MAU23302—Euclidean and Non-Euclidean Geometry
School of Mathematics, Trinity College
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Lecture Slides:
Definitions, Postulates and Common Notions

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1. Definitions, Postulates and Common Notions

Euclid's Definitions: Differences from Contemporary Definitions.

The term *line* (in Greek, *grammé*) encompasses straight lines, curved lines, and lines of which some portions may be straight and others curved. Accordingly *straight lines* are considered to be a particular type of line. (A straight line is referred to, in Greek, by the words *eutheia grammé*, or simply by the adjective *eutheia* treated as a substantive).

In Euclid's *Elements of Geometry*, *angles* are always taken to be positive and strictly less than two right angles. Thus (in contrast to the geometry course studied for the Irish Junior Certificate and Leaving Certificate), there are, in Euclid's *Elements of Goemetry*, no *null angles*, *straight angles*, *reflex angles* or *full angles*. The ancient Greeks accordingly classified *rectilineal angles* as being *acute* (when less than a right angle), *right*, and *obtuse* (when greater than a right angle but less than two right angles).

The ancient Greeks also had a more general concept of angle, applicable where curved lines meet straight lines or other curved lines. Such angles include the *angle of a semicircle*, which occurs where the semicircular arc forming part of the boundary a semicircle meets the diameter that constitutes the remainder of the boundary. The ancient Greeks and later mathematicians also discussed *horn angles*, that occur where the boundary of a circle meets a tangent line to the circle. Such non-rectilineal angles are referenced in Propositions 16 and 31 of Book III of Euclid's *Elements of Geometry*.

The Postulates

The *postulates* set out basic geometric assumptions that need to be accepted for the development of the geometrical theory in Euclid's *Elements of Geometry*. However Euclid's wording of the first three postulates is somewhat terse. Accordingly more lengthy statements of the basic assumptions corresponding to the first three postulates follow that attempt to set out in more detail what needs to be assumed in relation to these postulates to enable the development of the geometrical theory to proceed.

Postulate 1

Given two distinct points in a given plane, there exists a unique straight line segment in that plane which has as its endpoints those two given points.

Postulate 2

Any given straight line segment may be produced in a straight line beyond one of its endpoints far enough to ensure that the resulting straight line segment exceeds in length any other given straight line segment; and moreover, if two distinct line segments are each obtained by producing a given straight line segment in a straight line beyond the same endpoint in this fashion, then one of those straight line segments must be a part of the other.

Postulate 3

Given any two distinct points in a given plane, there exists a circle in that plane (with the properties specified in Euclid's *Definitions* included in Book I of the *Elements of Geometry*), where the centre of the circle is located at the first of the given points, and where the circle itself passes through the second of these points.

Postulate 4

All right angles are equal to one another.

(*This statement of the postulate quotes directly Euclid, as translated by Heath.*)

Postulate 5

If a straight line falling on two straight lines make the interior angles on the same side less than two right anngles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.

(*This statement of the postulate quotes directly Euclid, as translated by Heath.*)

The Common Notions

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

Further Unstated Assumptions

The points lying on an infinite straight line can be ordered in the usual fashion. There are two natural orderings of the points on any infinite line. They are total orderings, and each is the reverse of the other. When three points on the line are in sequence with respect to either ordering, the second lies between the first and the third.

An infinite straight line contained within a plane divides that plane into exactly two parts. These parts are each bounded by the straight line, and are referred to as the *sides* of the line. A straight line segment joining two points that both lie on the same side of the infinite straight line does not intersect that straight line at any point. A straight line segment or circular arc joining a point on one side of the infinite line to a point on the other side always intersects the infinite line at exactly one point.

(The Greek word for part, namely *meros*, is also used to refer to the sides of an infinite straight line.)

Given a circle in a given plane, a straight line segment or circular arc with one endpoint inside the circle and the other endpoint outside the circle always interects the circle at some point lying on the circle.

Suggested Encompassing Principles

Purely topological observations, where apparent, are valid.

Standard straightedge-and-compass constructions, when correctly designed, are effective in demonstrating the existence of the geometrical entities that they are designed to construct.