ hold simultaneously on taking $w = \dfrac{u'}{u}$.

Consequently all four properties (i), (ii), (iii) and (iv) have been established, as required. ∎

Lemma 1.4

Let p_1 and p_2 be elements of the Riemann sphere that are not both equal to ∞, and let u_1, u_2, v_1 and v_2 be complex numbers, where u_1 and v_1 are not both zero, u_2 and v_2 are not both zero, and v_1 and v_2 are not both zero, such that

$$p_1 = \frac{u_1}{v_1} \quad \textit{and} \quad p_2 = \frac{u_2}{v_2}.$$

Then the sum $p_1 + p_2$ of the elements p_1 and p_2 of the Riemann sphere is defined so as to ensure that

$$p_1 + p_2 = \frac{v_1 u_2 + v_2 u_1}{v_1 v_2}.$$

Proof

If $v_1 = 0$ then $v_2 u_1 \neq 0$, and consequently $v_1 u_2 + v_2 u_1 \neq 0$. Similarly if $v_2 = 0$ then $v_1 u_2 \neq 0$, and consequently $v_1 u_2 + v_2 u_1 \neq 0$. It follows that, in all cases, the complex numbers $v_1 u_2 + v_2 u_1$ and $v_1 v_2$ are not both zero, and consequently there is a well-defined element of the Riemann sphere that is determined by the fraction

$$\frac{v_1 u_2 + v_2 u_1}{v_1 v_2}.$$

If neither of p_1 and p_2 is the element ∞ of the Riemann sphere, then both p_1 and p_2 are complex numbers, and the above fraction represents the sum of those complex numbers, determined in the usual fashion within the algebra of complex numbers.

1. Möbius Transformations and Cross-Ratios (continued)

On the other hand, if exactly one of the elements p_1 and p_2 of the Riemann sphere concides with ∞ then exactly one of the complex numbers v_1 and v_2 is equal to zero, and the above fraction represents the element ∞ of the Riemann sphere. The result follows. $\blacksquare$

The following proposition follows directly from Proposition 1.2.

Proposition 1.5

Let $\varphi\colon \mathbb{P}^1 \to \mathbb{R}^3$ *be the mapping from the Riemann sphere* $\mathbb{P}^1$ *to* $\mathbb{R}^3$ *defined such that* $\varphi(\infty) = (0, 0, -1)$ *and*

$$\varphi(x + y\sqrt{-1}) = \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{1 - x^2 - y^2}{1 + x^2 + y^2} \right)$$

for all real numbers x *and* y. *Then the map* φ *sets up a one-to-one correspondence between points of the Riemann sphere* $\mathbb{P}^1$ *and points of the unit sphere* S^2 *in* $\mathbb{R}^3$. *To each point of the Riemann sphere* $\mathbb{P}^1$ *there corresponds exactly one point of the unit sphere* S^2 *in three-dimensional Euclidean space, and vice versa. Moreover if* (u, v, w) *is a point of the unit sphere* S^2 *distinct from* $(0, 0, -1)$ *then* $(u, v, w) = \varphi(x + y\sqrt{-1})$, *where*

$$x = \frac{u}{w + 1} \quad \text{and} \quad y = \frac{v}{w + 1}.$$

1.3. Möbius Transformations

Lemma 1.6

Let a, b, c and d be complex numbers satisfying $ad - bc \neq 0$. Then these complex numbers determine a well-defined function $\mu\colon \mathbb{P}^1 \to \mathbb{P}^1$ mapping the Riemann sphere $\mathbb{P}^1$ into itself that is characterized by the property that

$$\mu\left(\frac{u}{v}\right) = \frac{au + bv}{cu + dv}$$

for all complex numbers u and v that are not both zero.

Proof

Let u and v be complex numbers that are not both zero. Then

$$d(au + bv) - b(cu + dv) = (ad - bc)u$$

and

$$a(cu + dv) - c(au + bv) = (ad - bc)v$$

Now $ad - bc \neq 0$ and also u and v are not both zero. It must therefore be the case that $au + bv$ and $cu + dv$ are not both zero. It therefore follows that u and v determine a well-defined element of the Riemann sphere represented by the fraction $\dfrac{au + bv}{cu + dv}$.

Moreover if u, v, u' and v' are complex numbers, where u and v are not both zero, and where u' and v' are not both zero, and if $u/v = u'/v'$, then there exists some non-zero complex number w for which $u' = wu$ and $v' = wv$. But it then follows that

$$\frac{au + bv}{cu + dv} = \frac{au' + bv'}{cu' + dv'}.$$

It follows from what has been shown that a quadruple of complex numbers a, b, c and d satisfying the condition $ad - bc \neq 0$ does indeed determine a well-defined function μ mapping the Riemann sphere into itself that is characterized by the property that

$$\mu\left(\frac{u}{v}\right) = \frac{au + bv}{cu + dv}$$

for all complex numbers u and v that are not both zero, as claimed. ∎

A Möbius transformation of the Riemann sphere is determined by its coefficients. It is convenient to specify these coefficients in the form of a non-singular 2×2 matrix.

Accordingly let A be a non-singular 2×2 matrix. Then there exist complex numbers a, b, c and d for which

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Moreover the requirement that A be non-singular (i.e., invertible) ensures that $ad - bc \neq 0$. We denote by μ_A the Möbius transformation of the Riemann sphere defined so that

$$\mu_A\left(\frac{u}{v}\right) = \frac{au + bv}{cu + dv}$$

for all complex numbers u and v that are not both zero.

It then follows that

$$\mu_A(z) = \frac{az + b}{cz + d}$$

for all complex numbers z for which $cz + d \neq 0$. If $c \neq 0$ then

$$\mu_A\left(-\frac{d}{c}\right) = \infty \quad \text{and}$$

$$\mu_A(\infty) = \frac{a}{c}.$$

If $c = 0$ then $d \neq 0$ and accordingly $\mu_A(\infty) = \infty$ and $\mu_A(z) = (az + b)/d$ for all complex numbers z.

Proposition 1.7

The composition of any two Möbius transformations is a Möbius transformation. Specifically let A and B be non-singular 2×2 matrices with complex coefficients, and let μ_A and μ_B be the corresponding Möbius transformations of the Riemann sphere. Then the composition $\mu_A \circ \mu_B$ of these Möbius transformations is the Möbius transformation μ_{AB} of the Riemann sphere determined by the product AB of the matrices A and B.

Proof

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} f & g \\ h & k \end{pmatrix},$$

and let

$$AB = \begin{pmatrix} m & n \\ p & q \end{pmatrix}.$$

Then

$$m = af + bh, \quad n = ag + bk,$$

$$p = cf + dh \quad \text{and} \quad q = cg + dk.$$

Now let u and v be complex numbers that are not both zero. Then $fu + gv$ and $hu + kv$ are not both zero, because the matrix B is non-singular. The definition of the Möbius transformations μ_A, μ_B and μ_{AB} associated with the non-singular 2×2 matrices A, B and AB respectively ensures that

$$\begin{aligned}
\mu_A\left(\mu_B\left(\frac{u}{v}\right)\right) &= \mu_A\left(\frac{fu+gv}{hu+kv}\right) \\
&= \frac{a(fu+gv)+b(hu+kv)}{c(fu+gv)+d(hu+kv)} \\
&= \frac{mu+nv}{pu+qv} = \mu_{AB}\left(\frac{u}{v}\right).
\end{aligned}$$

The result follows. ∎

Corollary 1.8

Let a, b, c and d be complex numbers satisfying $ad - bc \neq 0$, let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

and let μ_A and μ_C be the corresponding Möbius transformations, defined so that

$$\mu_A\left(\frac{u}{v}\right) = \frac{au + bv}{cu + dv} \quad \text{and} \quad \mu_C(z) = \frac{du - bv}{-cu + av}$$

for all complex numbers u and v that are not both zero. Then the mapping $\mu_A\colon \mathbb{P}^1 \to \mathbb{P}^1$ is invertible, and its inverse is the Möbius transformation $\mu_C\colon \mathbb{P}^1 \to \mathbb{P}^1$.

Proof
Let

$$M = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}.$$

Then $AC = CA = M$. It follows from Proposition 1.7 that

$$\mu_A \circ \mu_C = \mu_C \circ \mu_A = \mu_M = \mathrm{Id}_{\mathbb{P}^1},$$

where $\mathrm{Id}_{\mathbb{P}^1}$ denotes the identity map of the Riemann sphere. The result follows. ∎

Proposition 1.9

Let a, b, c, d, f, g, h and k be complex numbers satisfying $ad \neq bc$ and $fk \neq gh$, and let μ_1 and μ_2 be the Möbius transformations of the Riemann sphere defined so that

$$\mu_1(z) = \frac{az+b}{cz+d}, \quad \mu_2(z) = \frac{fz+g}{hz+k}$$

for all complex numbers with $cz + d \neq 0$ and $hz_2 + k \neq 0$. Then the Möbius transformations μ_1 and μ_2 coincide if and only if there exists some non-zero complex number m such that $f = ma$, $g = mb$, $h = mc$ and $k = md$.

Proof

Clearly if there exists a complex number m with the stated properties then the Möbius transformations μ_1 and μ_2 coincide.

Conversely suppose that there is some Möbius transformation μ of the Riemann sphere with the property that

$$\mu(z) = \frac{az+b}{cz+d} = \frac{fz+g}{hz+k}$$

whenever $cz+d \neq 0$ and $hz+k \neq 0$.

First consider the case when $c = 0$. Then no real number is mapped by μ to the point ∞ of the Riemann sphere "at infinity" and therefore $h = 0$. But then $d \neq 0$, $k \neq 0$, $b/d = g/k$ and $a/d = f/k$. Therefore if we take $m = k/d$ in this case we find that $m \neq 0$, $f = ma$, $g = mb$, $h = mc$ and $k = md$. The existence of the required non-zero complex number m has therefore been verified in the case when $c = 0$.

Suppose then that $c \neq 0$. Then $h \neq 0$ and $\mu(-k/h) = \infty = \mu(-d/c)$, and therefore $k/h = d/c$. Let $m = h/c$. Then $k = md$. It then follows that

$$fz + g = (hz + k)\mu(z) = m(cz + d)\mu(z) = m(az + b)$$

for all complex numbers z distinct from $-d/c$, and therefore $f = ma$ and $g = mb$. The result follows. ∎

1.4. Straight Lines and Circles in the Complex Plane

We consider the forms of the equations that are commonly used to represent straight lines and circles in the complex plane.
Straight lines in the plane are represented with respect to standard Cartesian coordinates x and y by equations of the form $px + qy + h = 0$ where p, q and h are real numbers for which p and q are not both zero. If we represent the point (x, y) by the complex number $x + iy$, where $i = \sqrt{-1}$, then the equation of the line $px + qy + h = 0$ can be expressed, in the algebra of complex numbers, by the equation

$$2\mathrm{Re}[\overline{b}z] + h = 0,$$

where $b = \frac{1}{2}(p + iq)$. Moreover equations of this form, in which b is a non-zero complex number and h is a real number, determine straight lines in the complex plane.

Next we consider the form taken by the equation of a circle in the complex plane. If the centre of the circle is represented by the complex number m, and if the real number r represents the radius of the circle, where $r > 0$, then the circle consists of those complex numbers z that satisfy the equation $|z - m|^2 = r^2$. Expanding out, this equation can be presented in the form

$$|z|^2 - 2\mathrm{Re}[\overline{m}z] + |m|^2 - r^2 = 0.$$

It follows from this that, given an equation of the form

$$g|z|^2 + 2\mathrm{Re}[\overline{b}z] + h = 0,$$

in which g and h are real numbers, and b is a complex number, that equation represents a circle in the complex plane if and only if $g \neq 0$ and $|b|^2 > gh$. (In cases where $|b|^2 = gh$ the equation is satisfied only at a single point; and if $|b|^2 < gh$ then the equation is not satisfied anywhere in the complex plane.)

We conclude from this discussion that straight lines and circles in the complex plane are those loci (or subsets) of the complex plane that can be specified by equations of the form

$$g|z|^2 + 2\mathrm{Re}[\overline{b}z] + h = 0,$$

in which g and h are real numbers, b is a complex number, and $|b|^2 > gh$. The equation represents a circle if $g \neq 0$, but represents a straight line if $g = 0$.

Proposition 1.10

Any Möbius transformation maps straight lines and circles in the complex plane to straight lines and circles.

Proof

The equation of a line or circle in the complex plane can be expressed in the form

$$g|z|^2 + 2\mathrm{Re}[\overline{b}z] + h = 0,$$

where g and h are real numbers, b is a complex number, and $|b|^2 > gh$. Moreover a locus of points in the complex plane satisfying an equation of this form is a circle if $g \neq 0$ and is a line if $g = 0$.

Let g and h be real constants, let b be a complex constant, and let $z = 1/w$, where $w \neq 0$ and w satisfies the equation

$$g|w|^2 + 2\mathrm{Re}[\overline{b}w] + h = 0,$$

Then

$$g|w|^2 + \overline{b}w + b\overline{w} + h = 0,$$

and therefore

$$\begin{aligned} g + \mathrm{Re}[bz] + h|z|^2 &= g + \overline{b}\overline{z} + bz + h|z|^2 \\ &= \frac{1}{|w|^2}\left(g|w|^2 + \overline{b}w + b\overline{w} + h\right) = 0. \end{aligned}$$

We deduce from this that the Möbius transformation that sends z to $1/z$ for all non-zero complex numbers z maps lines and circles to lines and circles.

Let $\mu\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a Möbius transformation of the Riemann sphere. Then there exist complex numbers a, b, c and d satisfying $ad - bc \neq 0$ such that

$$\mu(z) = \frac{az + b}{cz + d}$$

for all complex numbers z for which $cz + d \neq 0$. The result is immediate when $c = 0$. We therefore suppose that $c \neq 0$. Then

1. Möbius Transformations and Cross-Ratios (continued)

$$\mu(z) = \frac{az+b}{cz+d} = \frac{a}{c} - \frac{ad-bc}{c} \times \frac{1}{cz+d}$$

when $cz + d \neq 0$. The Möbius transformation μ is thus the composition of three maps that each send circles and straight lines to circles and straight lines, namely the maps

$$z \mapsto cz + d, \quad z \mapsto \frac{1}{z} \quad \text{and} \quad z \mapsto \frac{a}{c} - \frac{(ad-bc)z}{c}.$$

Thus the Möbius transformation μ must itself map circles and straight lines to circles and straight lines, as required. ∎

1.5. Cross Ratios

Let p_1, p_2, p_3 and p_4 be elements of the Riemann sphere, and, for $j = 1, 2, 3, 4$, let u_j, v_j, u'_j and v'_j be complex numbers that are such as to ensure that u_j and v_j are not both zero, u'_j and v'_j are not both zero and

$$p_j = \frac{u_j}{v_j} = \frac{u'_j}{v'_j}$$

for $j = 1, 2, 3, 4$. Then there exist non-zero complex numbers w_1, w_2, w_3 and w_4 that are such as to ensure that $u'_j = w_j u_j$ and $v'_j = w_j v_j$ for $j = 1, 2, 3, 4$ (see Lemma 1.3). Let complex numbers ρ, ρ', σ and σ' be defined so that

$$\begin{aligned}
\rho &= (u_1 v_3 - u_3 v_1)(u_2 v_4 - u_4 v_2), \\
\sigma &= (u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1), \\
\rho' &= (u'_1 v'_3 - u'_3 v'_1)(u'_2 v'_4 - u'_4 v'_2), \\
\sigma' &= (u'_2 v'_3 - u'_3 v'_2)(u'_1 v'_4 - u'_4 v'_1).
\end{aligned}$$

Then $\rho' = w_1 w_2 w_3 w_4 \rho$ and Then $\sigma' = w_1 w_2 w_3 w_4 \sigma$.

It follows that $\rho' = 0$ if and only if $\rho = 0$, $\sigma' = 0$ if and only if $\sigma = 0$, and $\dfrac{\rho}{\sigma} = \dfrac{\rho'}{\sigma'}$ in all cases where ρ and σ are not both zero.

Now $\rho = 0$ if and only if either $p_1 = p_3$ or $p_2 = p_4$. (This follows on applying Lemma 1.3.) Moreover $p_1 = p_3$ and $\sigma = 0$ if and only if either $p_1 = p_2 = p_3$ or $p_1 = p_3 = p_4$. Also $p_2 = p_4$ and $\sigma = 0$ if and only if either $p_2 = p_3 = p_4$ or $p_1 = p_2 = p_4$. If follows that ρ and σ are both equal to zero if and only if three of the elements p_1, p_2, p_3, p_4 coincide with one another.

We conclude that, in all cases where no three of the elements p_1, p_2, p_3 and p_4 of the Riemann sphere coincide with one another, there exists a well-defined element $(p_1, p_2; p_3, p_4)$ of the Riemann sphere that is determined so as to ensure that if u_j and v_j are complex numbers determined for $j = 1, 2, 3, 4$ so as to ensure that u_j and v_j are not both zero and $p_j = \dfrac{u_j}{v_j}$, then

$$(p_1, p_2; p_3, p_4) = \frac{(u_1 v_3 - u_3 v_1)(u_2 v_4 - u_4 v_2)}{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)}.$$

This element $(p_1, p_2; p_3, p_4)$ of the Riemann sphere is referred to as the *cross-ratio* of p_1, p_2, p_3 and p_4.

Proposition 1.11

Let p_1, p_2, p_3 and p_4 be distinct elements of the Riemann sphere $\mathbb{P}^1$, and let $q = (p_1, p_2; p_3, p_4)$. Then

- $(p_1, p_2; p_3, p_4)$, $(p_2, p_1; p_4, p_3)$, $(p_3, p_4; p_1, p_2)$, $(p_4, p_3; p_2, p_1)$ *are all equal to* q;
- $(p_1, p_2; p_4, p_3)$, $(p_2, p_1; p_3, p_4)$, $(p_4, p_3; p_1, p_2)$, $(p_3, p_4; p_2, p_1)$ *are all equal to* $\dfrac{1}{q}$.
- $(p_1, p_3; p_2, p_4)$, $(p_3, p_1; p_4, p_2)$, $(p_2, p_4; p_1, p_3)$, $(p_4, p_2; p_3, p_1)$ *are all equal to* $1 - q$;
- $(p_1, p_4; p_2, p_3)$, $(p_4, p_1; p_3, p_2)$, $(p_2, p_3; p_1, p_4)$, $(p_3, p_2; p_4, p_1)$ *are all equal to* $\dfrac{q-1}{q}$;

- $(p_1, p_3; p_4, p_2)$, $(p_3, p_1; p_2, p_4)$, $(p_4, p_2; p_1, p_3)$, $(p_2, p_4; p_3, p_1)$ are all equal to $\dfrac{1}{1-q}$;
- $(p_1, p_4; p_3, p_2)$, $(p_4, p_1; p_2, p_3)$, $(p_3, p_2; p_1, p_4)$, $(p_2, p_3; p_4, p_1)$ are all equal to $\dfrac{q}{q-1}$;

Proof

Let u_1, v_1, u_2, v_2, u_3, v_3, u_4 and v_4 be complex numbers with the properties that u_j and v_j are not both zero and $p_j = u_j/v_j$ for $j = 1, 2, 3, 4$ (where $u_j/v_j = \infty$ in cases where $u_j \neq 0$ and $v_j = 0$). Then

$$q = (p_1, p_2; p_3, p_4) = \frac{(u_1 v_3 - u_3 v_1)(u_2 v_4 - u_4 v_2)}{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)}.$$

It follows directly that

$$(p_1, p_2; p_3, p_4), \ (p_2, p_1; p_4, p_3), \ (p_3, p_4; p_1, p_2) \text{ and } (p_4, p_3; p_2, p_1)$$

are all equal to q. Also

$$(p_1, p_2; p_4, p_3) = \frac{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)}{(u_1 v_3 - u_3 v_1)(u_2 v_4 - u_4 v_2)} = \frac{1}{q}.$$

Next we note that

$$(p_4, p_2; p_3, p_1) = \frac{(u_4 v_3 - u_3 v_4)(u_2 v_1 - u_1 v_2)}{(u_2 v_3 - u_3 v_2)(u_4 v_1 - u_1 v_4)}.$$

It follows that

$$\begin{aligned}
1 &- (p_4, p_2; p_3, p_1) \\
&= \frac{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1) + (u_4 v_3 - u_3 v_4)(u_2 v_1 - u_1 v_2)}{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)} \\
&= \frac{u_1 u_2 v_3 v_4 - v_1 u_2 v_3 u_4 - u_1 v_2 u_3 v_4 + v_1 v_2 u_3 u_4}{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)} \\
&\qquad + \frac{v_1 u_2 v_3 u_4 - v_1 u_2 u_3 v_4 - u_1 v_2 v_3 u_4 + u_1 v_2 u_3 v_4}{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)} \\
&= \frac{u_1 u_2 v_3 v_4 + v_1 v_2 u_3 u_4 - v_1 u_2 u_3 v_4 - u_1 v_2 v_3 u_4}{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)} \\
&= \frac{(u_1 v_3 - u_3 v_1)(u_2 v_4 - u_4 v_2)}{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)} \\
&= q.
\end{aligned}$$

Consequently

$$(p_4, p_2; p_3, p_1) = 1 - q.$$

It then follows that

$$(p_4, p_2; p_1, p_3) = \frac{1}{1 - q}.$$

Furthermore

$$(p_3, p_2; p_1, p_4) = 1 - (p_4, p_2; p_1, p_3) = 1 - \frac{1}{1 - q} = \frac{q}{q - 1},$$

and therefore

$$(p_3, p_2; p_4, p_1) = \frac{q - 1}{q}.$$

The remaining identities follow directly. ∎

Lemma 1.12

Let z_1, z_2 z_3 and z_4 be distinct complex numbers. Then

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}$$

Proof

This follows directly from the definition of cross ratios of quadruples of elements of the Riemann sphere on representing the complex number z_j as the fraction u_j/v_j with $u_j = z_j$ and $v_j = 1$ for $j = 1, 2, 3, 4$. ∎

Lemma 1.13

Let z_1, z_2 and, z_3 be distinct complex numbers. Then

$$(z_1, z_2; z_3, \infty) = \frac{z_1 - z_3}{z_2 - z_3}$$

Proof

Let $u_1 = z_1$, $u_2 = z_2$, $u_3 = z_3$, $u_4 = 1$, $v_1 = v_2 = v_3 = 1$ and $v_4 = 0$. Then $z_j = u_j/v_j$ for $j = 1, 2, 3$ and $\infty = u_4/v_4$. It follows from the definition of cross-ratios that

$$(z_1, z_2; z_3, \infty) = \frac{(u_1 v_3 - u_3 v_1)(u_2 v_4 - u_4 v_2)}{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)} = \frac{z_1 - z_3}{z_2 - z_3},$$

as required. ∎

Lemma 1.14

Let p_1, p_2, p_3, p_4 be a quadruple of points of the Riemann sphere satisfying the condition that no three of the points all coincide with one another. Then the following identities hold when two of the points coincide with one another:

$$(p_1, p_2; p_3, p_4) = \infty \text{ whenever } p_2 = p_3 \text{ or } p_1 = p_4;$$

$$(p_1, p_2; p_3, p_4) = 0 \text{ whenever } p_1 = p_3 \text{ or } p_2 = p_4;$$

$$(p_1, p_2; p_3, p_4) = 1 \text{ whenever } p_1 = p_2 \text{ or } p_3 = p_4.$$

Proof

Let complex numbers u_j and v_j be chosen for $j = 1, 2, 3, 4$ such that u_j and v_j are not both zero and $p_j = u_j/v_j$ for $j = 1, 2, 3, 4$. The definition of cross-ratios ensures that

$$(p_1, p_2; p_3, p_4) = \frac{(u_1 v_3 - u_3 v_1)(u_2 v_4 - u_4 v_2)}{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)}.$$

Now, for distinct integers j and k between 1 and 4, $p_j = p_k$ if and only if $u_j v_k = u_k v_j$. Also there exists a non-zero complex number w for which $u_2 = w u_1$ and $v_2 = w v_1$ if and only if $p_1 = p_2$, and there exists a non-zero complex number w for which $u_4 = w u_3$ and $v_4 = w v_3$ if and only if $p_3 = p_4$. The required identities therefore follow directly. $\blacksquare$

1.6. The Action of Möbius Transformations on the Riemann Sphere

Lemma 1.15

Let p_1, p_2 and p_3 be distinct elements of the Riemann sphere, and let $\mu^*_{p_1,p_2,p_3}\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the function mapping the Riemann sphere into itself defined such that

$$\mu^*_{p_1,p_2,p_3}(p) = (p_1, p_2; p_3, p)$$

for all elements p of the Riemann sphere. Then $\mu^*_{p_1,p_2,p_3}$ is Möbius transformation, and moreover $\mu^*_{p_1,p_2,p_3}(p_1) = \infty$, $\mu^*_{p_1,p_2,p_3}(p_2) = 0$ and $\mu^*_{p_1,p_2,p_3}(p_3) = 1$.

Proof

Let $p_j = u_j/v_j$ for $j = 1, 2, 3$, where, for each of these values of j, the elements u_j and v_j are complex numbers that are not both zero. It then follows from the definition of cross-ratio that

$$\mu^*_{p_1,p_2,p_3}\left(\frac{u}{v}\right) = \frac{(u_1v_3 - u_3v_1)(u_2v - uv_2)}{(u_2v_3 - u_3v_2)(u_1v - uv_1)}.$$

Consequently $\mu^*_{p_1,p_2,p_3}$ is the Möbius transformation corresponding to the coefficient matrix

$$\begin{pmatrix} -(u_1v_3 - u_3v_1)v_2 & (u_1v_3 - u_3v_1)u_2 \\ -(u_2v_3 - u_3v_2)v_1 & (u_2v_3 - u_3v_2)u_1 \end{pmatrix}.$$

It then follows from Lemma 1.14 that $\mu^*_{p_1,p_2,p_3}(p_1) = \infty$, $\mu^*_{p_1,p_2,p_3}(p_2) = 0$ and $\mu^*_{p_1,p_2,p_3}(p_3) = 1$. as required. ∎

Proposition 1.16

Let p_1, p_2, p_3 be distinct points of the Riemann sphere $\mathbb{P}^1$, and let q_1, q_2, q_3 also be distinct points of $\mathbb{P}^1$. Then there exists a unique Möbius transformation $\mu\colon \mathbb{P}^1 \to \mathbb{P}^1$ of the Riemann sphere with the property that $\mu(p_j) = q_j$ for $j = 1, 2, 3$.

Proof

Let $\mu^*_{p_1,p_2,p_3}\colon \mathbb{P}^1 \to \mathbb{P}^1$ and $\mu^*_{q_1,q_2,q_3}\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the Möbius transformations of the Riemann sphere defined so that

$$\mu^*_{p_1,p_2,p_3}(p) = (p_1, p_2; p_3, p) \quad \text{and} \quad \mu^*_{q_1,q_2,q_3}(p) = (q_1, q_2; q_3, p)$$

for all elements p of the Riemann sphere. Then

$$\begin{aligned}
\mu^*_{p_1,p_2,p_3}(p_1) = \mu^*_{q_1,q_2,q_3}(q_1) &= \infty, \\
\mu^*_{p_1,p_2,p_3}(p_2) = \mu^*_{q_1,q_2,q_3}(q_2) &= 0, \\
\mu^*_{p_1,p_2,p_3}(p_3) = \mu^*_{q_1,q_2,q_3}(q_3) &= 1.
\end{aligned}$$

It follows that $\mu(p_j) = q_j$ for $j = 1, 2, 3$, where $\mu\colon \mathbb{P}^1 \to \mathbb{P}^1$ is the Möbius transformation of the Riemann sphere defined such that

$$\mu(p) = \mu^{*-1}_{q_1,q_2,q_3}(\mu^*_{p_1,p_2,p_3}(p))$$

for all elements p of the Riemann sphere.

Now let $\hat{\mu}\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a Möbius transformation of the Riemann sphere with the property that $\hat{\mu}(p_j) = q_j$ for $j = 1, 2, 3$, and let

$$\lambda(p) = \mu^*_{q_1,q_2,q_3}(\hat{\mu}(\mu^{*-1}_{p_1,p_2,p_3}(p)))$$

for all elements p of the Riemann sphere. Then

$$\begin{aligned} \lambda(\infty) &= \mu^*_{q_1,q_2,q_3}(\hat{\mu}(\mu^{*-1}_{p_1,p_2,p_3}(\infty))) = \mu^*_{q_1,q_2,q_3}(\hat{\mu}(p_1)) \\ &= \mu^*_{q_1,q_2,q_3}(q_1) = \infty, \end{aligned}$$

and similarly $\lambda(0) = 0$ and $\lambda(1) = 1$.

Now λ is a Möbius transformation. It follows that there exist complex coefficients a, b, c and d, where $ad - bc \neq 0$, such that

$$\lambda\left(\frac{u}{v}\right) = \frac{au + bv}{cu + dv}$$

for all complex numbers u and v that are not both zero. Then the identity $\lambda(\infty) = \infty$ implies that $c = 0$, the identity $\lambda(0) = 0$ implies that $b = 0$, and consequently the identity $\lambda(1) = 1$ implies that $a = d$. Consequently $\lambda(p) = p$ for all elements p of the Riemann sphere.

It follows from this that

$$\hat{\mu}(p) = \mu^{*-1}_{q_1,q_2,q_3}(\mu^{*}_{p_1,p_2,p_3}(p)) = \mu(p),$$

for all elements p of the Riemann sphere. Thus the Möbius transformation μ is the unique Möbius transformation of the Riemann sphere that sends p_j to q_j for $j = 1, 2, 3$, as asserted. ∎

The following corollary follows immediately from Proposition 1.16).

Corollary 1.17

Two distinct Möbius transformations cannot coincide at three or more points of the Riemann sphere

Proposition 1.18

Let p_1, p_2, p_3, p_4 be distinct elements of the Riemann sphere $\mathbb{P}^1$, and let q_1, q_2, q_3, q_4 also be distinct elements of $\mathbb{P}^1$. Then a necessary and sufficient condition for the existence of a Möbius transformation $\mu\colon \mathbb{P}^1 \to \mathbb{P}^1$ of the Riemann sphere with the property that $\mu(p_j) = q_j$ for $j = 1, 2, 3, 4$ is that

$$(p_1, p_2; p_3, p_4) = (q_1, q_2; q_3, q_4).$$

Proof

Let $\mu^*_{p_1,p_2,p_3}\colon \mathbb{P}^1 \to \mathbb{P}^1$ and $\mu^*_{q_1,q_2,q_3}\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the Möbius transformations of the Riemann sphere defined so that

$$\mu^*_{p_1,p_2,p_3}(p) = (p_1, p_2; p_3, p) \quad \text{and} \quad \mu^*_{q_1,q_2,q_3}(p) = (q_1, q_2; q_3, p)$$

for all elements p of the Riemann sphere, and let $\mu\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the Möbius transformation of the Riemann sphere defined such that

$$\mu(p) = \mu^{*-1}_{q_1,q_2,q_3}(\mu^*_{p_1,p_2,p_3}(p))$$

for all elements p of the Riemann sphere. Then (as shown in the proof of Proposition 1.16) the Möbius transformation μ is the unique Möbius transformation that satisfies $\mu(p_j) = q_j$ for $j = 1, 2, 3$. Now $\mu(p_4) = \mu(q_4)$ if and only if $\mu^*_{p_1,p_2,p_3}(p_4) = \mu^*_{q_1,q_2,q_3}(q_4)$, and this is the case if and only if

$$(p_1, p_2; p_3, p_4) = (q_1, q_2; q_3, q_4).$$

The result follows. ∎

Proposition 1.19

Four distinct complex numbers z_1, z_2, z_3 and z_4 lie on a single line or circle in the complex plane if and only if their cross-ratio $(z_1, z_2; z_3, z_4)$ is a real number.

Proof

Let $\mu\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the Möbius transformation of the Riemann sphere defined such that $\mu(p) = (z_1, z_2; z_3, p)$ for all $p \in \mathbb{P}^1$. Then $\mu(z_1) = \infty$, $\mu(z_2) = 0$ and $\mu(z_3) = 1$. Möbius transformations map lines and circles to lines and circles (Propostion 1.10). It follows that a complex number z distinct from z_1, z_2 and z_3 lies on the circle in the complex plane passing through the points z_1, z_2 and z_3 if and only if $\mu(z)$ lies on the unique line in the complex plane that passes through 0 and 1, in which case $\mu(z)$ is a real number. The result follows. ∎

1.7. Cross-Ratios and Angles

We recall some basic properties of the algebra of complex numbers. Any complex number z can be written in the form

$$z = |z| \, (\cos\theta + \sqrt{-1}\sin\theta)$$

where $|z|$ is the modulus of z and θ is the angle in radians, measured anticlockwise, between the positive real axis and the line segment whose endpoints are represented by the complex numbers 0 and z. Moreover

$$\frac{1}{\cos\alpha + \sqrt{-1}\,\sin\alpha} = \cos\alpha - \sqrt{-1}\,\sin\alpha$$

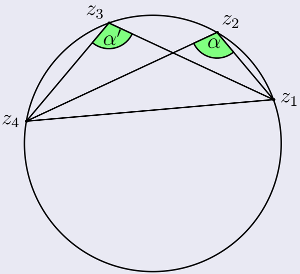
and

$$\begin{aligned}(\cos\alpha + \sqrt{-1}\,\sin\alpha)&(\cos\beta + \sqrt{-1}\,\sin\beta) \\ &= \cos(\alpha+\beta) + \sqrt{-1}\,\sin(\alpha+\beta)\end{aligned}$$

for all real numbers α and β.

Proposition 1.20

Let z_1, z_2, z_3 and z_4 be distinct complex numbers lying on a circle in the complex plane, listed in anticlockwise around the circle. Then the angle between the lines joining z_2 to z_4 and z_1 is equal to the angle between the lines joining z_3 to z_4 and z_1.



Proof

Let α denote the angle between the lines joining z_2 to z_4 and z_1, and let α' be the angle between the lines joining z_3 to z_4 and z_1. We must show that $\alpha = \alpha'$. Now it follows from the standard properties of complex numbers that

$$\begin{aligned}\frac{z_1 - z_2}{z_4 - z_2} &= \frac{|z_1 - z_2|}{|z_4 - z_2|}(\cos\alpha + \sqrt{-1}\sin\alpha),\\ \frac{z_1 - z_3}{z_4 - z_3} &= \frac{|z_1 - z_3|}{|z_4 - z_3|}(\cos\alpha' + \sqrt{-1}\sin\alpha').\end{aligned}$$

It now follows from the definition of cross-ratio that

$$\begin{aligned}(z_2, z_3; z_1, z_4) &= \frac{(z_1 - z_2)(z_4 - z_3)}{(z_1 - z_3)(z_4 - z_2)} = \frac{z_1 - z_2}{z_4 - z_2} \div \frac{z_1 - z_3}{z_4 - z_3}\\ &= \frac{|z_1 - z_2|\,|z_4 - z_3|}{|z_1 - z_3|\,|z_4 - z_2|} \times \frac{\cos\alpha + \sqrt{-1}\,\sin\alpha}{\cos\alpha' + \sqrt{-1}\,\sin\alpha'}.\end{aligned}$$

Now

$$\frac{1}{\cos\alpha' + \sqrt{-1}\sin\alpha'} = \cos\alpha' - \sqrt{-1}\sin\alpha',$$

and therefore

$$\begin{aligned} &\frac{\cos\alpha + \sqrt{-1}\ \sin\alpha}{\cos\alpha' + \sqrt{-1}\ \sin\alpha'} \\ &\qquad = (\cos\alpha + \sqrt{-1}\ \sin\alpha)(\cos\alpha' - \sqrt{-1}\ \sin\alpha') \\ &\qquad = \cos(\alpha - \alpha') + \sqrt{-1}\ \sin(\alpha - \alpha'). \end{aligned}$$

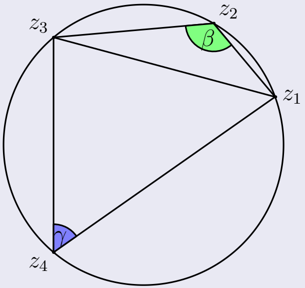
Consequently

$$\begin{aligned} &(z_2, z_3; z_1, z_4) \\ &\quad = |(z_2, z_3; z_1, z_4)|(\cos(\alpha - \alpha') + \sqrt{-1}\ \sin(\alpha - \alpha')). \end{aligned}$$

But the cross ratio $(z_2, z_3; z_1, z_4)$ is a real number, because the complex numbers z_1, z_2, z_3 and z_4 lie on a circle (see Proposition 1.19), and consequently $\alpha - \alpha'$ must be an integer multiple of π. Also $0 < \alpha < \pi$ and $0 < \alpha' < \pi$, and therefore $-\pi < \alpha - \alpha' < \pi$. It follows that $\alpha - \alpha' = 0$, and thus $\alpha = \alpha'$, as required. ∎

Proposition 1.21

Let z_1, z_2, z_3 and z_4 be distinct complex numbers lying on a circle in the complex plane, listed in anticlockwise around the circle, let β be the angle between the lines joining z_2 to z_3 and z_1, and let γ be the angle between the lines joining z_4 to z_1 and z_3. Then $\beta + \gamma = \pi$.



Proof

It follows from the standard properties of complex numbers that

$$\begin{aligned}\frac{z_1 - z_2}{z_3 - z_2} &= \frac{|z_1 - z_2|}{|z_3 - z_2|}(\cos\beta + \sqrt{-1}\sin\beta),\\ \frac{z_3 - z_4}{z_1 - z_4} &= \frac{|z_3 - z_4|}{|z_1 - z_4|}(\cos\gamma + \sqrt{-1}\sin\gamma).\end{aligned}$$

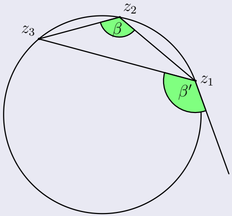
It now follows from the definition of cross-ratio that

$$\begin{aligned}&(z_2, z_4; z_1, z_3)\\ &= \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = \frac{z_1 - z_2}{z_3 - z_2} \times \frac{z_3 - z_4}{z_1 - z_4}\\ &= \frac{|z_1 - z_2|\,|z_3 - z_4|}{|z_1 - z_4|\,|z_3 - z_2|}(\cos\beta + \sqrt{-1}\,\sin\beta)(\cos\gamma + \sqrt{-1}\,\sin\gamma)\\ &= |(z_2, z_4; z_1, z_3)|\,(\cos(\beta+\gamma) + \sqrt{-1}\,\sin(\beta+\gamma)).\end{aligned}$$

But the cross ratio $(z_2, z_4; z_1, z_3)$ is a real number, because the complex numbers z_1, z_2, z_4 and z_3 lie on a circle (see Proposition 1.19), and consequently $\beta + \gamma$ must be an integer multiple of π. Also $0 < \beta < \pi$ and $0 < \gamma < \pi$, and therefore $0 < \beta + \gamma < 2\pi$. It follows that $\beta + \gamma = \pi$, as required. ∎

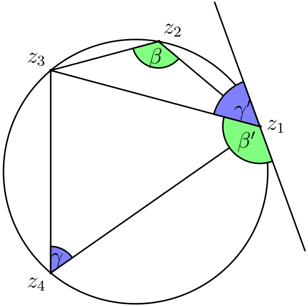
Proposition 1.22

Let z_1, z_2 and z_3 distinct complex numbers lying on a circle in the complex plane, listed in anticlockwise around the circle. Then the angle between the lines joining z_2 to z_3 and z_1 is equal to the angle between the line joining z_3 to z_1 and the ray tangent to the circle at z_1 that is directed so that the point z_2 and the tangent ray lie on opposite sides of the line that passes through the points z_1 and z_3.



Proof

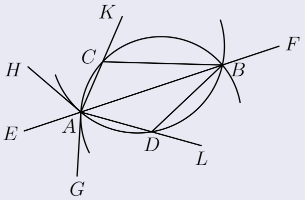
Let β denote the angle between the lines joining z_2 to z_3 and z_1. Also let a point z_4 be taken on the circle so that z_1, z_2, z_3 and z_4 are distinct and moreover the points z_2 and z_4 lie on opposite sides of the line that passes through z_1 and z_3, and let γ denote the angle between the lines joining z_4 to z_1 and z_3. It follows from Proposition 1.21 that $\beta + \gamma = \pi$.



Now suppose that the point z_4 moves along the circle towards the point z_1. As the point z_4 approaches z_1 the direction of the chord of the circle from z_4 to z_1 approaches the direction of the ray tangent to the circle at z_1 that points into the side of the line through z_1 and z_3 in which z_2 lies. But the angle between the rays joining z_4 to z_1 and z_3 remains constant as z_4 approaches z_1. Consequently the angle γ' between the tangent ray at z_1 pointing into the side of the chord joining z_1 to z_3 and that chord itself is equal to the angle γ. The angle β' between the chord joining z_1 and z_3 and the tangent ray pointing into the side of that chord opposite to z_2 is then the supplement of the angle γ', where $\gamma' = \gamma$, and therefore $\beta' + \gamma = \pi = \beta + \gamma$. Consequently $\beta' = \beta$. The result follows. ∎

Proposition 1.23

Let a geometrical configuration be as depicted in the accompanying figure. Thus let ACB and ADB be circular arcs that cut at the points A and B. Let the line joining points A and B be produced beyond A and B to E and F respectively. Let AG and AH be tangent to the circular arcs BCA and BDA respectively at A, where C and H lie on one side of AB and D and G lie on the other. Also let the lines AC and AD be produced to K and L respectively. Then the angle GAH is the sum of the angles KCB and LDB.

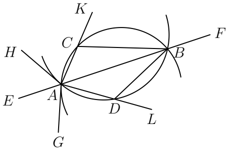


Proof

Applying results of previous propositions, together with standard geometrical results, we find that

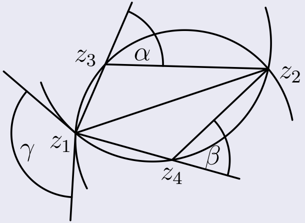
$$\begin{aligned}
\angle GAB &= \angle ACB && \text{(Proposition 1.22)} \\
\Rightarrow \angle EAG &= \angle KCB && \text{(supplementary angles)} \\
\angle HAB &= \angle ADB && \text{(Proposition 1.22)} \\
\Rightarrow \angle EAH &= \angle LDB && \text{(supplementary angles)} \\
\Rightarrow \angle GAH &= \angle EAG + \angle EAH && \\
&= \angle KCB + \angle LDB, &&
\end{aligned}$$

as required. ∎



Proposition 1.24

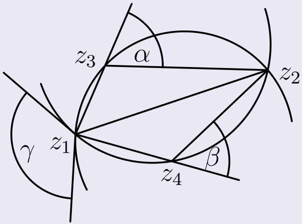
Let two circles in the complex plane intersect at points represented by complex numbers z_1 *and* z_2*, and let points represented by complex numbers* z_3 *and* z_4 *be taken on arcs of the respective circles joining* z_1 *and* z_2 *so that the point representing* z_3 *lies on the left hand side of the directed line from* z_1 *and* z_2 *and the point represented by the point* z_4 *lies on the right hand side of that line (as depicted in the accompanying figure).*



Then

$$(z_1, z_2; z_3, z_4) = \frac{|z_3 - z_1|\,|z_4 - z_2|}{|z_3 - z_2|\,|z_4 - z_1|}\,(\cos\gamma + \sqrt{-1}\,\sin\gamma),$$

where γ is the angle between the tangent lines to the two circles at the intersection point represented by the complex number z_1.



Proof

The configuration of the points z_1, z_2, z_3 and z_4 ensures that direction of the line from z_1 to z_3 is transformed into the direction of the line from z_3 to z_2 by rotation clockwise through an angle α less than two right angles. Similarly the direction of the line from z_1 to z_4 is transformed into the direction of the line from z_4 to z_2 by rotation anticlockwise through an angle β less than two right angles. Basic properties of complex numbers therefore ensure that

$$\begin{aligned}
\frac{z_2 - z_3}{z_3 - z_1} &= \frac{|z_2 - z_3|}{|z_3 - z_1|}(\cos\alpha - \sqrt{-1}\,\sin\alpha). \\
\frac{z_2 - z_4}{z_4 - z_1} &= \frac{|z_2 - z_4|}{|z_4 - z_1|}(\cos\beta + \sqrt{-1}\,\sin\beta).
\end{aligned}$$

Now

$$\begin{aligned}
&\frac{\cos\beta + \sqrt{-1}\,\sin\beta}{\cos\alpha - \sqrt{-1}\,\sin\alpha} \\
&\quad = \quad (\cos\alpha + \sqrt{-1}\,\sin\alpha)(\cos\beta + \sqrt{-1}\,\sin\beta) \\
&\quad = \quad \cos(\alpha+\beta) + \sqrt{-1}\,\sin(\alpha+\beta).
\end{aligned}$$

Moreover the geometry of the configuration ensures that $\alpha + \beta = \gamma$ (Proposition 1.23). Thus

$$\begin{aligned}
&\frac{z_2 - z_4}{z_4 - z_1} \times \frac{z_3 - z_1}{z_2 - z_3} \\
&\quad = \quad \frac{|z_2 - z_4|\,|z_3 - z_1|}{|z_4 - z_1||z_2 - z_3|}\,(\cos\gamma + \sqrt{-1}\sin\gamma).
\end{aligned}$$

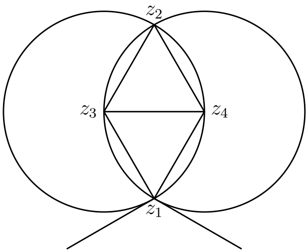
But

$$\frac{z_2 - z_4}{z_4 - z_1} \times \frac{z_3 - z_1}{z_2 - z_3} = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} = (z_1, z_2; z_3, z_4).$$

The result follows. ∎

Example

The circles in the complex plane of radius 2 centred on -1 and 1 intersect at the points $\pm\sqrt{3}\,i$, where $i = \sqrt{-1}$. In this situation, take $z_1 = -\sqrt{3}\,i$, $z_2 = \sqrt{3}\,i$, $z_3 = -1$ and $z_4 = 1$.



Then

$$\begin{aligned}(z_1, z_2; z_3, z_4) &= \frac{(-1+\sqrt{3}\,i)(1-\sqrt{3}\,i)}{(-1-\sqrt{3}\,i)(1+\sqrt{3}\,i)} = \frac{2+2\sqrt{3}\,i}{2-2\sqrt{3}i} \\ &= \frac{(2+2\sqrt{3}\,i)^2}{(2-2\sqrt{3}i)(2+2\sqrt{3}i)} \\ &= \frac{1}{2}(-1+\sqrt{3}\,i)\end{aligned}$$

It follows that $(z_1, z_2; z_3, z_4) = \cos\gamma + \sqrt{-1}\sin\gamma$, where $\gamma = \frac{2}{3}\pi$. Thus the angle between the tangent lines to the circles at the intersection point z_1 is thus $\frac{4}{3}$ of a right angle. This is what one would expect from the basic geometry of the configuration, given that the triangle with vertices z_1, z_3 and z_4 is equilateral and the tangent lines to the circles are perpendicular to the lines joining the point of intersection to the centres of those circles.

Proposition 1.25

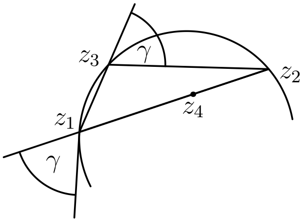
Let z_1 and z_2 be complex numbers representing the endpoints of a circular arc in the complex plane. Also, in the case where the circular arc lies on the left hand side of the directed line from z_1 to z_2, let points z_3 and z_4 be taken between z_1 and z_2 on the circular arc and the straight line segment respectively, and, in the case where the circular arc lies on the right hand side of the directed line from z_1 to z_2, let points z_3 and z_4 be taken between z_1 and z_2 on the straight line segment and the the circular arc respectively. Then

$$(z_1, z_2; z_3, z_4) = \frac{|z_3 - z_1|\,|z_4 - z_2|}{|z_3 - z_2|\,|z_4 - z_1|}\,(\cos\gamma + \sqrt{-1}\,\sin\gamma),$$

where γ is the angle between the tangent line to the circle at the intersection point represented by the complex number z_1 and the line obtained by producing the chord joining z_2 and z_1 beyond z_1.

Proof

We consider the configuration in which the circular arc lies on the left hand side of the directed line from z_1 to z_2. In that case the configuration is as depicted in the accompanying figure.



In this configuration the angle made at z_3 by the lines from z_1 and z_2 is equal to the angle between the chord from z_1 to z_2 and the depicted tangent line. The complements of those angles are then also equal to one another; these equal complements have been labelled γ in the figure.

Also the direction of the line from z_3 to z_2 is obtained from the direction of the line from z_1 to z_3 by rotation clockwise through an angle γ less than two right angles. It follows that

$$\frac{z_2 - z_3}{z_3 - z_1} = \frac{|z_2 - z_3|}{|z_3 - z_1|}(\cos\gamma - \sqrt{-1}\sin\gamma).$$

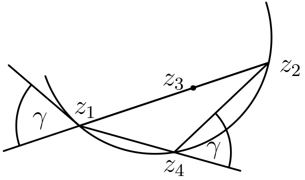
Also the direction of $z_2 - z_4$ is the same as that of $z_4 - z_1$, and therefore

$$\frac{z_2 - z_4}{z_4 - z_1} = \frac{|z_2 - z_4|}{|z_4 - z_1|}.$$

It follows that

$$\begin{aligned}
(z_1, z_2; z_3, z_4) &= \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} \\
&= \frac{z_2 - z_4}{z_4 - z_1} \times \frac{z_3 - z_1}{z_2 - z_3} \\
&= \frac{|z_3 - z_1|\,|z_4 - z_2|}{|z_3 - z_2|\,|z_4 - z_1|}(\cos\gamma + \sqrt{-1}\sin\gamma).
\end{aligned}$$

We consider now the case in which the circular arc from z_1 to z_2 lies on the right hand side of the directed line from z_1 to z_2. In this case the complex numbers z_3 and z_4 represent points between z_1 and z_2 on the line and the circular arc respectively, as depicted in the following figure.



In this configuration, the angle sought is the angle γ, which in this case is equal both to the angle between the depicted tangent line to the circle at z_1 and the line that produces the chord joining z_2 to z_1 beyond z_1.

Moreover, in this case

$$\frac{z_2 - z_4}{z_4 - z_1} = \frac{|z_2 - z_4|}{|z_4 - z_1|} (\cos\gamma + \sqrt{-1}\,\sin\gamma)$$

and

$$\frac{z_2 - z_3}{z_3 - z_1} = \frac{|z_2 - z_3|}{|z_3 - z_1|}.$$

It follows in this case also that

$$\begin{aligned}(z_1, z_2; z_3, z_4) &= \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} \\ &= \frac{z_2 - z_4}{z_4 - z_1} \times \frac{z_3 - z_1}{z_2 - z_3} \\ &= \frac{|z_3 - z_1|\,|z_4 - z_2|}{|z_3 - z_2|\,|z_4 - z_1|} (\cos\gamma + \sqrt{-1}\,\sin\gamma).\end{aligned}$$

This completes the proof. ∎

Proposition 1.26

Let two lines in the complex plane intersect at at point represented by the complex number z_1, and let points represented by z_3 and z_4 be taken distinct from z_1, one on each of the two lines, where these points are labelled so that the direction of $z_3 - z_1$ is obtained from the direction of $z_4 - z_1$ by rotation anticlockwise through an angle γ less than two right angles. Then

$$(z_1, \infty; z_3, z_4) = \frac{|z_3 - z_1|}{|z_4 - z_1|} (\cos\gamma + \sqrt{-1}\,\sin\gamma).$$

Proof

The cross-ratio in this situation is defined so that

$$(z_1, \infty; z_3, z_4) = \frac{z_3 - z_1}{z_4 - z_1}.$$

Furthermore

$$\frac{z_3 - z_1}{z_4 - z_1} = \frac{|z_3 - z_1|}{|z_4 - z_1|}\,(\cos\gamma + \sqrt{-1}\,\sin\gamma).$$

The result follows directly. ∎

Lines in the complex plane correspond to circles on the Riemann sphere that pass through the point at infinity. With that in mind, it can seen that Propositions 1.24, 1.25 and 1.26 conform to a common pattern, and show that, where two curves intersect at a point, each of those curves being either a circle or a straight line, the angle between the tangent lines to those curves at the point of intersection may be expressed in terms of the argument of an appropriate cross-ratio.

Indeed, to determine the angle the tangent lines to two circles on the Riemann sphere at a point p_1 where they intersect, one can determine the other point of intersection p_2, a point p_3 on one circular arc between p_1 to p_2, and a point p_4 on the other circular arc between p_1 and p_2. A positive real number R and a real number γ satisfying $-\pi < \gamma < \pi$ can then be determined so that

$$(p_1, p_2; p_3, p_4) = R(\cos\gamma + \sqrt{-1}\,\sin\gamma).$$

Then the angle between the tangent lines to those circles at the point p_1 of intersection, measured in radians, is then the absolute value $|\gamma|$ of γ.

Proposition 1.27

Möbius transformations of the Riemann sphere $\mathbb{P}^1$ *are angle-preserving. Thus if two circles on the Riemann sphere intersect at a point p of the Riemann sphere, and if a Möbius transformation* μ *maps p to a point q of the Riemann sphere, then the angle between the tangent lines to the original circles at the point p is equal to the angle between the tangent lines to the corresponding circles at the point q, the corresponding circles being the images of the original circles under the Möbius transformation.*

Proof

The angle between the tangent lines to the original circles at p is determined by the value of a cross ratio of the form $(p_1, p_2; p_3, p_4)$, where p_1 and p_2 are the points of intersection of the original circles, and p_3 and p_4 lie on the circular arcs joining p_1 to p_2, with p_4 on the right hand side as the circle through p_3 is traversed in the direction from p_1 through p_3 to p_2. The angle between the tangent lines to the corresponding circles at q is determined in the analogous fashion by the value of the cross ratio $(q_1, q_2; q_3, q_4)$, where q_j is the image of p_j under the Möbius transformation sending the original circles to the corresponding circles. Proposition 1.18 ensures that $(p_1, p_2; p_3, p_4) = (q_1, q_2; q_3, q_4)$. The result follows. ∎

1.8. The Orientation-Preserving Property of Möbius Transformations

A subset X of the complex plane $\mathbb{C}$ is said to be *open* if, given any complex number w belonging to X, there exists an open disk in the complex plane of sufficiently small radius centred on w that is wholly contained within the set X.

Definition

An invertible function $\varphi\colon X \to Y$ between open subsets X and Y of the complex plane is said to be *orientation-preserving* if, given any point w of X, paths that traverse circles of sufficiently small radius centred on w once in the anticlockwise direction are mapped by φ to paths that wind around $\varphi(w)$ once in the anticlockwise direction.

Definition

An invertible function $\varphi\colon X \to Y$ between open subsets X and Y of the complex plane is said to be *orientation-reversing* if, given any point w of X, paths that traverse circles of sufficiently small radius centred on w once in the anticlockwise direction are mapped by φ to paths that wind around $\varphi(w)$ once in the clockwise direction.

The transformation of the complex plane that maps each complex number to its complex conjugate is an example of an orientation-reversing transformation of the complex plane.

The composition of two orientation-preserving transformations between open subsets of the complex plane is orientation-preserving, as is the composition of two orientation-reversing transformations between such subsets. A transformation obtained on composing an orientation-preserving transformation with an orientation-reversing transformation is orientation-reversing, as is a transformation obtained on composing an orientation-reversing transformation with an orientation-preserving transformation.

Proposition 1.28

A Möbius transformation of the Riemann sphere is orientation-preserving over the open subset of the complex plane consisting of those complex numbers that are not mapped to the element ∞ of the Riemann sphere.

Proof

Given complex numbers a and b, where $a \neq 0$, let $\tau_{a,b}$ denote the Möbius transformation of the Riemann sphere that maps ∞ to ∞ and maps each complex number z to $az + b$. Also let κ denote the Möbius transformation of the Riemann sphere that interchanges 0 and ∞ and maps z to $1/z$ for all non-zero complex numbers z. Then any Möbius transformation of the Riemann sphere can be expressed as a composition of Möbius transformations that are either of the form $\tau_{a,b}$ for appropriate coefficients a and b or else coincide with the Möbius transformation κ. (See the proof of Proposition 1.10.)

It is not difficult to see that the transformations $\tau_{a,b}$ restrict to orientation-preserving transformations of the complex plane. The required result therefore follows from the observation that compositions of orientation-preserving transformations as orientation-preserving, once we establish that the Möbius transformation κ, when restricted to the non-zero complex numbers, is also an orientation-preserving transformation.

Consider a circle of radius s in the complex plane centred on 1, where $s < 1$. If that circle is traversed in the anticlockwise direction, starting at $1 + s$ and passing successively through $1 + s\sqrt{-1}$, $1 - s$ and $1 - s\sqrt{-1}$ before returning to $1 + s$, then then that path is mapped by the Möbius transformation κ to a path traversing a circle surrounding 1 and passing successively through the points

$$\frac{1}{1+s}, \quad \frac{1 - s\sqrt{-1}}{1+s^2}, \quad \frac{1}{1-s}, \quad \frac{1 + s\sqrt{-1}}{1+s^2}, \quad \frac{1}{1+s}.$$

This latter path is traversed in an anticlockwise direction. Thus if a circle centred on 1 of sufficiently small radius is traversed in an anticlockwise direction, then its image under the Möbius transformation κ will also be traversed in an anticlockwise direction.

A path traversing a sufficiently small circles centred on any non-zero complex number w in the anticlockwise direction will then be mapped to a path traversing a circle centred on w^{-1} in an anticlockwise direction, because κ is equal to the composition of the successive orientation-preserving transformations $z \mapsto w^{-1}z$, $z \mapsto z^{-1}$ and $z \mapsto w^{-1}z$. Consequently κ restricts to an orientation-preserving transformation defined over the set of non-zero complex numbers. We can therefore conclude that any Möbius transformation of the Riemann sphere is indeed orientation-preserving when restricted to the open subset of the complex plane consisting of those complex numbers that are not mapped to the element ∞ of the Riemann sphere, as required. $\blacksquare$