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Part II, Section 2:
Möbius Transformations of the Riemann Sphere

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2. Möbius Transformations of the Riemann Sphere

2.1. The Riemann Sphere

The *Riemann sphere* $\mathbb{P}^1$ may be defined as the set $\mathbb{C} \cup \{\infty\}$ obtained by augmenting the system $\mathbb{C}$ of complex numbers with an additional element, denoted by ∞, where ∞ is not itself a complex number, but is an additional element added to the set, with the additional conventions that

$$z + \infty = \infty, \quad \infty \times \infty = \infty, \quad \frac{z}{\infty} = 0 \quad \text{and} \quad \frac{\infty}{z} = \infty$$

for all complex numbers z, and

$$z \times \infty = \infty, \qquad \text{and} \quad \frac{z}{0} = \infty$$

for all non-zero complex numbers z. The symbol ∞ cannot be added to, or subtracted from, itself. Also 0 and ∞ cannot be divided by themselves.

Note that, because the sum of two elements of $\mathbb{P}^1$ is not defined for every single pair of elements of $\mathbb{P}^1$, this set cannot be regarded as constituting a group under the operation of addition. Similarly its non-zero elements cannot be regarded as constituting a group under multiplication. In particular, the Riemann sphere cannot be regarded as constituting a field.

The following proposition follows directly from Proposition 1.3.

Proposition 2.1

Let $\sigma\colon \mathbb{P}^1 \to \mathbb{R}^3$ *be the mapping from the Riemann sphere* $\mathbb{P}^1$ *to* $\mathbb{R}^3$ *defined such that* $\sigma(\infty) = (0, 0, -1)$ *and*

$$\sigma(x + y\sqrt{-1}) = \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{1 - x^2 - y^2}{1 + x^2 + y^2}\right)$$

for all real numbers x *and* y*. Then the map* σ *sets up a one-to-one correspondence between points of the Riemann sphere* $\mathbb{P}^1$ *and points of the unit sphere* S^2 *in* $\mathbb{R}^3$*. To each point of the Riemann sphere* $\mathbb{P}^1$ *there corresponds exactly one point of the unit sphere* S^2 *in three-dimensional Euclidean space, and vice versa. Moreover if* (u, v, w) *is a point of the unit sphere* S^2 *distinct from* $(0, 0, -1)$ *then* $(u, v, w) = \sigma(x + y\sqrt{-1})$*, where*

$$x = \frac{u}{w + 1} \quad \textit{and} \quad y = \frac{v}{w + 1}.$$

2.2. Möbius Transformations

Definition

Let a, b, c and d be complex numbers satisfying $ad - bc \neq 0$. The *Möbius transformation* $\mu_{a,b,c,d} \colon \mathbb{P}^1 \to \mathbb{P}^1$ with coefficients a, b, c and d is defined to be the function from the Riemann sphere $\mathbb{P}^1$ to itself determined by the following properties:

$$\mu_{a,b,c,d}(z) = \frac{az + b}{cz + d}$$

for all complex numbers z for which $cz + d \neq 0$;
$\mu_{a,b,c,d}(-d/c) = \infty$ and $\mu_{a,b,c,d}(\infty) = a/c$ if $c \neq 0$;
$\mu_{a,b,c,d}(\infty) = \infty$ if $c = 0$.

Note that the requirement in the above definition of a Möbius transformation that its coefficients a, b, c and d satisfy the condition $ad - bc \neq 0$ ensures that there is no complex number for which $az + b$ and $cz + d$ are both zero.

Let A be a non-singular 2×2 matrix whose coefficients are complex numbers, and let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We denote by μ_A the Möbius transformation $\mu_{a,b,c,d}$ with coefficients a, b, c, d, defined so that

$$\mu_A(z) = \begin{cases} \dfrac{az+b}{cz+d} & \text{if } cz + d \neq 0; \\ \infty & \text{if } c \neq 0 \text{ and } z = -d/c; \end{cases}$$

$$\mu_A(\infty) = \begin{cases} \dfrac{a}{c} & \text{if } c \neq 0; \\ \infty & \text{if } c = 0. \end{cases}$$

The following result exemplifies the reason for representing the coefficients of a Möbius transformation in the form of a matrix.

Proposition 2.2

The composition of any two Möbius transformations is a Möbius transformation. Specifically let A and B be non-singular 2×2 matrices with complex coefficients, and let μ_A and μ_B be the corresponding Möbius transformations of the Riemann sphere. Then the composition $\mu_A \circ \mu_B$ of these Möbius transformations is the Möbius transformation μ_{AB} of the Riemann sphere determined by the product AB of the matrices A and B.

Proof

Let

$$A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix},$$

and let

$$AB = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}.$$

Then

$$a_3 = a_1 a_2 + b_1 c_2, \quad b_3 = a_1 b_2 + b_1 d_2,$$

$$c_3 = c_1 a_2 + d_1 c_2 \quad \text{and} \quad d_3 = c_1 b_2 + d_1 d_2.$$

The definitions of Möbius transformations determined by non-singular 2×2 matrices ensure that

$$\mu_A(z) = \frac{a_1 z + b_1}{c_1 z + d_1}$$

whenever $c_1 z + d_1 \neq 0$ and

$$\mu_B(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$$

whenever $c_2 z + d_2 \neq 0$.

First suppose that z is a complex number for which $c_2 z + d_2 \neq 0$. Then

$$\begin{aligned}(a_1\mu_B(z) + b_1)(c_2 z + d_2) &= a_1(a_2 z + b_2) + b_1(c_2 z + d_2)\\ &= a_3 z + b_3,\\ (c_1\mu_B(z) + d_1)(c_2 z + d_2) &= c_1(a_2 z + b_2) + d_1(c_2 z + d_2)\\ &= c_3 z + d_3.\end{aligned}$$

It follows that if $c_2 z + d_2 \neq 0$ and $c_1\mu_B(z) + d_1 \neq 0$ then

$$\mu_A(\mu_B(z)) = \frac{a_1\mu_B(z) + b_1}{c_1\mu_B(z) + d_1} = \frac{a_3 z + b_3}{c_3 z + d_3} = \mu_{AB}(z).$$

If $c_2 z + d_2 \neq 0$ but $c_1\mu_B(z) + d_1 = 0$ then $c_3 z + d_3 = 0$ and

$$\mu_A(\mu_B(z)) = \infty = \mu_{AB}(z).$$

We conclude that $\mu_A(\mu_B(z)) = \mu_{AB}(z)$ for all complex numbers z satisfying $c_2 z + d_2 \neq 0$.

Next suppose that z is a complex number for which $c_2 z + d_2 = 0$. Now the definition of Möbius transformations requires that $a_2 d_2 - b_2 c_2 \neq 0$. It follows that c_2 and d_2 cannot both be equal to zero. Thus if $c_2 z + d_2 = 0$ then either $z = d_2 = 0$ and $c_2 \neq 0$ or else z, c_2 and d_2 are all non-zero. Thus, in all cases where $c_2 z + d_2 = 0$, the coefficient c_2 of the Möbius transformation is non-zero and $z = -d_2/c_2$. Also the equations $a_2 z + b_2 = 0$ and $c_2 z + d_2 = 0$ cannot both be satisfied, because $a_2 d_2 - b_2 c_2 \neq 0$, and therefore $a_2 z + b_2 \neq 0$.

Now the equations determining a_3, b_3, c_3 and d_3 ensure that if $c_2 z + d_2 = 0$ then

$$
\begin{aligned}
c_2(a_3 z + b_3) &= -d_2 a_3 + c_2 b_3 \\
&= c_2(a_1 b_2 + b_1 d_2) - d_2(a_1 a_2 + b_1 c_2) \\
&= a_1(b_2 c_2 - a_2 d_2) \\
&= a_1 c_2(a_2 z + b_2) \\
c_2(c_3 z + d_3) &= -d_2 c_3 + c_2 d_3 \\
&= c_2(c_1 b_2 + d_1 d_2) - d_2(c_1 a_2 + d_1 c_2) \\
&= c_1(b_2 c_2 - a_2 d_2) \\
&= c_1 c_2(a_2 z + b_2),
\end{aligned}
$$

and therefore

$$a_3 z + b_3 = a_1(a_2 z + b_2) \quad \text{and} \quad c_3 z + d_3 = c_1(a_2 z + b_2),$$

Thus if $c_2 z + d_2 = 0$ and $c_1 \neq 0$ then $c_3 z + d_3 \neq 0$ and

$$\mu_{AB}(z) = \frac{a_3 z + b_3}{c_3 z + d_3} = \frac{a_1}{c_1} = \mu_A(\infty) = \mu_A(\mu_B(z)).$$

And if $c_2 z + d_2 = 0$ and $c_1 = 0$ then $c_3 z + d_3 = 0$ and

$$\mu_{AB}(z) = \infty = \mu_A(\infty) = \mu_A(\mu_B(z)).$$

Thus $\mu_{AB}(z) = \mu_A(\mu_B(z))$ in all cases for which $c_2 z + d_2 = 0$.

It remains to show that $\mu_{AB}(\infty) = \mu_A(\mu_B(\infty))$. If $c_2 \neq 0$ (so that $\mu_B(\infty) = a_2/c_2$) and $c_1\mu_B(\infty) + d_2 \neq 0$ then

$$\mu_A(\mu_B(\infty)) = \frac{a_1\mu_B(\infty) + b_1}{c_1\mu_B(\infty) + d_1} = \frac{a_1a_2 + b_1c_2}{c_1a_2 + d_1c_2} = \frac{a_3}{c_3} = \mu_{AB}(\infty).$$

If $c_2 \neq 0$ and $c_1\mu_B(\infty) + d_2 = 0$ then $c_3 = c_1a_2 + d_1c_2 = 0$, because $\mu_B(\infty) = a_2/c_2$, and therefore

$$\mu_A(\mu_B(\infty)) = \infty = \mu_{AB}(\infty).$$

If $c_1 = c_2 = 0$ then $\mu_B(\infty) = \infty$ and therefore

$$\mu_A(\mu_B(\infty)) = \mu_A(\infty) = \infty = \mu_{AB}(\infty).$$

If $c_2 = 0$ and $c_1 \neq 0$ then $a_3 = a_1a_2$, $c_3 = c_1a_2$ and $a_2 \neq 0$ (because $a_2d_2 - b_2c_2 \neq 0$), and therefore

$$\mu_A(\mu_B(\infty)) = \mu_A(\infty) = \frac{a_1}{c_1} = \frac{a_3}{c_3} = \mu_{AB}(\infty).$$

We conclude that $\mu_A(\mu_B(\infty)) = \mu_{AB}(\infty)$ in all cases. This completes the proof. ∎

Corollary 2.3

Let a, b, c and d be complex numbers satisfying $ad - bc \neq 0$, and let $\mu_{a,b,c,d}\colon \mathbb{P}^1 \to \mathbb{P}^1$ denote the Möbius transformation of the Riemann sphere $\mathbb{P}^1$ defined such that $\mu_{a,b,c,d}(z) = \dfrac{az+b}{cz+d}$ if $z \in \mathbb{C}$ and $cz + d \neq 0$, $\mu_{a,b,c,d}(-d/c) = \infty$ and $\mu_{a,b,c,d}(\infty) = a/c$ if $c \neq 0$, and $\mu_{a,b,c,d}(\infty) = \infty$ if $c = 0$. Then the mapping $\mu_{a,b,c,d}\colon \mathbb{P}^1 \to \mathbb{P}^1$ is invertible, and its inverse is the Möbius transformation $\mu_{d,-b,-c,a}\colon \mathbb{P}^1 \to \mathbb{P}^1$, where $\mu_{d,-b,-c,a}(z) = \dfrac{dz-b}{a-cz}$ if $z \in \mathbb{C}$ and $a - cz \neq 0$, $\mu_{d,-b,-c,a}(a/c) = \infty$ and $\mu_{d,-b,-c,a}(\infty) = -d/c$ if $c \neq 0$, and $\mu_{d,-b,-c,a}(\infty) = \infty$ if $c = 0$.

Proof

If the coefficients a, b, c and d of a Möbius transformation are all multiplied by a non-zero complex number then this does not change the Möbius transformation represented by those coefficients. It follows that we may assume, without loss of generality, that $ad - bc = 1$. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $ad - bc = 1$. Then

$$A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The result therefore follows directly on applying Proposition 2.2. ∎

2.3. Inversions of the Riemann Sphere in Great Circles

Let S^2 denote the unit sphere in $\mathbb{R}^3$, defined so that

$$S^2 = \{(u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 = 1\},$$

and let us refer to the points $(0, 0, 1)$ and $(0, 0, -1)$ as the *North Pole* and *South Pole* respectively. Let E denote the *Equatorial Plane* in $\mathbb{R}^3$, consisting of those points whose Cartesian coordinates are of the form $(x, y, 0)$, where x and y are real numbers.

Stereographic projection from the South Pole maps each point (u, v, w) of the unit sphere S^2 distinct from the South Pole to the point $(x, y, 0)$ of the equatorial plane E for which

$$x = \frac{u}{w+1} \quad \text{and} \quad y = \frac{v}{w+1}.$$

Moreover a point $(x, y, 0)$ of the Equatorial Plane E is the image under stereographic projection from the South Pole of the point (u, v, w) of the unit sphere S^2 for which

$$u = \frac{2x}{1+x^2+y^2}, \quad v = \frac{2y}{1+x^2+y^2}, \quad w = \frac{1-x^2-y^2}{1+x^2+y^2}.$$

We can also stereographically project from the North Pole. Note that, given a point in the Equatorial Plane, reflection in that Equatorial Plane will interchange the points of the sphere corresponding to it under stereographic projection from the North and South Poles. Thus a point (u, v, w) of the unit sphere S^2 distinct from the North Pole corresponds under stereographic projection to the point $(x, y, 0)$ of the Equatorial Plane E for which

$$x = \frac{u}{1 - w} \quad \text{and} \quad y = \frac{v}{1 - w}.$$

In the other direction, a point $(x, y, 0)$ of the Equatorial Plane E corresponds under stereographic projection from the North Pole to the point (u, v, w) of the unit sphere S^2 for which

$$u = \frac{2x}{1 + x^2 + y^2}, \quad v = \frac{2y}{1 + x^2 + y^2}, \quad w = \frac{x^2 + y^2 - 1}{1 + x^2 + y^2}.$$

Proposition 2.4

Let O denote the origin $(0,0,0)$ of the Equatorial Plane E, where

$$E = \{(x,y,z) \in \mathbb{R}^3 : z = 0\},$$

and let A be a point $(x,y,0)$ of E distinct from the origin O. Let C be the point on the unit sphere S^2 that corresponds to A under stereographic projection from the North Pole $(0,0,1)$, and let B be the point of the Equatorial Plane E that corresponds to C under stereographic projection from the South Pole. Then $B = (p,q,0)$, where

$$p = \frac{x}{x^2+y^2} \quad \textit{and} \quad q = \frac{y}{x^2+y^2}.$$

Thus the points O, A and B are collinear, and the points A and B lie on the same side of the origin O. Also the distances $|OA|$ and $|OB|$ of the points A and B from the origin satisfy $|OA| \times |OB| = 1$.

Proof

Let $(x, y, 0)$ be a point of the Equatorial plane E distinct from the origin. This point is the image, under stereographic projection from the North Pole $(0, 0, 1)$ of the point (u, v, w) of the unit sphere S^2 for which

$$u = \frac{2x}{1 + x^2 + y^2}, \quad v = \frac{2y}{1 + x^2 + y^2}, \quad w = \frac{x^2 + y^2 - 1}{1 + x^2 + y^2}.$$

This point then gets mapped under stereographic projection from the South Pole to the point $(p, q, 0)$ of the Equatorial Plane E for which

$$p = \frac{u}{w + 1} \quad \text{and} \quad q = \frac{v}{w + 1}.$$

Now

$$w + 1 = \frac{2(x^2 + y^2)}{1 + x^2 + w^2}.$$

It follows that

$$p = \frac{x}{x^2 + y^2} \quad \text{and} \quad q = \frac{y}{x^2 + y^2}.$$

Finally we note that O, A and B are collinear, where $0 = (0, 0, 0)$, $A = (x, y, 0)$ and $B = (p, q, 0)$, and the points A and B lie on the same side of the origin O. Also

$$|OA| = \sqrt{x^2 + y^2}, \quad \text{and} \quad |OB| = \frac{1}{\sqrt{x^2 + y^2}},$$

and therefore $|OA| \times |OB| = 1$, as required. ∎

2.4. The Action of Möbius Transformations on the Riemann Sphere

Proposition 2.5

Let $\zeta_1, \zeta_2, \zeta_3$ be distict points of the Riemann sphere $\mathbb{P}^1$, and let $\omega_1, \omega_2, \omega_3$ also be distinct points of $\mathbb{P}^1$. Then there exists a Möbius transformation $\mu\colon \mathbb{P}^1 \to \mathbb{P}^1$ of the Riemann sphere with the property that $\mu(\zeta_j) = \omega_j$ for $j = 1, 2, 3$.

Proof

The composition of any two Möbius transformations of the Riemann sphere $\mathbb{P}^1$ is itself a Möbius transformation of $\mathbb{P}^1$ (Proposition 2.2). Also the inverse of any Möbius transformation of the Riemann sphere is itself a Möbius transformation (Corollary 2.3). It follows that the Möbius transformations of the Riemann sphere constitute a group under the operation of composition of transformations.

Next we note that permutation of the elements 0, 1 and ∞ of the Riemann sphere can be effected by a suitable Möbius transformation. Indeed the Möbius transformation $z \mapsto 1 - z$ transposes 0 and 1 whilst fixing ∞, and the Möbius transformation $z \mapsto -1/(z-1)$ cyclicly permutes 0, 1 and ∞. It follows that any permutation of 0, 1 and ∞ may be effected by the action of some Möbius transformation.

Next we show that there exists a Möbius transformation $\mu_1 \colon \mathbb{P}^1 \to \mathbb{P}^1$ with the property that $\mu_1(\zeta_1) = 0$, $\mu_1(\zeta_2) = 1$ and $\mu_1(\zeta_3) = \infty$. Suppose first that at least one of the distinct points $\zeta_1, \zeta_2, \zeta_3$ of $\mathbb{P}^1$ is the point ∞. Because we have shown that there exist Möbius transformations permuting 0, 1 and ∞ amongst themselves, we may assume in this case, without loss of generality, that $\zeta_3 = \infty$. Let $\zeta_1 = z_1$ and $\zeta_2 = z_2$, where z_1 and z_2 are complex numbers, and let

$$\mu_1(z) = \frac{z - z_1}{z_2 - z_1}.$$

Then $\mu_1(\zeta_1) = \mu_1(z_1) = 0$, $\mu_1(\zeta_2) = \mu_1(z_2) = 1$ and $\mu_1(\infty) = \infty$. The existence of the Möbius transformation μ_1 has thus been verified in the case where at least one of $\zeta_1, \zeta_2, \zeta_3$ is the point ∞ of the Riemann sphere.

Next we consider the case where $\zeta_j = z_j$ for $j = 1, 2, 3$, where z_1, z_2, z_3 are complex numbers. In this case let μ_1 be the Möbius transformation defined so that

$$\mu_1(z) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

for all complex numbers z. Then $\mu_1(\zeta_1) = \mu_1(z_1) = 0$, $\mu_1(\zeta_2) = \mu_1(z_2) = 1$ and $\mu_1(\zeta_3) = \mu_1(z_3) = \infty$. We conclude therefore that, given any distinct points $\zeta_1, \zeta_2, \zeta_3$ of the Riemann sphere, there exists a Möbius transformation μ_1 of the Riemann sphere for which $\mu_1(\zeta_1) = 0$, $\mu_1(\zeta_2) = 1$ and $\mu_1(\zeta_3) = \infty$.

Now let $\omega_1, \omega_2, \omega_3$ also be points of the Riemann sphere that are distinct from one another. Then there exists a Möbius transformation μ_2 with the property that $\mu_2(\omega_1) = 0$, $\mu_2(\omega_2) = 1$ and $\mu_2(\omega_3) = \infty$. Let $\mu \colon \mathbb{P}^1 \to \mathbb{P}^1$ be the Möbius transformation of the Riemann sphere that is the composition $\mu_2^{-1} \circ \mu_1$ of μ_1 followed by the inverse of μ_2. Then $\mu(\zeta_j) = \omega_j$ for $j = 1, 2, 3$, as required. $\blacksquare$

Proposition 2.6

Any Möbius transformation of the Riemann sphere maps straight lines and circles to straight lines and circles.

Proof

The equation of a line or circle in the complex plane can be expressed in the form

$$g|z|^2 + 2\mathrm{Re}[\overline{b}z] + h = 0,$$

where g and h are real numbers, and b is a complex number. Moreover a locus of points in the complex plane satisfying an equation of this form is a circle if $g \neq 0$ and is a line if $g = 0$.

Let g and h be real constants, let b be a complex constant, and let $z = 1/w$, where $w \neq 0$ and w satisfies the equation

$$g|w|^2 + 2\mathrm{Re}[\overline{b}w] + h = 0,$$

Then

$$g|w|^2 + \overline{b}w + b\overline{w} + h = 0,$$

Then

$$\begin{aligned} g + \mathrm{Re}[bz] + h|z|^2 &= g + \overline{b}\overline{z} + bz + h|z|^2 \\ &= \frac{1}{|w|^2}\left(g|w|^2 + \overline{b}w + b\overline{w} + h\right) = 0. \end{aligned}$$

We deduce from this that the Möbius transformation that sends z to $1/z$ for all non-zero complex numbers z maps lines and circles to lines and circles.

Let $\mu\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a Möbius transformation of the Riemann sphere. Then there exist complex numbers a, b, c and d satisfying $ad - bc \neq 0$ such that

$$\mu(z) = \frac{az + b}{cz + d}$$

for all complex numbers z for which $cz + d \neq 0$. The result is immediate when $c = 0$. We therefore suppose that $c \neq 0$. Then

$$\mu(z) = \frac{az+b}{cz+d} = \frac{a}{c} - \frac{ad-bc}{c} \times \frac{1}{cz+d}$$

when $cz + d \neq 0$. The Möbius transformation μ is thus the composition of three maps that each send circles and straight lines to circles and straight lines and preserve angles between lines and circles, namely the maps

$$z \mapsto cz + d, \quad z \mapsto \frac{1}{z} \quad \text{and} \quad z \mapsto \frac{a}{c} - \frac{(ad-bc)z}{c}.$$

Thus the Möbius transformation μ must itself map circles and straight lines to circles and straight lines, as required. ∎

2.5. Cross-Ratios of Points of the Riemann Sphere

Definition

The *cross-ratio* $(z_1, z_2; z_3, z_4)$ of four distinct complex numbers z_1, z_2, z_3 and z_4 is defined so that

$$(z_1, z_2; z_3, z_4) = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)}.$$

In addition $(\infty, z_2; z_3, z_4)$ is defined, when z_2, z_3 and z_4 are distinct, so that

$$(\infty, z_2; z_3, z_4) = \frac{z_4 - z_2}{z_3 - z_2},$$

$(z_1, \infty; z_3, z_4)$ is defined, when z_1, z_3 and z_4 are distinct, so that

$$(z_1, \infty; z_3, z_4) = \frac{z_3 - z_1}{z_4 - z_1},$$

$(z_1, z_2; \infty, z_4)$ is defined, when z_1, z_2 and z_4 are distinct, so that

$$(z_1, z_2; \infty, z_4) = \frac{z_4 - z_2}{z_4 - z_1},$$

and $(z_1, z_2; z_3, \infty)$ is defined, when z_1, z_2 and z_3 are distinct, so that

$$(z_1, z_2; z_3, \infty) = \frac{z_3 - z_1}{z_3 - z_2}.$$

These definitions ensure that the cross-ratio of any four distinct points on the Riemann sphere $\mathbb{P}^1$ is always defined, and that

$$\begin{aligned}
\lim_{z_1 \to \infty} (z_1, z_2; z_3, z_4) &= (\infty, z_2; z_3, z_4), \\
\lim_{z_2 \to \infty} (z_1, z_2; z_3, z_4) &= (z_1, \infty; z_3, z_4), \\
\lim_{z_3 \to \infty} (z_1, z_2; z_3, z_4) &= (z_1, z_2; \infty, z_4), \\
\lim_{z_4 \to \infty} (z_1, z_2; z_3, z_4) &= (z_1, z_2; z_3, \infty)
\end{aligned}$$

whenever the three complex numbers distinct from ∞ occurring in the relevant cross-ratio are distinct from one another.

We now show that, given four distinct elements ζ_1, ζ_2, ζ_3, ζ_4 of the Riemann sphere, the value of the cross-ratio $(\zeta_1, \zeta_2; \zeta_3, \zeta_4)$ taken with respect to any one particular ordering of those four elements determines the value of the cross-ratio taken with respect to any other ordering of those elements.

Proposition 2.7

Let ζ_1, ζ_2, ζ_3 and ζ_4 be distinct elements of the Riemann sphere $\mathbb{P}^1$, and let $\rho = (\zeta_1, \zeta_2; \zeta_3, \zeta_4)$. Then

- $(\zeta_1, \zeta_2; \zeta_3, \zeta_4)$, $(\zeta_2, \zeta_1; \zeta_4, \zeta_3)$, $(\zeta_3, \zeta_4; \zeta_1, \zeta_2)$, $(\zeta_4, \zeta_3; \zeta_2, \zeta_1)$ *are all equal to* ρ*;*
- $(\zeta_1, \zeta_2; \zeta_4, \zeta_3)$, $(\zeta_2, \zeta_1; \zeta_3, \zeta_4)$, $(\zeta_4, \zeta_3; \zeta_1, \zeta_2)$, $(\zeta_3, \zeta_4; \zeta_2, \zeta_1)$ *are all equal to* $\dfrac{1}{\rho}$*.*
- $(\zeta_1, \zeta_3; \zeta_2, \zeta_4)$, $(\zeta_3, \zeta_1; \zeta_4, \zeta_2)$, $(\zeta_2, \zeta_4; \zeta_1, \zeta_3)$, $(\zeta_4, \zeta_2; \zeta_3, \zeta_1)$ *are all equal to* $1 - \rho$*;*
- $(\zeta_1, \zeta_4; \zeta_2, \zeta_3)$, $(\zeta_4, \zeta_1; \zeta_3, \zeta_2)$, $(\zeta_2, \zeta_3; \zeta_1, \zeta_4)$, $(\zeta_3, \zeta_2; \zeta_4, \zeta_1)$ *are all equal to* $\dfrac{\rho - 1}{\rho}$*;*

- $(\zeta_1, \zeta_3; \zeta_4, \zeta_2)$, $(\zeta_3, \zeta_1; \zeta_2, \zeta_4)$, $(\zeta_4, \zeta_2; \zeta_1, \zeta_3)$, $(\zeta_2, \zeta_4; \zeta_3, \zeta_1)$ are all equal to $\dfrac{1}{1-\rho}$;
- $(\zeta_1, \zeta_4; \zeta_3, \zeta_2)$, $(\zeta_4, \zeta_1; \zeta_2, \zeta_3)$, $(\zeta_3, \zeta_2; \zeta_1, \zeta_4)$, $(\zeta_2, \zeta_3; \zeta_4, \zeta_1)$ are all equal to $\dfrac{\rho}{\rho-1}$;

Proof

Let z_1, z_2, z_3 and z_4 be distinct complex numbers. Then inspection of the expression determining cross-ratios involving these four complex numbers establishes that

$$(z_1, z_2; z_3, z_4) = (z_2, z_1; z_4, z_3) = (z_3, z_4; z_1, z_2) = (z_4, z_3; z_2, z_1)$$

when z_1, z_2, z_3 and z_4 are distinct complex numbers. Further inspection shows that

$$\begin{aligned}(\infty, z_2; z_3, z_4) &= (z_2, \infty; z_4, z_3) = (z_3, z_4; \infty, z_2) = (z_4, z_3; z_2, \infty) \\ &= \frac{z_4 - z_2}{z_3 - z_2}\end{aligned}$$

Making appropriate substitutions in these identities, we find also that

$$\begin{aligned}(z_1, \infty; z_3, z_4) &= (\infty, z_1; z_4, z_3) = (z_3, z_4; z_1, \infty) = (z_4, z_3; \infty, z_1) \\ &= \frac{z_3 - z_1}{z_4 - z_1}, \\ (z_1, z_2; \infty, z_4) &= (z_2, z_1; z_4, \infty) = (\infty, z_4; z_1, z_2) = (z_4, \infty; z_2, z_1) \\ &= \frac{z_2 - z_4}{z_1 - z_4}\end{aligned}$$

and

$$\begin{aligned}(z_1, z_2; z_3, \infty) &= (z_2, z_1; \infty, z_3) = (z_3, \infty; z_1, z_2) = (\infty, z_3; z_2, z_1) \\ &= \frac{z_1 - z_3}{z_2 - z_3}.\end{aligned}$$

These verifications establish that, given any four distinct elements ζ_1, ζ_2, ζ_3 and ζ_4 of the Riemann sphere, the cross-ratios $(\zeta_1, \zeta_2; \zeta_3, \zeta_4)$, $(\zeta_2, \zeta_1; \zeta_4, \zeta_3)$, $(\zeta_3, \zeta_4; \zeta_1, \zeta_2)$, $(\zeta_4, \zeta_3; \zeta_2, \zeta_1)$ are all equal to one another.

Next let z_1, z_2, z_3 and z_4 be distinct complex numbers. Then

$$(z_1, z_2; z_4, z_3) = \frac{(z_4 - z_1)(z_3 - z_2)}{(z_4 - z_2)(z_3 - z_1)} = \frac{1}{(z_1, z_2; z_3, z_4)}.$$

Also

$$\begin{aligned}
(\infty, z_2; z_4, z_3) &= \frac{z_3 - z_2}{z_4 - z_2} = \frac{1}{(\infty, z_2; z_3, z_4)}, \\
(z_1, \infty; z_4, z_3) &= \frac{z_4 - z_1}{z_3 - z_1} = \frac{1}{(z_1, \infty; z_3, z_4)}, \\
(z_1, z_2; z_4, \infty) &= \frac{z_1 - z_4}{z_2 - z_4} = \frac{1}{(z_1, z_2; \infty, z_4)}, \\
(z_1, z_2; \infty, z_3) &= \frac{z_2 - z_3}{z_1 - z_3} = \frac{1}{(z_1, z_2; z_3, \infty)}.
\end{aligned}$$

It follows from these identities that

$$(\zeta_1, \zeta_2; \zeta_4, \zeta_3) = \frac{1}{(\zeta_1, \zeta_2; \zeta_3, \zeta_4)}$$

for all distinct elements ζ_1, ζ_2, ζ_3 and ζ_4 of the Riemann sphere $\mathbb{P}^1$.

Next note that

$$\begin{aligned}(z_1, z_3; z_2, z_4) &= \frac{(z_2 - z_1)(z_4 - z_3)}{(z_2 - z_3)(z_4 - z_1)} \\ &= \frac{z_4 - z_3}{z_4 - z_1} + \frac{z_3 - z_1}{z_4 - z_1} + \frac{(z_3 - z_1)(z_4 - z_2)}{(z_2 - z_3)(z_4 - z_1)} \\ &= 1 - \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} \\ &= 1 - (z_1, z_2; z_3, z_4)\end{aligned}$$

whenever z_1, z_2, z_3 and z_4 are distinct complex numbers. Also

$$\begin{aligned}(\infty, z_3; z_2, z_4) &= \frac{z_4 - z_3}{z_2 - z_3} = 1 + \frac{z_4 - z_2}{z_2 - z_3} = 1 - \frac{z_4 - z_2}{z_3 - z_2} \\ &= 1 - (\infty, z_2; z_3, z_4),\end{aligned}$$

$$\begin{aligned}
(z_1, z_3; \infty, z_4) &= \frac{z_3 - z_4}{z_1 - z_4} = 1 + \frac{z_3 - z_1}{z_1 - z_4} = 1 - \frac{z_3 - z_1}{z_4 - z_1} \\
&= 1 - (z_1, \infty; z_3, z_4), \\
(z_1, \infty; z_2, z_4) &= \frac{z_2 - z_1}{z_4 - z_1} = 1 + \frac{z_2 - z_4}{z_4 - z_1} = 1 - \frac{z_2 - z_4}{z_1 - z_4} \\
&= 1 - (z_1, z_2; \infty, z_4), \\
(z_1, z_3; z_2, \infty) &= \frac{z_1 - z_2}{z_3 - z_2} = 1 + \frac{z_1 - z_3}{z_3 - z_2} = 1 - \frac{z_1 - z_3}{z_2 - z_3} \\
&= 1 - (z_1, z_2; z_3, \infty).
\end{aligned}$$

It follows from these identities that

$$(\zeta_1, \zeta_3; \zeta_2, \zeta_4) = 1 - (\zeta_1, \zeta_2; \zeta_3, \zeta_4)$$

for all distinct elements ζ_1, ζ_2, ζ_3 and ζ_4 of the Riemann sphere $\mathbb{P}^1$.

The remaining identities follow by repeated substitution in those already established. Indeed let ζ_1, ζ_2, ζ_3 and ζ_4 be distinct elements of the Riemann sphere, and let $\rho = (\zeta_1, \zeta_2; \zeta_3, \zeta_4)$. We have established that

$$(\zeta_2, \zeta_1; \zeta_4, \zeta_3) = (\zeta_3, \zeta_4; \zeta_1, \zeta_2) = (\zeta_4, \zeta_3; \zeta_2, \zeta_1) = \rho,$$

$$(\zeta_1, \zeta_2, \zeta_4, \zeta_3) = \frac{1}{\rho},$$

$$(\zeta_1, \zeta_3, \zeta_2, \zeta_4) = 1 - \rho,$$

Straightforward substitutions in these identities yield the identities

$$(\zeta_2, \zeta_1, \zeta_3, \zeta_4) = (\zeta_4, \zeta_3, \zeta_1, \zeta_2) = (\zeta_3, \zeta_4, \zeta_2, \zeta_1) = \frac{1}{\rho}$$

and

$$(\zeta_3, \zeta_1, \zeta_4, \zeta_2) = (\zeta_2, \zeta_4, \zeta_1, \zeta_3) = (\zeta_4, \zeta_2, \zeta_3, \zeta_1) = 1 - \rho.$$

Furthermore

$$\begin{aligned}(\zeta_1, \zeta_4; \zeta_2, \zeta_3) &= 1 - (\zeta_1, \zeta_2; \zeta_4, \zeta_3) \\ &= 1 - \frac{1}{\rho} = \frac{\rho - 1}{\rho},\end{aligned}$$

and therefore

$$(\zeta_4, \zeta_1, \zeta_3, \zeta_2) = (\zeta_2, \zeta_3, \zeta_1, \zeta_4) = (\zeta_3, \zeta_2, \zeta_4, \zeta_1) = \frac{\rho - 1}{\rho}.$$

Also

$$\begin{aligned}(\zeta_1, \zeta_3; \zeta_4, \zeta_2) &= \frac{1}{(\zeta_1, \zeta_3; \zeta_2, \zeta_4)} = \frac{1}{1 - (\zeta_1, \zeta_2; \zeta_3, \zeta_4)} \\ &= \frac{1}{1 - \rho},\end{aligned}$$

and therefore

$$(\zeta_3, \zeta_1; \zeta_2, \zeta_4) = (\zeta_4, \zeta_2, \zeta_1, \zeta_3) = (\zeta_2, \zeta_4, \zeta_3, \zeta_1) = \frac{1}{1 - \rho}.$$

Finally

$$(\zeta_1, \zeta_4; \zeta_3, \zeta_2) = 1 - (\zeta_1, \zeta_3; \zeta_4, \zeta_2) = 1 - \frac{1}{1-\rho} = \frac{\rho}{\rho - 1},$$

and therefore

$$(\zeta_4, \zeta_1; \zeta_2, \zeta_3) = (\zeta_3, \zeta_2; \zeta_1, \zeta_4) = (\zeta_2, \zeta_3; \zeta_4, \zeta_1) = \frac{\rho}{\rho - 1}.$$

All the required identities have now been verified. ■

Lemma 2.8

Let ζ_1, ζ_2, ζ_3 and ζ_4 be distinct elements of the Riemann sphere $\mathbb{P}^1$. Then $(\zeta_1, \zeta_2; \zeta_3, \zeta_4) \neq 1$.

Proof

Let z_1, z_2, z_3 and z_4 be complex numbers. Then

$$\begin{aligned} &(z_3 - z_1)(z_4 - z_2) - (z_3 - z_2)(z_4 - z_1) \\ &\qquad = -z_1 z_4 - z_3 z_2 + z_3 z_1 + z_2 z_4 \\ &\qquad = (z_2 - z_1) z_4 + (z_1 - z_2) z_3 \\ &\qquad = (z_2 - z_1)(z_4 - z_3). \end{aligned}$$

It follows that if z_1, z_2, z_3 and z_4 are distinct then

$$(z_3 - z_1)(z_4 - z_2) \neq (z_3 - z_2)(z_4 - z_1),$$

and therefore $(z_1, z_2; z_3, z_4) \neq 1$. Furthermore, examination of the expressions defining $(\zeta_1, \zeta_2; \zeta_3, \zeta_4)$ in cases where exactly one of ζ_1, ζ_2, ζ_3 and ζ_4 is equal to ∞ and the rest are distinct complex numbers shows that $(\zeta_1, \zeta_2; \zeta_3, \zeta_4) \neq 1$ in these cases also. The result follows. ∎

Proposition 2.9

Let ζ_1, ζ_2 and ζ_3 be distinct elements of the Riemann sphere, and let w be a complex number distinct from 0 and 1. Then there exists a unique element ζ_4 of the Riemann sphere distinct from ζ_1, ζ_2 and ζ_3 for which $(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = w$.

Proof

First suppose that $\zeta_1 = z_1$, $\zeta_2 = z_2$ and $\zeta_3 = z_3$, where z_1, z_2 and z_3 are distinct complex numbers. In the case where

$$\frac{z_1 - z_3}{z_2 - z_3} = w,$$

the identity $(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = w$ is satisfied when $\zeta_4 = \infty$.

Moreover, in this case,

$$(z_1, z_2, z_3, z_4) = w \, \frac{z_2 - z_4}{z_1 - z_4}$$

and $z_1 \neq z_2$, and consequently there is no complex number z_4 for which $(z_1, z_2; z_3, z_4) = w$. Therefore, in the case under consideration, the identity $(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = w$ is satisfied if and only if $\zeta_4 = \infty$.

Now suppose that

$$\frac{z_1 - z_3}{z_2 - z_3} \neq w.$$

Then the equation $(z_1, z_2; z_3, \zeta_4) = w$ is not satisfied when $\zeta_4 = \infty$, and we must therefore show the existence of a unique complex number z_4 for which $(z_1, z_2; z_3, z_4) = w$. We must therefore determine a complex number z_4 for which

$$\frac{z_4 - z_2}{z_4 - z_1} = k,$$

$$k = w\,\frac{z_3 - z_2}{z_3 - z_1}.$$

Note that the value of k is then distinct from both 0 and 1 in the case we are considering.

Now the required equation is satisfied if and only if $z_4 - z_2 = kz_4 - kz_1$. The equation $(z_1, z_2; z_3, z_4)$ is therefore satisfied if and only if

$$z_4 = \frac{z_2 - kz_1}{1 - k}.$$

Moreover, when this equation is satisfied, z_4 cannot be equal to z_1, because $z_1 - z_2 \neq 0$. Also z_4 cannot be equal to z_2, because $k(z_2 - z_1) \neq 0$. And z_4 cannot be equal to z_3 because

$$z_3 - z_2 \neq k(z_3 - z_1).$$

The result follows. ∎

We have defined the value of the cross-ratio $(\zeta_1, \zeta_2; \zeta_3, \zeta_4)$ in all cases where ζ_1, ζ_2, ζ_3 and ζ_4 are distinct elements of the Riemann sphere. This cross-ratio itself may be regarded as an element of the Riemann sphere. The value of the cross-ratio $(\zeta_1, \zeta_2; \zeta_3, \zeta_4)$ cannot equal 0, 1 or ∞ in any case where ζ_1, ζ_2, ζ_3 and ζ_4 are distinct. The definition of cross-ratio can be extended in a natural fashion to define $(\zeta_1, \zeta_2; \zeta_3, \zeta_4)$ in cases when exactly two of the elements $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ of the Riemann sphere coincide with one another and are distinct from the other elements in the list, those other elements also being distinct from one another. Moreover, in view of the identities established in Proposition 2.7, it suffices to determine the appropriate value of the cross-ratio $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ in cases where ζ_1, ζ_2 and ζ_3 are distinct from one another and ζ_4 coincides with ζ_1, ζ_2 or ζ_3.

So now let z_1, z_2 and z_3 be distinct complex numbers. If z is a complex number distinct from z_1, z_2 and z_3 then

$$(z_1, z_2; z_3, z) = \frac{(z_3 - z_1)(z - z_2)}{(z_3 - z_2)(z - z_1)}.$$

We should ensure, if possible, that the cross-ratio $(z_1, z_2; z_3, z)$ is a continuous function of z on the Riemann sphere. Accordingly we should define

$$(z_1, z_2; z_3, z_2) = 0, \quad (z_1, z_2; z_3, z_3) = 1$$

and

$$(z_1, z_2; z_3, z_1) = \infty.$$

The function from the Riemann sphere to itself that sends each element ζ of the Riemann sphere to $(z_1, z_2; z_3, \zeta)$ is then the unique Möbius transformation of the Riemann sphere that sends z_1, z_2 and z_3 to ∞, 0 and 1 respectively.

Next we note that if z is a complex number distinct from z_2 and z_3 then

$$(\infty, z_2; z_3, z) = \frac{z - z_2}{z_3 - z_2}.$$

To ensure continuity of the cross-ratio, we should therefore define

$$(\infty, z_2; z_3, \infty) = \infty, \quad (\infty, z_2; z_3, z_2) = 0$$

and

$$(\infty, z_2; z_3, z_3) = 1.$$

The function from the Riemann sphere to itself that sends each element ζ of the Riemann sphere to $(\infty, z_2; z_3, \zeta)$ is then the unique Möbius transformation of the Riemann sphere that sends ∞, z_2 and z_3 to ∞, 0 and 1 respectively.

In a similar fashion, if z is a complex number distinct from z_1 and z_3 then

$$(z_1, \infty; z_3, z) = \frac{z_3 - z_1}{z - z_1}.$$

To ensure continuity of the cross-ratio, we should therefore define

$$(z_1, \infty; z_3, z_1) = \infty, \quad (z_1, \infty; z_3, \infty) = 0$$

and

$$(z_1, \infty; z_3, z_3) = 1.$$

The function from the Riemann sphere to itself that sends each element ζ of the Riemann sphere to $(z_1, \infty; z_3, \zeta)$ is then the unique Möbius transformation of the Riemann sphere that sends z_1, ∞ and z_3 to ∞, 0 and 1 respectively.

Again, in a similar fashion, if z is a complex number distinct from z_1 and z_2 then

$$(z_1, z_2; \infty, z) = \frac{z_2 - z}{z_1 - z}.$$

To ensure continuity of the cross-ratio, we should therefore define

$$(z_1, z_2; \infty, z_1) = \infty, \quad (z_1, z_2; \infty, z_2) = 0$$

and

$$(z_1, z_2; \infty, \infty) = 1.$$

The function from the Riemann sphere to itself that sends each element ζ of the Riemann sphere to $(z_1, z_2; \infty, \zeta)$ is then the unique Möbius transformation of the Riemann sphere that sends z_1, z_2 and ∞ to ∞, 0 and 1 respectively.

The discussion above has established the truth of the following proposition.

Proposition 2.10

Let ζ_1, ζ_2 and ζ_3 be distinct elements of the Riemann sphere, and let the cross-ratio $(\zeta_1, \zeta_2; \zeta_3, \zeta)$ be defined for all elements ζ of the Riemann sphere in the natural fashion that ensures that this cross-ratio depends continuously on ζ. Then the function from the Riemann sphere to itself that sends each element ζ of the Riemann sphere to $(\zeta_1, \zeta_2; \zeta_3, \zeta)$ is the unique Möbius transformation of the Riemann sphere that sends ζ_1, ζ_2 and ζ_3 to ∞, 0 and 1 respectively.

Corollary 2.11

Given elements ζ_1, ζ_2, ζ_3 and ω of the Riemann sphere, where ζ_1, ζ_2 and ζ_3 are distinct from one another, there exists a unique element ζ_4 of the Riemann sphere for which $(\zeta_1, \zeta_2; \zeta_3, \zeta_4) = \omega$.

Proof

(Corollary 2.3). The result follows immediately on combining the results of Proposition 2.9 and Proposition 2.10. This result is also an immediate consequence of the fact that every Möbius transformation of the Riemann sphere is invertible. ∎

Proposition 2.12

Let $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ be distinct elements of the Riemann sphere $\mathbb{P}^1$, and let $\omega_1, \omega_2, \omega_3, \omega_4$ also be distinct elements of $\mathbb{P}^1$. Then a necessary and sufficient condition for the existence of a Möbius transformation $\mu\colon \mathbb{P}^1 \to \mathbb{P}^1$ of the Riemann sphere with the property that $\mu(\zeta_j) = \omega_j$ for $j = 1, 2, 3, 4$ is that

$$(\zeta_1, \zeta_2; \zeta_3, \zeta_4) = (\omega_1, \omega_2; \omega_3, \omega_4).$$

Proof

Let $\mu_1 \colon \mathbb{P}^1 \to \mathbb{P}^1$ and $\mu_2 \colon \mathbb{P}^1 \to \mathbb{P}^1$ be the functions from the Riemann sphere $\mathbb{P}^1$ to itself defined such that

$$\mu_1(\zeta_1) = \infty, \quad \mu_1(\zeta_2) = 0, \quad \mu_1(\zeta_3) = 1,$$

$$\mu_2(\omega_1) = \infty, \quad \mu_2(\omega_2) = 0, \quad \mu_2(\omega_3) = 1,$$

$$\mu_1(\zeta) = (\zeta_1, \zeta_2; \zeta_3, \zeta) \quad \text{for all } \zeta \in \mathbb{P}^1 \setminus \{\zeta_1, \zeta_2, \zeta_3\},$$

$$\mu_2(\zeta) = (\omega_1, \omega_2; \omega_3, \zeta) \quad \text{for all } \zeta \in \mathbb{P}^1 \setminus \{\omega_1, \omega_2, \omega_3\}.$$

Then $\mu_1 \colon \mathbb{P}^1 \to \mathbb{P}^1$ and $\mu_2 \colon \mathbb{P}^1 \to \mathbb{P}^1$ are Möbius transformations of the Riemann sphere (Proposition 2.10).

Now the composition of any two Möbius transformations of the Riemann sphere $\mathbb{P}^1$ is itself a Möbius transformation of $\mathbb{P}^1$ (Proposition 2.2). Also the inverse of any Möbius transformation of the Riemann sphere is itself a Möbius transformation (Corollary 2.3). Let the function $\mu\colon \mathbb{P}^1 \to \mathbb{P}^1$ be defined so that $\mu = \mu_2^{-1} \circ \mu_1$. Then μ is the function obtained by composing the Möbius transformation μ_1 with the inverse of the Möbius transformation μ_2, and is thus itself a Möbius transformation. Moreover $\mu_1(\zeta) = \mu_2(\mu(\zeta))$ for all elements ζ of the Riemann sphere $\mathbb{P}^1$. In particular

$$\begin{aligned}
\mu_2(\mu(\zeta_1)) &= \mu_1(\zeta_1) = \infty = \mu_2(\omega_1),\\
\mu_2(\mu(\zeta_2)) &= \mu_1(\zeta_2) = 0 = \mu_2(\omega_2),\\
\mu_2(\mu(\zeta_3)) &= \mu_1(\zeta_3) = 1 = \mu_2(\omega_3).
\end{aligned}$$

It follows from the invertibility of the Möbius transformation μ_2 that $\mu(\zeta_j) = \omega_j$ for $i = 1, 2, 3$.

Now μ is the unique Möbius transformation that maps ζ_j to ω_j for $i = 1, 2, 3$. It follows that if there exists a Möbius transformation which sends ζ_j to ω_j for $i = 1, 2, 3, 4$, then this Möbius transformation must coincide with the Möbius transformation μ. Thus there exists a Möbius transfomation mapping ζ_j to ω_j for $j = 1, 2, 3, 4$ if and only if $\mu(\zeta_4) = \omega_4$. Moreover $\mu(\zeta_4) = \omega_4$ if and only if $\mu_2(\mu(\zeta_4)) = \mu_2(\omega_4)$. But

$$\mu_2(\mu(\zeta_4)) = \mu_1(\zeta_4) = (\zeta_1, \zeta_2; \zeta_3, \zeta_4)$$

and

$$\mu_2(\omega_4) = (\omega_1, \omega_2; \omega_3, \omega_4).$$

It follows that there exists a Möbius transfomation mapping ζ_j to ω_j for $j = 1, 2, 3, 4$ if and only if

$$(\zeta_1, \zeta_2; \zeta_3, \zeta_4) = (\omega_1, \omega_2; \omega_3, \omega_4),$$

as claimed. ∎

Proposition 2.13

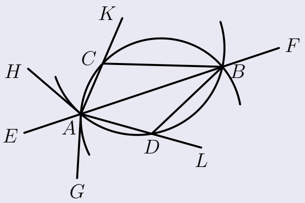
Four distinct complex numbers z_1, z_2, z_3 and z_4 lie on a single line or circle in the complex plane if and only if their cross-ratio $(z_1, z_2; z_3, z_4)$ is a real number.

Proof

Let $\mu\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the Möbius transformation of the Riemann sphere defined such that $\mu(\zeta) = (z_1, z_2; z_3, \zeta)$ for all $\zeta \in \mathbb{P}^1$. Then $\mu(z_1) = \infty$, $\mu(z_2) = 0$ and $\mu(z_3) = 1$. Möbius transformations map lines and circles to lines and circles (Propostion 2.6). It follows that a complex number z distinct from z_1, z_2 and z_3 lies on the circle in the complex plane passing through the points z_1, z_2 and z_3 if and only if $\mu(z)$ lies on the unique line in the complex plane that passes through 0 and 1, in which case $\mu(z)$ is a real number. The result follows. ∎

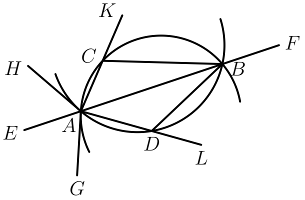
Proposition 2.14

Let a geometrical configuration be as depicted in the accompanying figure. Thus let ACB and ADB be circular arcs that cut at the points A and B. Let the line joining points A and B be produced beyond A and B to E and F respectively. Let AG and AH be tangent to the circular arcs BCA and BDA respectively at A, where C and H lie on one side of AB and D and G lie on the other. Also let the lines AC and AD be produced to K and L respectively. Then the angle GAH is the sum of the angles KCB and LDB.



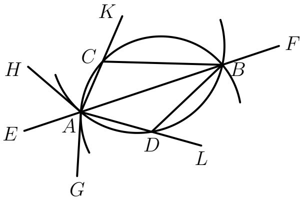
Proof

The angle GAB that the tangent line AG makes with the chord AB of the circle passing through A, C and B is equal to the angle ACB in the alternate segment of that circle (Euclid, *Elements*, III. 32). Similarly the angle HAB that the tangent line AH makes with the chord AB of the circle passing through A, D and B is equal to the angle ADB.



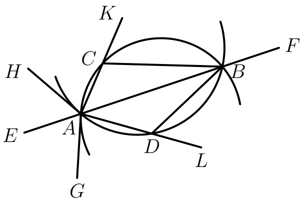
Now the angles EAG and GAB add up to two right angles, as do the angles EAH and HAB. Also the angles KCB and ACB add up to two right angles, as do the angles LDB and ADB.

Therefore the angles EAG and KCB are equal, and the angles EAH and LDB are equal. Moreover the angle GAH is the sum of the angles EAG and EAH. It follows that the angle GAH is equal to the sum of the angles KCB and LDB, as required. ∎

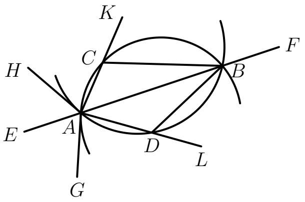


The argument presented in the proof of Proposition 2.14 may be summarized more concisely and symbolically as follows.

$$\begin{aligned} \angle GAB &= \angle ACB \quad \text{(Euclid, \textit{Elements}, III. 32)};\\ \angle HAB &= \angle ADB \quad \text{(Euclid, \textit{Elements}, III. 32)};\\ \angle EAG + \angle GAB &= \text{two right angles};\\ \angle KCB + \angle ACB &= \text{two right angles};\\ \Longrightarrow \angle EAG &= \angle KCB. \end{aligned}$$



$$\begin{aligned} \text{Similarly} \quad \angle EAH &= \angle LDB; \\ \text{Therefore} \quad \angle GAH &= \angle EAG + \angle EAH; \\ &= \angle KCB + \angle LDB, \quad \text{Q.E.D.} \end{aligned}$$



We now summarize some basic properties of the algebra of complex numbers. Any complex number z can be written in the form

$$z = |z| \left(\cos\theta + \sqrt{-1} \sin\theta \right)$$

where $|z|$ is the modulus of z and θ is the angle in radians, measured anticlockwise, between the positive real axis and the line segment whose endpoints are represented by the complex numbers 0 and z. Moreover

$$\frac{1}{\cos\alpha + \sqrt{-1}\,\sin\alpha} = \cos\alpha - \sqrt{-1}\,\sin\alpha$$

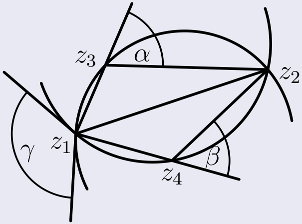
and

$$\begin{aligned}(\cos\alpha + \sqrt{-1}\,\sin\alpha)(\cos\beta + \sqrt{-1}\,\sin\beta) \\ = \quad \cos(\alpha + \beta) + \sqrt{-1}\,\sin(\alpha + \beta)\end{aligned}$$

for all real numbers α and β.

Proposition 2.15

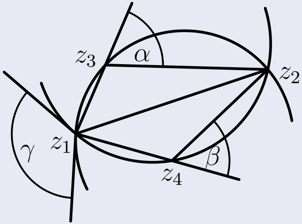
Let two circles in the complex plane intersect at points represented by complex numbers z_1 and z_2, and let points represented by complex numbers z_3 and z_4 be taken on arcs of the respective circles joining z_1 and z_2 so that the point representing z_3 lies on the left hand side of the directed line from z_1 and z_2 and the point represented by the point z_4 lies on the right hand side of that line, as depicted in the accompanying figure.



Then

$$(z_1, z_2; z_3, z_4) = \frac{|z_3 - z_1|\,|z_4 - z_2|}{|z_3 - z_2|\,|z_4 - z_1|}\,(\cos\gamma + \sqrt{-1}\,\sin\gamma),$$

where γ is the angle between the tangent lines to the two circles at the intersection point represented by the complex number z_1.



Proof

The configuration of the points z_1, z_2, z_3 and z_4 ensures that direction of the line from z_1 to z_3 is transformed into the direction of the line from z_3 to z_2 by rotation clockwise through an angle α less than two right angles. Similarly the direction of the line from z_1 to z_4 is transformed into the direction of the line from z_4 to z_2 by rotation anticlockwise through an angle β less than two right angles. Basic properties of complex numbers therefore ensure that

$$\begin{aligned} \frac{z_2 - z_3}{z_3 - z_1} &= \frac{|z_2 - z_3|}{|z_3 - z_1|}(\cos\alpha - \sqrt{-1}\,\sin\alpha). \\ \frac{z_2 - z_4}{z_4 - z_1} &= \frac{|z_2 - z_4|}{|z_4 - z_1|}(\cos\beta + \sqrt{-1}\,\sin\beta). \end{aligned}$$

Now

$$\begin{aligned} & \frac{\cos\beta + \sqrt{-1}\,\sin\beta}{\cos\alpha - \sqrt{-1}\,\sin\alpha} \\ & \quad = \quad (\cos\alpha + \sqrt{-1}\,\sin\alpha)(\cos\beta + \sqrt{-1}\,\sin\beta) \\ & \quad = \quad \cos(\alpha+\beta) + \sqrt{-1}\,\sin(\alpha+\beta). \end{aligned}$$

Moreover the geometry of the configuration ensures that $\alpha + \beta = \gamma$ (Proposition 2.14). Thus

$$\begin{aligned} & \frac{z_2 - z_4}{z_4 - z_1} \times \frac{z_3 - z_1}{z_2 - z_3} \\ & \quad = \quad \frac{|z_2 - z_4|\,|z_3 - z_1|}{|z_4 - z_1||z_2 - z_3|}\,(\cos\gamma + \sqrt{-1}\sin\gamma). \end{aligned}$$

2. Möbius Transformations of the Riemann Sphere (continued)

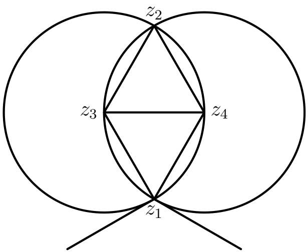
But

$$\frac{z_2 - z_4}{z_4 - z_1} \times \frac{z_3 - z_1}{z_2 - z_3} = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} = (z_1, z_2; z_2, z_4).$$

The result follows. $\blacksquare$

Example

The circles in the complex plane of radius 2 centred on -1 and 1 intersect at the points $\pm\sqrt{3}\,i$, where $i = \sqrt{-1}$. In this situation, take $z_1 = -\sqrt{3}\,i$, $z_2 = \sqrt{3}\,i$, $z_3 = -1$ and $z_4 = 1$.



Then

$$\begin{aligned}(z_1, z_2; z_3, z_4) &= \frac{(-1+\sqrt{3}\,i)(1-\sqrt{3}\,i)}{(-1-\sqrt{3}\,i)(1+\sqrt{3}\,i)} = \frac{2+2\sqrt{3}\,i}{2-2\sqrt{3}i} \\ &= \frac{(2+2\sqrt{3}\,i)^2}{(2-2\sqrt{3}i)(2+2\sqrt{3}i)} \\ &= \frac{1}{2}(-1+\sqrt{3}\,i)\end{aligned}$$

It follows that $(z_1, z_2; z_3, z_4) = \cos\gamma + \sqrt{-1}\,\sin\gamma$, where $\gamma = \frac{2}{3}\pi$. Thus the angle between the tangent lines to the circles at the intersection point z_1 is thus $\frac{4}{3}$ of a right angle. This is what one would expect from the basic geometry of the configuration, given that the triangle with vertices z_1, z_3 and z_4 is equilateral and the tangent lines to the circles are perpendicular to the lines joining the point of intersection to the centres of those circles.

Proposition 2.16

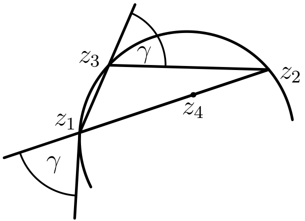
Let z_1 and z_2 be complex numbers representing the endpoints of a circular arc in the complex plane. Also, in the case where the circular arc lies on the left hand side of the directed line from z_1 to z_2, let points z_3 and z_4 be taken between z_1 and z_2 on the circular arc and the straight line segement respectively, and, in the case where the circular arc lies on the right hand side of the directed line from z_1 to z_2, let points z_3 and z_4 be taken between z_1 and z_2 on the straight line segment and the the circular arc respectively. Then

$$(z_1, z_2; z_3, z_4) = \frac{|z_3 - z_1|\,|z_4 - z_2|}{|z_3 - z_2|\,|z_4 - z_1|}\,(\cos\gamma + \sqrt{-1}\,\sin\gamma),$$

where γ is the angle between the tangent line to the circle at the intersection point represented by the complex number z_1 and the line obtained by producing the chord joining z_2 and z_1 beyond z_1.

Proof

We consider the configuration in which the circular arc lies on the left hand side of the directed line from z_1 to z_2. In that case the configuration is as depicted in the accompanying figure.



In this configuration the angle made at z_3 by the lines from z_1 and z_2 is equal to the angle between the chord from z_1 to z_2 and the depicted tangent line. The complements of those angles are then also equal to one another; these equal complements have been labelled γ in the figure.

Also the direction of the line from z_3 to z_2 is obtained from the direction of the line from z_1 to z_3 by rotation clockwise through an angle γ less than two right angles. It follows that

$$\frac{z_2 - z_3}{z_3 - z_1} = \frac{|z_2 - z_3|}{|z_3 - z_1|} (\cos\gamma - \sqrt{-1}\,\sin\gamma).$$

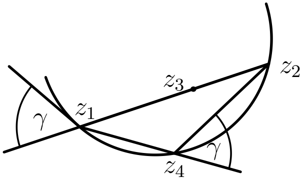
Also the direction of $z_2 - z_4$ is the same as that of $z_4 - z_1$, and therefore

$$\frac{z_2 - z_4}{z_4 - z_1} = \frac{|z_2 - z_4|}{|z_4 - z_1|}.$$

It follows that

$$\begin{aligned}(z_1, z_2; z_3, z_4) &= \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} \\ &= \frac{z_2 - z_4}{z_4 - z_1} \times \frac{z_3 - z_1}{z_2 - z_3} \\ &= \frac{|z_3 - z_1|\,|z_4 - z_2|}{|z_3 - z_2|\,|z_4 - z_1|} (\cos\gamma + \sqrt{-1}\,\sin\gamma).\end{aligned}$$

We consider now the case in which the circular arc from z_1 to z_2 lies on the right hand side of the directed line from z_1 to z_2. In this case the complex numbers z_3 and z_4 represent points between z_1 and z_2 on the line and the circular arc respectively, as depicted in the following figure.



In this configuration, the angle sought is the angle γ, which in this case is equal both to the angle between the depicted tangent line to the circle at z_1 and the line that produces the chord joining z_2 to z_1 beyond z_1.

Moreover, in this case

$$\frac{z_2 - z_4}{z_4 - z_1} = \frac{|z_2 - z_4|}{|z_4 - z_1|} (\cos\gamma + \sqrt{-1}\,\sin\gamma)$$

and

$$\frac{z_2 - z_3}{z_3 - z_1} = \frac{|z_2 - z_3|}{|z_3 - z_1|}.$$

It follows in this case also that

$$\begin{aligned}
(z_1, z_2; z_3, z_4) &= \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} \\
&= \frac{z_2 - z_4}{z_4 - z_1} \times \frac{z_3 - z_1}{z_2 - z_3} \\
&= \frac{|z_3 - z_1|\,|z_4 - z_2|}{|z_3 - z_2|\,|z_4 - z_1|} (\cos\gamma + \sqrt{-1}\,\sin\gamma).
\end{aligned}$$

This completes the proof. ∎

Proposition 2.17

Let two lines in the complex plane intersect at at point represented by the complex number z_1, and let points represented by z_3 and z_4 be taken distinct from z_1, one on each of the two lines, where these points are labelled so that the direction of $z_3 - z_1$ is obtained from the direction of $z_4 - z_1$ by rotation anticlockwise through an angle γ less than two right angles. Then

$$(z_1, \infty; z_3, z_4) = \frac{|z_3 - z_1|}{|z_4 - z_1|} (\cos\gamma + \sqrt{-1}\,\sin\gamma).$$

Proof

The cross-ratio in this situation is defined so that

$$(z_1, \infty; z_3, z_4) = \frac{z_3 - z_1}{z_4 - z_1}.$$

Furthermore

$$\frac{z_3 - z_1}{z_4 - z_1} = \frac{|z_3 - z_1|}{|z_4 - z_1|} (\cos\gamma + \sqrt{-1}\,\sin\gamma).$$

The result follows directly. ∎

Lines in the complex plane correspond to circles on the Riemann sphere that pass through the point at infinity. With that in mind, it can seen that Propositions 2.15, 2.16 and 2.17 conform to a common pattern, and show that, where two curves intersect at a point, each of those curves being either a circle or a straight line, the angle between the tangent lines to those curves at the point of intersection may be expressed in terms of the argument of an appropriate cross-ratio.

Indeed, to determine the angle the tangent lines to two circles on the Riemann sphere at a point ζ_1 where they intersect, one can determine the other point of intersection ζ_2, a point ζ_3 on one circular arc between ζ_1 to ζ_2, and a point ζ_4 on the other circular arc between ζ_1 and ζ_2. A positive real number R and a real number γ satisfying $-\pi < \gamma < \pi$ can then be determined so that

$$(\zeta_1, \zeta_2; \zeta_3, \zeta_4) = R(\cos\gamma + \sqrt{-1}\,\sin\gamma).$$

Then the angle between the tangent lines to those circles at the point ζ_1 of intersection, measured in radians, is then the absolute value $|\gamma|$ of γ.

Proposition 2.18

Möbius transformations of the Riemann sphere $\mathbb{P}^1$ are angle-preserving. Thus if two circles on the Riemann sphere intersect at a point ζ of the Riemann sphere, and if a Möbius transformation μ maps ζ to a point ω of the Riemann sphere, then the angle between the tangent lines to the original circles at the point ζ is equal to the angle between the tangent lines to the corresponding circles at the point ω, the corresponding circles being the images of the original circles under the Möbius transformation.

Proof

The angle between the tangent lines to the original circles at ζ is determined by the value of a cross ratio of the form $(\zeta_1, \zeta_2; \zeta_3, \zeta_4)$, where ζ_1 and ζ_2 are the points of intersection of the original circles, and ζ_3 and ζ_4 lie on the circular arcs joining ζ_1 to ζ_2, with ζ_4 on the right hand side as the circle through ζ_3 is traversed in the direction from ζ_1 through ζ_3 to ζ_2. The angle between the tangent lines to the corresponding circles at ω is determined in the analogous fashion by the value of the cross ratio $(\omega_1, \omega_2; \omega_3, \omega_4)$, where ω_j is the image of ζ_j under the Möbius transformation sending the original circles to the corresponding circles. Proposition 2.12 ensures that $(\zeta_1, \zeta_2; \zeta_3, \zeta_4) = (\omega_1, \omega_2; \omega_3, \omega_4)$. The result follows. ∎