MAU23302—Euclidean and Non-Euclidean Geometry
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Part II, Section 3:
The Disk Model of the Hyperbolic Plane

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2.1. Inversion of the Riemann Sphere in the Unit Circle

Let D denote the open unit disk in the complex plane $\mathbb{C}$, and in the Riemann sphere, defined so that

$$D = \{z \in \mathbb{C} : |z| < 1\}$$

and let S denote the unit circle in the complex plane $\mathbb{C}$, and in the Riemann sphere, defined so that

$$S = \{z \in \mathbb{C} : |z| = 1\}$$

We define the *inversion* Ω of the Riemann sphere in the circle S bounding the open unit disk D to be the transformation of the Riemann sphere defined so that $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex numbers z.

Then $\Omega(z) = z$ for all $z \in S$, and the composition $\Omega \circ \Omega$ of the inversion Ω with itself is the identity transformation of the Riemann sphere. Moreover Ω maps the open unit disk D into the region of the Riemann sphere that lies outside the unit circle S.

Lemma 2.1

Let μ be a Möbius transformation of the Riemann sphere, and let Ω be the inversion of the Riemann sphere in the unit circle, defined so that $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex numbers z. Also let a, b, c and d be complex coefficients determined so that

$$\mu(z) = \frac{az + b}{cz + d}$$

for all complex numbers z for which $cz + d \neq 0$. Then $\Omega \circ \mu \circ \Omega$ is also a Möbius transformation, and moreover

$$\Omega(\mu(\Omega(z))) = \frac{\overline{c} + \overline{d}z}{\overline{a} + \overline{b}z}$$

for all complex numbers $z \in \mathbb{C}$ for which $\overline{a} + \overline{b}z \neq 0$ and $\overline{c} + \overline{d}z \neq 0$.

Proof

It follows from the definition of Möbius transformations that there exist complex numbers a, b, c and d such that

$$\mu(z) = \frac{az + b}{cz + d}$$

for all complex numbers z for which $cz + d \neq 0$. Then

$$\Omega(\mu(\Omega(z))) = \Omega\left(\frac{a\dfrac{1}{\overline{z}} + b}{c\dfrac{1}{\overline{z}} + d}\right) = \Omega\left(\frac{a + b\overline{z}}{c + d\overline{z}}\right) = \frac{\overline{c} + \overline{d}z}{\overline{a} + \overline{b}z}$$

for all $z \in \mathbb{C}$ for which $z \neq 0$, $\overline{c} + \overline{d}z \neq 0$ and $\overline{a} + \overline{b}z \neq 0$.

Also

$$\Omega(\mu(\Omega(0))) = \Omega(\mu(\infty)) = \Omega\left(\frac{a}{c}\right) = \frac{\overline{c}}{\overline{a}}$$

provided that $a \neq 0$ and $c \neq 0$,

$$\Omega(\mu(\Omega(0))) = \Omega(\mu(\infty)) = \Omega(\infty) = 0 \quad \text{when} \quad c = 0,$$

$$\Omega(\mu(\Omega(0))) = \Omega(\mu(\infty)) = \Omega(0) = \infty \quad \text{when} \quad a = 0,$$

$$\Omega\left(\mu\left(\Omega\left(-\frac{\overline{c}}{\overline{d}}\right)\right)\right) = \Omega\left(\mu\left(-\frac{d}{c}\right)\right) = \Omega(\infty) = 0,$$

provided that $c \neq 0$ and $d \neq 0$, and

$$\Omega\left(\mu\left(\Omega\left(-\frac{\overline{a}}{\overline{b}}\right)\right)\right) = \Omega\left(\mu\left(-\frac{b}{a}\right)\right) = \Omega(0) = \infty$$

provided that $a \neq 0$ and $b \neq 0$.

Now the definition of Möbius transformations requires that $ad - bc \neq 0$. Consequently $c \neq 0$ when $d = 0$, and $a \neq 0$ when $b = 0$. We have therefore determined the image of each element of the Riemann sphere under the composition map $\Omega \circ \mu \circ \Omega$, and can now conclude that this composition map $\Omega \circ \mu \circ \Omega$ is indeed a Möbius transformation, and that it is characterized by the property that

$$\Omega(\mu(\Omega(z))) = \frac{\overline{c} + \overline{d}z}{\overline{a} + \overline{b}z}$$

for all complex numbers $z \in \mathbb{C}$ for which $\overline{a} + \overline{b}z \neq 0$ and $\overline{c} + \overline{d}z \neq 0$, as required. $\blacksquare$

Proposition 2.2

Let μ *be a Möbius transformation of the Riemann sphere, let* D *be the open unit disk in the complex plane, where*

$$D = \{z \in \mathbb{C} : |z| < 1\}$$

and let Ω *be the inversion of the Riemann sphere in the unit circle that is defined so that*

$$\Omega(0) = \infty, \quad \Omega(\infty) = 0 \quad \textit{and} \quad \Omega(z) = \frac{1}{\overline{z}} \textit{ for all } z \in \mathbb{C} \setminus \{0\}.$$

Then the Möbius transformation μ *maps the unit disk* D *onto itself if and only if both of the following two conditions are satisfied:*

(i) $\Omega \circ \mu = \mu \circ \Omega$;

(ii) *there exists at least one* $z \in D$ *for which* $\mu(z) \in D$.

Proof

First suppose that the Möbius transformation μ maps the unit disk D onto itself. Let z be a complex number satisfying $|z| = 1$. Then $tz \in D$ for all real numbers t satisfying $0 \leq t < 1$, and consequently $|\mu(tz)| < 1$ for all real numbers t satisfying $0 \leq t < 1$. The continuity of Möbius transformations then ensures that $|\mu(z)| \leq 1$. Now if it were the case that $|\mu(z)| < 1$ then there would exist $w \in D$ for which $\mu(w) = \mu(z)$, because the Möbius transformation μ maps the unit disk D onto itself. But this is not possible, because if it were, then two distinct z and w complex numbers would be mapped by μ to the same complex number, contradicting the fact that Möbius transformations are invertible transformations of the Riemann sphere. Thus the Möbius transformation μ maps the unit circle into itself.

Now let $\hat{\mu} = \Omega \circ \mu \circ \Omega$. Then $\hat{\mu}$ is a Möbius transformation of the Riemann sphere (Lemma 2.1). Moreover $\Omega(z) = z$ and $|\mu(z)| = 1$ for all complex numbers z satisfying $|z| = 1$, and therefore $\hat{\mu}(z) = \mu(z)$ for all complex numbers z satisfying $|z| = 1$. Now two distinct Möbius transformations cannot coincide at three or more points of the Riemann sphere. (see Proposition 1.9). It follows therefore that $\hat{\mu} = \mu$. Consequently $\Omega \circ \mu = \mu \circ \Omega$. It now follows directly that any Möbius transformation that maps the unit disk D onto itself must satisfy conditions (i) and (ii) in the statement of the proposition.

Conversely, suppose that Möbius transformation μ of the Riemann sphere satisfies conditions (i) and (ii) in the statement of the proposition. Then $\Omega \circ \mu = \mu \circ \Omega$. Let z be a complex number satisfying $|z| \neq 1$. Then $\Omega(z) \neq z$. It follows that $\mu(\Omega(z)) \neq \mu(z)$, because Möbius transformations are invertible transformations of the Riemann sphere, and therefore $\Omega(\mu(z)) \neq \mu(z)$, from which it follows that $|\mu(z)| \neq 1$. Consequently no complex number belonging to the open unit disk D is mapped by the Möbius transformation D to a point that lies on the unit circle. It follows that if one endpoint of a straight line segment or circular arc contained in the open disk D is mapped by μ into D, then the same must be true of the other endpoint of that straight line segment or circular arc.

Now the complex numbers belonging to the unit disk D can be joined to one another by straight line segments. Moreover condition (ii) in the statement of the proposition ensures that at least one complex number belonging to the unit disk D is mapped by the Möbius transformation μ into the unit disk D. Consequently the unit disk is mapped into itself by the Möbius transformation μ. Moreover if the Möbius transformation μ has the property that $\Omega \circ \mu = \mu \circ \Omega$ then

$$\Omega \circ \mu^{-1} = \mu^{-1} \circ \mu \circ \Omega \circ \mu^{-1} = \mu^{-1} \circ \Omega \circ \mu \circ \mu^{-1} = \mu^{-1} \circ \Omega,$$

and consequently the inverse μ^{-1} of the Möbius transformation μ also satisfies (i) and (ii) in the statement of the proposition, and therefore maps the open unit disk D into itself. It follows that if the Möbius transformation μ satisfies conditions (i) and (ii) then it must map the open unit disk D onto itself, as required. ∎

Corollary 2.3

Let μ be a Möbius transformation of the Riemann sphere, and let S be the unit circle consisting of all complex numbers z for which $|z| = 1$. Suppose that $\mu(S) \subset S$ and that $|\mu(0)| < 1$. Then the Möbius transformation μ maps the open unit disk onto itself. Moreover $\Omega \circ \mu = \mu \circ \Omega$, where Ω is the inversion of the Riemann sphere in the unit circle S, defined so that $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex numbers z.

Proof

Let $\hat{\mu} = \Omega \circ \mu \circ \Omega$. Then $\hat{\mu}$ is a Möbius transformation of the Riemann sphere (Lemma 2.1), and moreover $\hat{\mu}(z) = \mu(z)$ for all $z \in S$, because $\mu(S) \subset S$ and $\Omega(z) = z$ for all $z \in S$. Now two distinct Möbius transformations cannot coincide at three or more points of the Riemann sphere. (see Proposition 1.9). It follows that $\hat{\mu} = \mu$, and therefore $\Omega \circ \mu = \mu \circ \Omega$. The required result now follows on applying Proposition 2.2. ∎

Lemma 2.4

Given distinct complex numbers z_1 and z_2, where $|z_1| = |z_2| = 1$, there exists a Möbius transformation μ of the Riemann sphere mapping the unit disk D onto itself for which $\mu(z_1) = -1$ and $\mu(z_2) = 1$.

Proof

Choose a complex number z_3 distinct from z_1 and z_2 for which $|z_3| = 1$. Then there exists a unique Möbius transformation μ_1 with the properties that $\mu_1(z_1) = -1$, $\mu_1(z_2) = 1$ and $\mu_1(z_3) = i$. Möbius transformations map circles to circles, and, given any three distinct complex numbers that are not collinear, there exists exactly one circle in the complex plane passing through all three of these complex numbers. Consequently the Möbius transformation μ_1 must map the unit circle onto itself.

If $|\mu_1(0)| < 0$ let the Möbius transformation μ be identical to μ_1; if $|\mu_1(0)| > 1$ or $\mu_1(0) = \infty$ let the Möbius transformation μ be defined so that $\mu(z) = 1/\mu_1(z)$ for all complex numbers z for which $\mu_1(z) \neq 0$. Then μ maps the unit circle onto itself, $\mu(z_1) = -1$, $\mu(z_2) = 1$ and $|\mu(0)| < 1$. Then $\mu(D)$ must map the open unit disk onto itself (see Corollary 2.3). The Möbius transformation μ then has the required properties. ∎

Proposition 2.5

Let a and b be complex numbers satisfying $|b| < |a|$, and let μ be the Möbius transformation of the Riemann sphere defined so that

$$\mu(z) = \frac{az + b}{\overline{b}\,z + \overline{a}} \quad \text{whenever } \overline{b}\,z + \overline{a} \neq 0,$$

$\mu(-\overline{a}/\overline{b}) = \infty$ and $\mu(\infty) = a/\overline{b}$ in cases where $b \neq 0$ and $\mu(\infty) = \infty$ in cases where $b = 0$. Then $|\mu(z)| < 1$ whenever $|z| < 1$, $|\mu(z)| = 1$ whenever $|z| = 1$, and $|\mu(z)| > 1$ whenever $|z| > 1$ and $\overline{b}z + \overline{a} \neq 0$. Moreover the Möbius transformation μ maps the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ onto itself.

Proof

Calculating, we find that

$$\begin{aligned} |\overline{b}z+\overline{a}|^2-|az+b|^2 &= (\overline{b}z+\overline{a})(b\overline{z}+a)-(az+b)(\overline{a}\,\overline{z}+\overline{b}) \\ &= |b|^2|z|^2+|a|^2+a\,\overline{b}z+\overline{a}b\,\overline{z} \\ &\quad -|a|^2|z|^2-|b|^2-a\,\overline{b}z-\overline{a}b\,\overline{z} \\ &= (|a|^2-|b|^2)(1-|z|^2)>0. \end{aligned}$$

Consequently $|\mu(z)|<1$ whenever $|z|<1$, $|\mu(z)|=1$ whenever $|z|=1$ and $|\mu(z)>1$ whenever $|z|>1$ and $\overline{b}z+\overline{a}\neq 0$.

Now the inverse μ^{-1} of the Möbius transformation μ is characterized by the property that

$$\mu^{-1}(z) = \frac{\overline{a}z - b}{-\overline{b}z + a}$$

for all complex numbers z for which $-\overline{b}z + a \neq 0$ (see Corollary 1.6). Because the coefficients of this Möbius transformation μ^{-1} have properties analogous to those of the Möbius transformation μ, we can conclude that μ^{-1} maps the open unit disk into itself, and therefore μ maps the open unit disk onto itself, as required. ∎

Corollary 2.6

Let w be a complex number satisfying $|w| < 1$, and let μ_w be the Möbius transformation of the Riemann sphere defined so that $\mu_w(1/\overline{w}) = \infty$, $\mu(\infty) = -1/\overline{w}$ and

$$\mu_w(z) = \frac{z - w}{1 - \overline{w}\, z}$$

for all complex numbers z distinct from $1/\overline{w}$. Then the Möbius transformation μ_w maps the open unit disk onto itself. Moreover

$$\mu_w(tw) = \frac{t - 1}{1 - |w|^2 t}\, w$$

for all real numbers t distinct from $1/|w|^2$, and consequently the diameter of the unit circle passing through 0 and w is mapped onto itself by the Möbius transformation μ_w. In particular $\mu_w(w) = 0$ and $\mu_w(0) = -w$.

Proposition 2.7

Let μ be a Möbius transformation of the Riemann sphere that maps the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ into itself, whilst mapping the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ into itself. Then there exist complex numbers a and b, where $|b| < |a|$, such that

$$\mu(z) = \frac{az + b}{\overline{b}z + \overline{a}} \quad \text{for all } z \in \mathbb{C} \text{ for which } \overline{a}z + \overline{b} \neq 0.$$

Proof

The Möbius transformation μ maps the unit circle into itself. It follows from Proposition 2.2 that $\Omega \circ \mu = \mu \circ \Omega$, where $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex numbers z. Consequently $\mu = \Omega \circ \Omega \circ \mu = \Omega \circ \mu \circ \Omega$ because the composition of the inversion Ω with itself is the identity transformation of the Riemann sphere. Let a_1, b_1, c_1 and d_1 be complex coefficients determined so that

$$\mu(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \quad \text{whenever } c_1 z + d_1 \neq 0.$$

Then the identity $\mu = \Omega \circ \mu \circ \Omega$ ensures that

$$\frac{a_1 z + b_1}{c_1 z + d_1} = \frac{\overline{d}_1 z + \overline{c}_1}{\overline{b}_1 z + \overline{a}_1}$$

for all complex numbers z for which $a_1 z + b_1 \neq 0$, $\overline{a}_1 + \overline{b}_1 z \neq 0$, $c_1 z + d_1 \neq 0$, and $\overline{c}_1 + \overline{d}_1 z \neq 0$ (see Lemma 2.1).

Consequently there exists some non-zero complex number ω with the property that $\overline{a}_1 = \omega d_1$, $\overline{b}_1 = \omega c_1$, $\overline{c}_1 = \omega b_1$ and $\overline{d}_1 = \omega a_1$ (see Proposition 1.10). It then follows that

$$\overline{a}_1 \, \overline{d}_1 = \omega^2 a_1 d_1.$$

But

$$|\overline{a}_1 \, \overline{d}_1| = |a_1 d_1|.$$

It follows that $|\omega^2| = 1$, and therefore $|\omega| = 1$. Accordingly a real number θ can be found so that

$$\omega = \cos 2\theta + \sqrt{-1} \sin 2\theta.$$

Let

$$\eta = \cos \theta + \sqrt{-1} \sin \theta.$$

It then follows from De Moivre's Theorem that $\eta^2 = \omega$. Now $\overline{\eta}^2 \, \eta^2 = |\eta|^4 = 1$. It follows that $\overline{\eta}^2 \omega = 1$.

Let $a = \eta a_1$ and $b = \eta b_1$, $c = \eta c_1$ and $d = \eta d_1$. Then

$$\mu(z) = \frac{az+b}{cz+d} \quad \text{whenever } cz + d \neq 0.$$

Also $a_1 = \overline{\eta} a$, $b_1 = \overline{\eta} b$, $c_1 = \overline{\eta} c$ and $d_1 = \overline{\eta} d$. Consequently

$$\overline{d} = \overline{\eta}\,\overline{d}_1 = \overline{\eta}\omega a_1 = \overline{\eta}^2 \omega a = a$$

and

$$\overline{c} = \overline{\eta}\,\overline{c}_1 = \overline{\eta}\omega b_1 = \overline{\eta}^2 \omega b = b.$$

Accordingly

$$\mu(z) = \frac{az+b}{\overline{b}\,z+\overline{a}} \quad \text{whenever } \overline{b}\,z + \overline{a} \neq 0.$$

Moreover $|\mu(0)| < 1$, because μ maps the unit disk into itself. consequently $|b| < |a|$, as required. ∎

2.2. The Poincaré Distance Function on the Unit Disk

Definition

Let D be the open unit disk in the complex plane $\mathbb{C}$, defined so that

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

The *Poincaré distance function* ρ on D is defined so that

$$\rho(z, w) = \log\left(\frac{|1 - \overline{w}\,z| + |z - w|}{|1 - \overline{w}\,z| - |z - w|}\right)$$

for all complex numbers z and w satisfying $|z| < 1$ and $|w| < 1$.

Note that

$$\frac{|z - w|}{|1 - \overline{w}\, z|} < 1$$

for all complex numbers z and w satisfying $|z| < 1$ and $|w| < 1$. (This follows directly from Corollary 2.6). Consequently the Poincaré distance $\rho(z, w)$ between any two points z and w of the unit disk is a well-defined positive real-number.

Lemma 2.8

Let s and t be real numbers satisfying $-1 < s < t < 1$. Then the Poincaré distance, in the unit disk, between s and t is given by the formula

$$\rho(s,t) = \log\left(\frac{1+t}{1-t}\right) - \log\left(\frac{1+s}{1-s}\right).$$

Proof

Evaluating, and noting that $1 - st > 0$ (because $|s| < 1$ and $|t| < 1$) and $|t - s| = t - s$ (since $s < t$ by assumption), we find that

$$\begin{aligned}
\rho(s,t) &= \log\left(\frac{|1-st| + |t-s|}{|1-st| - |t-s|}\right) \\
&= \log\left(\frac{1-st+t-s}{1-st+s-t}\right) \\
&= \log\left(\frac{(1-s)(1+t)}{(1+s)(1-t)}\right) \\
&= \log\left(\frac{1+t}{1-t}\right) - \log\left(\frac{1+s}{1-s}\right),
\end{aligned}$$

as required. ∎

Proposition 2.9

Let ρ be the Poincaré distance function on the open unit disk D, and let δ be a positive real number. Then

$$\{z \in D : \rho(z, 0) = \delta\} = \{z \in D : |z| = R\},$$

where

$$R = \frac{e^\delta - 1}{e^\delta + 1}.$$

Proof

It follows from the definition of Poincaré distance function that all complex numbers z satisfying $\rho(z, 0) = \delta$ are equidistant from zero. They therefore constitute a circle centred on zero. It remains to determine the radius of that circle. Now it follows, on applying Lemma 2.8, that

$$\delta = \log\left(\frac{1+R}{1-R}\right).$$

Consequently

$$e^{\delta} - 1 = \frac{2R}{1-R}, \quad e^{\delta} + 1 = \frac{2}{1-R},$$

and therefore

$$R = \frac{e^{\delta} - 1}{e^{\delta} + 1},$$

as required. ∎

The Poincaré distance function ρ on the unit disk D has the property that $\rho(z, w) = \rho(w, z)$ for all $z, w \in D$. It therefore follows immediately from Lemma 2.8 that

$$\rho(s, t) = \left| \log \left(\frac{1 + t}{1 - t} \right) - \log \left(\frac{1 + s}{1 - s} \right) \right|$$

for all real numbers s and t satisfying $-1 < s < 1$ and $-1 < t < 1$.

Lemma 2.10

Let z and w be complex numbers, and let Ω be the inversion of the Riemann sphere in the unit circle, defined so that $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex numbers z. Then

$$(z, \Omega(z); w, \Omega(w)) = \left| \frac{z - w}{1 - \overline{w}z} \right|^2$$

for all complex numbers z and w with the exception of those pairs z, w for which $|z| = 1$ and $z = w$.

Proof

Let z and w be complex numbers. Suppose that it is not the case that $|z| = 1$ and $z = w$. Examination of possible cases shows that it is not then possible for three of the complex numbers z, $\Omega(z)$, w and $\Omega(w)$ to coincide with one another. Indeed if $|z| \neq 1$ and $|w| \neq 1$ then exactly two of the points $z, \Omega(z), w, \Omega(w)$ will lie in the unit disk consisting of those complex numbers whose modulus is less than one, and therefore it is not possible for any three of the four points to coincide with one another. If $|z| = 1$, it would only be possible for three of the points $z, \Omega(z), w, \Omega(w)$ to coincide with one another if it were also the case that $w = z$. Consequently the cross-ratio $(z, \Omega(z), w, \Omega(w))$ is defined in all cases with the exception of those where $|z| = 1$ and $w = z$.

Now let $u_1 = z$, $v_1 = 1$, $u_2 = 1$, $v_2 = \overline{z}$, $u_3 = w$, $v_3 = 1$, $u_4 = 1$, $v_4 = \overline{w}$. Then $u_1/v_1 = z$, $u_2/v_2 = \Omega(z)$, $u_3/v_3 = w$ and $u_4/v_4 = \Omega(w)$. The definition of cross-ratio then ensures that

$$\begin{aligned}
(z, \Omega(z); w, \Omega(w)) &= \frac{(u_1 v_3 - u_3 v_1)(u_2 v_4 - u_4 v_2)}{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)} \\
&= \frac{(z - w)(\overline{w} - \overline{z})}{(1 - w\overline{z})(z\overline{w} - 1)} \\
&= \left| \frac{z - w}{1 - \overline{w} z} \right|^2,
\end{aligned}$$

as required. ∎

Proposition 2.11

Let z and w be complex numbers satisfying $|z| < 1$ and $|w| < 1$, and let $\rho(z, w)$ denote the Poincaré distance between z and w. Then

$$\rho(z, w) = \log \left(\frac{1 + \sqrt{(z, \Omega(z); w, \Omega(w))}}{1 - \sqrt{(z, \Omega(z); w, \Omega(w))}} \right),$$

where $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex numbers z.

Proof

Evaluating, and applying the result of Lemma 2.10, we find that

$$\begin{aligned}\rho(z,w) &= \log\left(\frac{|1-\overline{w}\,z| + |z-w|}{|1-\overline{w}\,z| - |z-w|}\right)\\ &= \log\left(\frac{1+\dfrac{|z-w|}{|1-\overline{w}\,z|}}{1-\dfrac{|z-w|}{|1-\overline{w}\,z|}}\right)\\ &= \log\left(\frac{1+\sqrt{(z,\Omega(z);w,\Omega(w))}}{1-\sqrt{(z,\Omega(z);w,\Omega(w))}}\right),\end{aligned}$$

as required. ∎

Corollary 2.12

Let z and w be complex numbers satisfying $|z| < 1$ and $|w| < 1$, and let $\rho(z, w)$ denote the Poincaré distance between z and w. Then the cross-ratio $(z, \Omega(z); w, \Omega(w))$ is expressed in terms of the Poincaré distance according to the formula

$$(z, \Omega(z); w, \Omega(w)) = \left(\frac{e^{\rho(z,w)} - 1}{e^{\rho(z,w)} + 1} \right)^2 .$$

Proof

Let $q = (z, \Omega(z); w, \Omega(w))$ and $s = \rho(z, w)$. It follows from Proposition 2.11 that

$$s = \log\left(\frac{1+\sqrt{q}}{1-\sqrt{q}}\right).$$

Consequently

$$e^s - 1 = \frac{2\sqrt{q}}{1-\sqrt{q}}, \quad e^s + 1 = \frac{2}{1-\sqrt{q}},$$

and thus

$$q = \left(\frac{e^s - 1}{e^s + 1}\right)^2.$$

The result follows. ∎

Definition

A transformation φ that maps the open unit disk D in the complex plane onto itself is said to be an *isometry* (with respect to Poincaré distance) if

$$\rho\Big(\varphi(z), \varphi(w)\Big) = \rho(z, w)$$

for all complex numbers z and w in the open unit disk D, where ρ denotes the Poincaré distance function on D.

Proposition 2.13

Let D be the open unit disk in the complex plane, defined so that $D = \{z \in \mathbb{C} : |z| < 1\}$. Then every Möbius transformation of the Riemann sphere that maps the open unit disk D onto itself is an isometry with respect to the Poincaré distance function on D.

Proof
The Möbius transformation μ has the property that $\mu \circ \Omega = \Omega \circ \mu$, because it maps the unit disk onto itself (see Proposition 2.2). Moreover the values of cross-ratios are preserved under the action of Möbius transformations (Proposition 1.18). Consequently

$$\begin{aligned} &\Big(\mu(z), \Omega(\mu(z)); \mu(w), \Omega(\mu(w))\Big) \\ &\qquad = \Big(\mu(z), \mu(\Omega(z)); \mu(w), \mu(\Omega(w))\Big) \\ &\qquad = \Big(z, \Omega(z); w, \Omega(w)\Big). \end{aligned}$$

The required result therefore follows immediately from an identity previously established (Proposition 2.11) expressing the Poincaré distance $\rho(z, w)$ in terms of the cross-ratio $(z, \Omega(z); w, \Omega(w))$. ∎

Proposition 2.14

Let z_1, w_1, z_2 and w_2 be elements of the open unit disk D, where

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

Suppose that $\rho(z_1, w_1) = \rho(z_2, w_2)$, where ρ denotes the Poincaré distance function on D. Then there exists a Möbius transformation μ mapping the open unit disk D onto itself with the property that $\mu(z_1) = z_2$ and $\mu(w_1) = w_2$.

Proof

The values of the cross-ratios

$$(z_1, \Omega(z_1); w_1, \Omega(w_1)) \quad \text{and} \quad (z_2, \Omega(z_2); w_2, \Omega(w_2))$$

are determined by the values of the Poincaré distances $\rho(z_1, w_1)$ and $\rho(z_2, w_2)$ respectively (see Corollary 2.12). Now $\mu(z_1) = z_2$ and $\mu(w_1) = w_2$. Consequently

$$(z_1, \Omega(z_1); w_1, \Omega(w_1)) = (z_2, \Omega(z_2); w_2, \Omega(w_2)).$$

It follows from this that there exists a unique Möbius transformation μ with the properties that $\mu(z_1) = z_2$, $\mu(\Omega(z_1)) = \Omega(z_2)$, $\mu(w_1) = w_2$ and $\mu(\Omega(w_1)) = \Omega(w_2)$, (see Proposition 1.18).

Now let $\hat{\mu} = \Omega \circ \mu \circ \Omega$. Then $\hat{\mu}$ is itself a Möbius transformation (Lemma 2.1) Then

$$\begin{aligned}
\hat{\mu}(z_1) &= \Omega(\mu(\Omega(z_1))) = \Omega(\Omega(z_2)) = z_2, \\
\hat{\mu}(\Omega(z_1)) &= \Omega(\mu(\Omega(\Omega(z_1)))) = \Omega(\mu(z_1)) = \Omega(z_2), \\
\hat{\mu}(w_1) &= \Omega(\mu(\Omega(w_1))) = \Omega(\Omega(w_2)) = w_2, \\
\hat{\mu}(\Omega(w_1)) &= \Omega(\mu(\Omega(\Omega(w_1)))) = \Omega(\mu(w_1)) = \Omega(w_2).
\end{aligned}$$

Consequently the Möbius transformations μ and $\hat{\mu}$ both map z_1, $\Omega(z_1)$, w_1 and $\Omega(w_1)$ to z_2, $\Omega(z_2)$, w_2 and $\Omega(w_2)$ respectively. But two distinct Möbius transformations cannot coincide at three or more points of the Riemann sphere. (see Proposition 1.9). Consequently $\hat{\mu} = \mu$, and thus $\Omega \circ \mu = \mu \circ \Omega$. Moreover elements z_1 and z_2 of the open unit disk D are mapped into D. Applying Proposition 2.2, we conclude that the Möbius transformation μ maps the open unit disk D onto itself. This completes the proof. ∎

Lemma 2.15

Let ρ be the Poincaré distance function on the open unit disk D in the complex plane, let t be a real number satisfying $0 < t < 1$, and let w be a complex number distinct from 0 and t for which $|w| < 1$. Then

$$\rho(0, w) \leq \rho(0, t) + \rho(t, w).$$

Moreover $\rho(0, w) = \rho(0, t) + \rho(t, w)$ if and only if the complex number w is a positive real number for which $t < w < 1$.

Proof

We first note that

$$\rho(0, t) = \log\left(\frac{1+t}{1-t}\right)$$

(see Lemma 2.8).

Given a complex number w in the unit disk that is distinct from 0 and t, let real numbers s and u between -1 and 1 be determined so that

$$\log\left(\frac{1+t}{1-t}\right) - \log\left(\frac{1+s}{1-s}\right) = \rho(t,w)$$

and

$$\log\left(\frac{1+u}{1-u}\right) - \log\left(\frac{1+t}{1-t}\right) = \rho(t,w).$$

Then $-1 < -u < s < t < u < 1$ and

$$\rho(s,t) = \rho(t,u) = \rho(t,w)$$

(again applying Lemma 2.8).

Let μ_0 be the Möbius transformation of the Riemann sphere defined such that $\mu_0(-1/t) = \infty$, $\mu_0(\infty) = 1/t$ and $\mu_0(z) = (z + t)/(1 + tz)$ for all complex numbers z distinct from $-1/t$. Then the Möbius transformation μ_0 maps the unit disk onto itself (Corollary 2.6), is an isometry of the Poincaré distance function (Proposition 2.13), cannot map a circle contained within the unit disk onto any straight line, and therefore maps circles contained within the unit disk onto circles within that disk (Proposition 1.11). Moreover the Möbius transformation μ_0 has the property that $\mu_0(\overline{z}) = \overline{\mu_0(z)}$ for all complex numbers z and therefore must map circles within the unit disk that are centred on points of the real line to circles that are also centred on points of the real line.

Let

$$C_0 = \{z \in D : \rho(0, z) = \rho(t, w)\}.$$

Then C_0 is a circle contained in the unit disk (Proposition 2.9), and $\mu_0(C_0) = C$, where

$$C = \{z \in D : \rho(t, z) = \rho(t, w)\}.$$

Consequently the subset C of the unit disk D, being the image of a circle centred on zero under the Möbius transformation μ_0, must be a circle contained within the unit disk and centred on a point of the complex plane that belongs to the open interval in the real line bounded by -1 and -1.

Now $s \in C$ and $u \in C$. It follows that the centre of the circle C is $\frac{1}{2}(u+s)$, and the radius of the circle C is circle C is $\frac{1}{2}(u-s)$. Consequently all points of the circle C other than u lie inside the circle centred on the origin that passes through the point u. The latter circle is the circle

$$\{z \in \mathbb{C} : \rho(0,z) = \rho(0,t) + \rho(t,w)\}.$$

Moreover w lies on the circle C. It follows that

$$\rho(0,w) \leq \rho(0,t) + \rho(t,w).$$

Moreover

$$\rho(0,w) = \rho(0,t) + \rho(t,w)$$

if and only if w is a real number satisfying $t < w < 1$, as required. ∎

Proposition 2.16 (Triangle Inequality for Poincaré Distance)

The Poincaré distance function ρ on the open unit disk D has the property that

$$\rho(z_1, z_3) \leq \rho(z_1, z_2) + \rho(z_2, z_3)$$

for all complex numbers z_1, z_2 and z_3 belonging to the disk D.

Proof

This inequality follows directly in cases where any two of z_1, z_2 and z_3 coincide with one another. Accordingly it remains to prove that the inequality holds in cases where these three complex numbers are distinct.

Accordingly let z_1, z_2 and z_3 be any three distinct points of the unit disk D. Then there exists a Möbius transformation μ that maps the unit disk onto itself and satisfies $\mu(z_1) = 0$ and $\mu(z_2) = t$ for some real number t satisfying the inequalities $0 < t < 1$ (see Proposition 2.14). Let $w = \mu(z_3)$. We have already shown that

$$\rho(0, w) \leq \rho(0, t) + \rho(t, w).$$

But the Möbius transformation μ is an isometry of the Poincaré distance function (Proposition 2.13). Consequently

$$\rho(z_1, z_3) \leq \rho(z_1, z_2) + \rho(z_2, z_3).$$

as required. ∎

2.3. Geodesics in the Open Unit Disk

Definition

We say that an (open) straight line segment or circular arc within the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ is *complete* if it is the intersection of the open unit disk with the full circle or straight line in the complex plane of which it forms part.

A complete straight line segment or circular arc in the open unit disk has no endpoints in the open unit disk itself. However its closure has endpoints that lie on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ that constitutes the boundary of the open unit disk: the complete straight line segment or circular arc may be said to *join* the endpoints of its closure in the complex plane.

Definition

A straight line segment or circular arc Γ in the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ is said to be a *geodesic* if it has the property that

$$\rho(z_1, z_3) = \rho(z_1, z_2) + \rho(z_2, z_3)$$

for all complex numbers z_1, z_2 and z_3 positioned on the straight line segment or circular arc Γ so that z_2 occurs between z_1 and z_3.

Definition

A *complete geodesic* in the open unit disk is a geodesic in that disk which is the intersection of the open unit disk with a full straight line or circle in the complex plane.

Definition

A *geodesic ray* in the open unit disk is a geodesic in that disk which is the intersection of the open unit disk with a closed straight line segment or circular arc in the complex plane for which one endpoint lies in the open unit disk and the other lies outside the open unit disk.

Definition

A *geodesic segment* in the open unit disk is a geodesic that is also a closed straight line segment or circular arc contained in the open unit disk both of whose endpoints lie in the open unit disk.

Definition

Given a point η on the unit circle in the complex plane, the *diameter* of the unit disk that *joins* $-\eta$ and η is the open straight line segment consisting of those complex numbers that are of the form $t\eta$ for some real number t satisfying the inequalities $-1 < t < 1$.

Proposition 2.17

Let D be the open unit disk in the complex plane, Then the diameter of the disk D obtained on intersecting the disk D with the real axis of the complex plane is a complete geodesic.

Proof

Let I be the set of real numbers t satisfying $|t| < 1$ and let t_1, t_2 and t_3 be real numbers satisfying $-1 < t_1 < t_2 < t_3 < 1$. It follows from Lemma 2.8 that

$$\begin{aligned} \rho(t_1, t_3) &= \log\left(\frac{1+t_3}{1-t_3}\right) - \log\left(\frac{1+t_1}{1-t_1}\right) \\ &= \log\left(\frac{1+t_3}{1-t_3}\right) - \log\left(\frac{1+t_2}{1-t_2}\right) \\ &\qquad + \log\left(\frac{1+t_2}{1-t_2}\right) - \log\left(\frac{1+t_1}{1-t_1}\right) \\ &= \rho(t_1, t_2) + \rho(t_2, t_3). \end{aligned}$$

Thus I is indeed a geodesic in the open unit disk D. ∎

Proposition 2.18

Given any real number t satisfying $0 < t < 1$, the unique complete geodesic in the open unit disk that passes through both 0 and t is the diameter of the disk obtained on intersecting the disk with the real axis of the complex plane.

Proof

Let Γ be a complete geodesic in the open unit disk D that passes through 0 and t, and let z be chosen on Γ so that t lies between 0 and z. Then $\rho(0,z) = \rho(0,t) + \rho(t,z)$, where ρ denotes the Poincaré distance function on D. Applying Lemma 2.15, we see that z must be a real number between t and 1. Consequently the three points 0, t and z on Γ are real numbers. Now the geodesic Γ must be the intersection of the open unit disk D with a straight line or circle in the complex plane. It follows that Γ must coincide with the intersection of the open unit disk with the real axis of the complex plane. The result follows. ∎

Proposition 2.19

Möbius transformations mapping the open unit disk onto itself map geodesics onto geodesics.

Proof

Let Γ be a geodesic in the open unit disk D, where $D = \{z \in \mathbb{C} : |z| < 1\}$, and let μ be a Möbius transformation that maps the open unit disk D onto itself. Let w_1, w_2 and w_3 be complex numbers positioned on the image $\mu(\Gamma)$ of the geodesic Γ so that w_2 occurs on $\mu(\Gamma)$ between w_1 and w_3. Then there exist complex numbers z_1, z_2 and z_3 in the open unit disk D lying on the geodesic Γ for which $\mu(z_1) = w_1$, $\mu(z_2) = w_2$ and $\mu(z_3) = w_3$. Moreover z_2 is positioned on Γ between z_1 and z_3.

The definition of geodesics then ensures that

$$\rho(z_1, z_3) = \rho(z_1, z_2) + \rho(z_2, z_3)$$

Now $\rho(w_1, w_2)$, $\rho(w_2, w_3)$ and $\rho(w_1, w_3)$ are equal to $\rho(z_1, z_2)$, $\rho(z_2, z_3)$ and $\rho(z_1, z_3)$ respectively, because Möbius transformations that map the open unit disk onto itself are isometries with respect to Poincaré distance (see Proposition 2.13) Consequently

$$\rho(w_1, w_3) = \rho(w_1, w_2) + \rho(w_2, w_3).$$

Thus the line segment or circular arc $\mu(\Gamma)$ is a geodesic, as required. ∎

Proposition 2.20

Let A be a complete straight line segment or circular arc in the open unit disk D. Suppose that there are complex numbers z_1, z_2 and z_3 on A, where z_2 lies between z_1 and z_3, such that

$$\rho(z_1, z_3) = \rho(z_1, z_2) + \rho(z_2, z_3).$$

Then A is a complete geodesic in the open unit disk D, and moreover there exists a Möbius transformation μ with the property that $\mu(A)$ is the diameter of the open unit disk that joins -1 and 1.

Proof

Let

$$t = \frac{e^\delta - 1}{e^\delta + 1}, \quad \text{where } \delta = \rho(z_1, z_2).$$

Then $\rho(0, t) = \rho(z_1, z_2)$. (see Proposition 2.9). Then there exists a Möbius transformation μ of the Riemann sphere mapping the open unit disk D onto itself which has the properties that $\mu(z_1) = 0$ and $\mu(z_2) = t$. (see Proposition 2.14). Let $w = \mu(z_3)$. Then

$$\rho(0, w) = \rho(0, t) + \rho(t, w),$$

because the Möbius transformation μ is an isometry of the Poincaré distance function ρ. It now follows from Lemma 2.15 that w is a real number and $t \leq w < 1$. The complex numbers z_1, z_2 and z_3 therefore all lie on the straight line or circle in the complex plane that is the image of the real axis under the inverse μ^{-1} of the Möbius transformation μ.

But two distinct straight lines or circles cannot pass through the three points z_1, z_2 and z_3. Consequently the complete arc A is contained in the image of the real axis under μ^{-1}, and therefore the Möbius transformation μ must map the complete arc onto the diameter of the open unit disk that joins -1 and 1. Moreover A must itself be a geodesic, because Möbius transformations that map the open unit disk D onto itself map geodesics onto geodesics Proposition 2.19. This completes the proof. ∎

Corollary 2.21

A complete straight line segment or circular arc A in the open unit disk D is a complete geodesic if and only if there exists a Möbius transformation μ that maps the straight line segment or circular arc onto a diameter of the unit circle.

Proof

If A is a complete geodesic then a direct application of Proposition 2.20 ensures that existence of a Möbius transformation mapping that complete geodesic onto the diameter of the disk D that joins -1 and 1.

Conversely if some Möbius transformation maps a complete straight line segment or circular arc onto a diameter, then that Möbius transformation can be composed with a rotation of the open unit disk about zero so as to obtain a Möbius transformation mapping the complete straight line segment or circular arc onto the diameter of the disk that is the intersection of the disk with the real axis of the complex plane. That diameter is a geodesic (see Proposition 2.17), and Möbius transformations map geodesics onto geodesics (Proposition 2.19). Consequently A must itself be a geodesic, as required. ∎

Proposition 2.22

Given two complete geodesics in the open unit disk D, there exists a Möbius transformation of the Riemann sphere that maps the open unit disk D onto itself and maps one complete geodesic onto the other.

Proof

Let Γ_1 and Γ_2 be complete geodesics in the open unit disk D, and let I be the geodesic joining -1 and 1 that is the intersection of the disk D with the real axis of the complex plane. It follows from Proposition 2.20 that there exist Möbius transformations μ_1 and μ_2 of the Riemann sphere that map the open unit disk onto itself, where μ_1 maps Γ_1 onto I and μ_2 maps Γ_2 onto I. Then $\mu_2^{-1} \circ \mu_1$ is a Möbius transformation of the Riemann sphere that maps the open unit disk D onto itself and also maps the complete geodesic Γ_1 onto the complete geodesic Γ_2, as required. ∎

Proposition 2.23

Given two distinct complex numbers w_1 and w_2 belonging to the open unit disk in the complex plane, there exists a unique complete geodesic in the open unit disk that passes through both w_1 and w_2.

Proof

Let

$$t = \left| \frac{w_2 - w_1}{1 - \overline{w}_1 w_2} \right|.$$

Then there exists a complex number η satisfying $|\eta| = 1$ for which $t = \mu(w_2)$, where μ is the Möbius transformation of the Riemann sphere that satisfies

$$\mu(z) = \frac{\eta(z - w_1)}{1 - \overline{w}_1 z}.$$

for all complex numbers z satisfying $1 - \overline{w}_1 z \neq 0$. Then μ maps the open unit disk onto itself and also maps w_1 and w_2 to 0 and t respectively. Let $\Gamma = \{z \in D : \mu(z) \in I\}$, where I is the diameter of the open unit disk consisting of all real numbers lying between -1 and 1. The Möbius transformation μ maps Γ onto the diameter I of the disk. Consequently Γ must be a geodesic in the unit disk (Corollary 2.21). This geodesic passes through w_1 and w_2.

We now show that Γ is the unique complete geodesic in the open unit disk that passes through w_1 and w_2. Let Γ' be a complete geodesic in the open unit disk that passes through w_1 and w_2. Then $\mu(\Gamma')$ is also a complete geodesic in the open unit disk, because Möbius transformations that map the open unit disk onto itself map geodesics onto geodesics (Proposition 2.19). But the distinct real numbers 0 and t lie on $\mu(\Gamma')$. It follows from Proposition 2.18 that $\mu(\Gamma')$ is the diameter I of the open unit disk consisting of all real numbers between -1 and 1. Consequently $\Gamma' \subset \Gamma$. The completeness of Γ' then ensures that Γ' coincides with Γ. Thus the complete geodesic Γ is indeed uniquely determined by w_1 and w_2, as required. ∎

Proposition 2.24

A complete straight line segment or circular arc in the unit disk is a complete geodesic if and only if the straight line or circle in the complex plane of which it forms part intersects the unit circle at right angles.

Proof

A complete straight line segment or circular arc A in the open unit disk D is a complete geodesic if and only if there exists a Möbius transformation μ that maps the arc onto a diameter of the unit circle (see Corollary 2.21).

The diameters of a circle intersect the circle at right angles, and angles between intersecting straight lines and circles are preserved under the action of Möbius transformations (see Proposition 1.27). Consequently if a complete circular arc is a geodesic then it is part of a circle that intersects the unit circle at right angles.

Conversely suppose that a complete circular arc A in the unit circle forms part of a circle that intersects the unit circle at right angles at z_1 and z_2, where $|z_1| = 1$ and $|z_2| = 1$. There then exists a Möbius transformation μ mapping the unit disk D onto itself for which $\mu(z_1) = -1$ and $\mu(z_2) = 1$ (see Lemma 2.4). The image $\mu(A)$ of the circular arc A under μ then intersects the boundary circle at right angles at -1 and 1, because Möbius transformations are angle-preserving. But Möbius transformations map circular arcs to circular arcs or straight lines. It follows that $\mu(A)$ must be the diameter of the unit circle that is the intersection of the open unit disk with the real axis. Consequently the complete circular arc A must be a geodesic (Corollary 2.21). The result follows. ∎

Definition

Let X be a subset of the complex plane. A collection of invertible transformations of the set X is said to be a *transformation group* acting on the set X if the following conditions are satisfied:

- **(i)** the identity transformation belongs to the collection;
- **(ii)** any composition of transformations belonging to the collection must itself belong to the collection;
- **(iii)** the inverse of any transformation belonging to the collection must itself belong to the collection.

The collection of all Möbius transformations of the Riemann sphere that map the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ onto itself is a transformation group acting on the open unit disk. Indeed the identity transformation is a Möbius transformation mapping the open unit disk onto itself, the composition of any two Möbius transformations that each map the open unit disk onto itself must also map the open unit disk onto itself, and the inverse of any Möbius transformation that maps the open unit disk onto itself must also map the open unit disk onto itself.

Definition

Let D be the open unit disk in the complex plane, defined so that $D = \{z \in \mathbb{C} : |z| < 1\}$, and let $\kappa\colon D \to D$ be the transformation of the open unit disk defined so that $\kappa(z) = \overline{z}$ for all $z \in D$, where $\overline{z}$ denotes the complex conjugate of the complex number z. A transformation of the open unit disk is said to be a *hyperbolic motion* of the unit disk if either it is a Möbius transformation mapping the unit disk D onto itself or else it expressible as a composition of transformations of the form $\mu \circ \kappa$, where μ is a Möbius transformation mapping the open unit disk onto itself.

Möbius transformations give rise to orientation-preserving transformations of the complex plane (see Proposition 1.28 and the discussion of orientation-preserving and orientation-reversing transformations of the complex plane that follows the proof of that proposition). Also the transformation $\kappa\colon D \to D$ that maps each complex number z in D to its complex conjugate $\overline{z}$ is orientation-reversing. Consequently a composition of two transformations in which some Möbius transformation follows the complex conjugation transformation κ is orientation-reversing.

Orientation-preserving hyperbolic motions are the analogues, in hyperbolic geometry, of transformations of the flat Euclidean plane that can be represented as the composition of a rotation followed by a translation.

Orientation-reversing hyperbolic motions are the analogues, in hyperbolic geometry, of reflections and glide reflections of the flat Euclidean plane.

Proposition 2.25

Let D be the open unit disk in the complex plane, consisting of those complex numbers z that satisfy $|z| < 1$. Then, given any orientation-preserving hyperbolic motion φ of the open unit disk D, there exist complex numbers a and b, where $|b| < |a|$, such that

$$\varphi(z) = \frac{az + b}{\overline{b}\,z + \overline{a}} \quad \text{for all } z \in D.$$

Similarly, given any orientation-reversing hyperbolic motion φ of the open unit disk D, there exist complex numbers a and b, where $|b| < |a|$ such that

$$\varphi(z) = \frac{a\,\overline{z} + b}{\overline{b}\,\overline{z} + \overline{a}} \quad \text{for all } z \in D.$$

2. The Disk Model of the Hyperbolic Plane (continued)

Proof

This result follows directly on applying Proposition 2.7. ∎

Proposition 2.26

The collection of all hyperbolic motions of the open unit disk is a transformation group acting on the open unit disk.

Proof

The identity transformation is a Möbius transformation that maps the open unit disk onto itself and is thus a hyperbolic motion. Next let μ_1 and μ_2 be Möbius transformations that map the open unit disk onto itself, Then $\kappa \circ \mu_2 \circ \kappa$ is also a Möbius transformation that maps the open unit disk onto itself.

Indeed there exist complex numbers a_2 and b_2, where $|b_2| < |a_2|$, such that

$$\mu_2(z) = \frac{a_2 z + b_2}{\overline{b}_2\, z + \overline{a}_2}$$

for all complex numbers z for which $\overline{b}_2\, z + \overline{a}_2 \neq 0$ (see Proposition 2.7). Then

$$\kappa(\mu_2(\kappa(z))) = \frac{\overline{a}_2 z + \overline{b}_2}{b_2\, z + a_2},$$

and therefore $\kappa \circ \mu \circ \kappa$ is also a Möbius transformation that maps the open unit disk D onto itself. Now

$$\mu_1 \circ (\mu_2 \circ \kappa) = (\mu_1 \circ \mu_2) \circ \kappa, \quad (\mu_1 \circ \kappa) \circ \mu_2 = (\mu_1 \circ (\kappa \circ \mu_2 \circ \kappa)) \circ \kappa$$

and

$$(\mu_1 \circ \kappa) \circ (\mu_2 \circ \kappa) = \mu_1 \circ (\kappa \circ \mu_2 \circ \kappa).$$

Moreover $\mu_1 \circ \mu_2$ and $\mu_1 \circ (\kappa \circ \mu_2 \circ \kappa)$, being compositions of Möbius transformations that map the open unit disk onto itself, are themselves Möbius transformations that map the open unit disk onto itself. It follows from this observation that any composition of hyperbolic motions of the open unit disk is itself a hyperbolic motion of the open unit disk. Also

$$(\mu_2 \circ \kappa)^{-1} = \kappa \circ \mu_2^{-1} = (\kappa \circ \mu_2^{-1} \circ \kappa) \circ \kappa,$$

and the inverse of any Möbius transformation that maps the open unit disk onto itself must itself be a Möbius transformation that maps the open unit disk onto itself. Consequently the inverse of any hyperbolic motion is itself a hyperbolic motion. It follows that the collection of all hyperbolic motions of the open unit disk is indeed a transformation group acting on the open unit disk. ∎

Proposition 2.27

Let Γ be a complete geodesic in the open unit disk D. Then there exists an orientation-reversing hyperbolic motion φ with the property that $\varphi(z) = z$ for all complex numbers z that lie on the geodesic Γ and also those points of the open unit disk D that lie on one side of the geodesic Γ are mapped by points that lie on the other side of Γ.

Proof

Let I be the set of real numbers t that satisfy the inequalities $-1 < t < 1$. Then I is a complete geodesic in the open unit disk D. There then exists a Möbius transformation μ that maps the geodesic I onto the geodesic Γ. (see Proposition 2.20 or Proposition 2.22). Let $\varphi = \mu \circ \kappa \circ \mu^{-1}$, where $\kappa(z) = \overline{z}$ for all $z \in D$. Then the orientation-reversing hyperbolic motion Γ has the required properties. ∎

Proposition 2.28

Let z_1, w_1, z_2 and w_2 be complex numbers belonging to the open unit disk D. Suppose that $\rho(z_1, w_1) = \rho(z_2, w_2)$, and suppose also that one of the sides of the geodesic Γ_1 in D passing through z_1 and w_1 has been chosen, and that one of the sides of the geodesic Γ_2 in D passing through z_2 and w_2 has also been chosen. Then there exists a hyperbolic motion φ with the following properties: $\varphi(z_1) = z_2$; $\varphi(w_1) = w_2$; φ maps complex numbers on the chosen side of the geodesic Γ_1 to complex numbers on the chosen side of the geodesic Γ_2.

Proof

It follows from Proposition 2.14 there exists a Möbius transformation that maps the open unit disk onto itself and also maps z_1 and w_1 to z_2 and w_2 respectively. If this Möbius transformation does not itself map the chosen side of Γ_1 to the chosen side of Γ_2, then it may be composed with an orientation-reversing hyperbolic motion that fixes all complex numbers of the geodesic Γ_2 whilst mapping complex numbers on one side of Γ_2 to complex numbers on the other side. The result follows. ∎

Proposition 2.29

Let w be a complex number belonging to the open unit disk D in the complex plane, and let ρ denote the Poincaré distance function on D. Let δ be a positive real number. Then

$$\{z \in D : \rho(z, w) < \delta\} = \left\{ z \in D : \left| \frac{z - w}{1 - \overline{w}\, z} \right| < R \right\},$$

where

$$R = \frac{e^\delta - 1}{e^\delta + 1}.$$

Proof
Let
$$\mu_w(z) = \frac{z - w}{1 - \overline{w}z}$$
for all complex numbers z. Then μ_w is a Möbius transformation mapping the open unit disk onto itself for which $\mu_w(w) = 0$ (see Corollary 2.6). Now Möbius transformations mapping the open unit disk onto itself are isometries with regard to the Poincaré distance function (see Proposition 2.13). Consequently

$$\{z \in D : \rho(z, w) < \delta\} = \{z \in D : \rho(\mu_w(z), 0) < \delta\}.$$

The required result now follows on applying Proposition 2.9. ∎

Definition

Let D be the open unit disk in the complex plane that consists of those complex numbers z satisfying $|z| < 1$, and let C be a circle in the complex plane that is contained within D. A complex number w is said to be the *hyperbolic centre* of the circle C if the Poincaré distance between z and w is the same for all points z that lie on the circle C.

Proposition 2.30

Let C be a circle in the complex plane that is contained within the open unit disk D. Suppose that the circle C intersects the real axis at real numbers u and v, where $-1 < u < v < 1$. Suppose also that the hyperbolic centre of the circle C lies on the real axis, and is located at t, where $u < t < v$. Then

$$\left(\frac{1+t}{1-t}\right)^2 = \frac{(1+u)(1+v)}{(1-u)(1-v)}.$$

Proof

Applying Lemma 2.8, we find that t, u and v must satisfy the identity

$$\log\left(\frac{1+v}{1-v}\right) - \log\left(\frac{1+t}{1-t}\right) = \log\left(\frac{1+t}{1-t}\right) - \log\left(\frac{1+u}{1-u}\right).$$

Consequently

$$2\log\left(\frac{1+t}{1-t}\right) = \log\left(\frac{1+u}{1-u}\right) + \log\left(\frac{1+v}{1-v}\right).$$

The required result then follows on taking the exponential of both sides of this identity. ∎