Study Note—Euclid's *Elements*, Book I, Proposition 47

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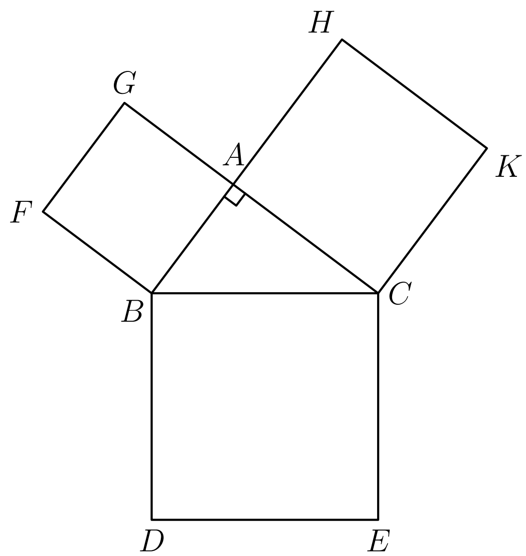
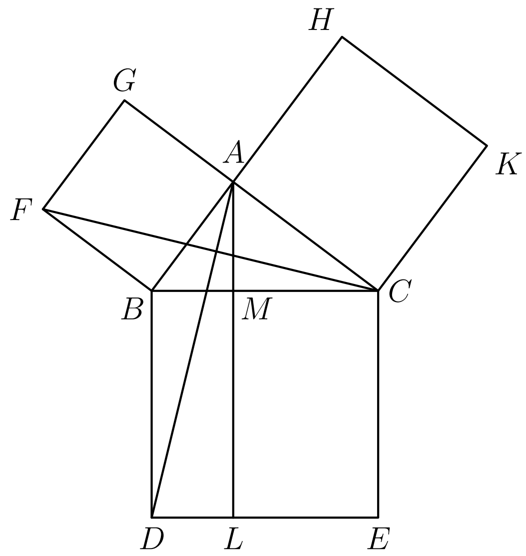
A Free Presentation of the Proof of Proposition 47 in Book I of Euclid's *Elements of Geometry*

Let ABC be a right-angled triangle, with the right angle located at the vertex A, and let squares $BDEC$, $HKCA$ and $GFBA$ be constructed on the sides BC, CA and AB respectively.



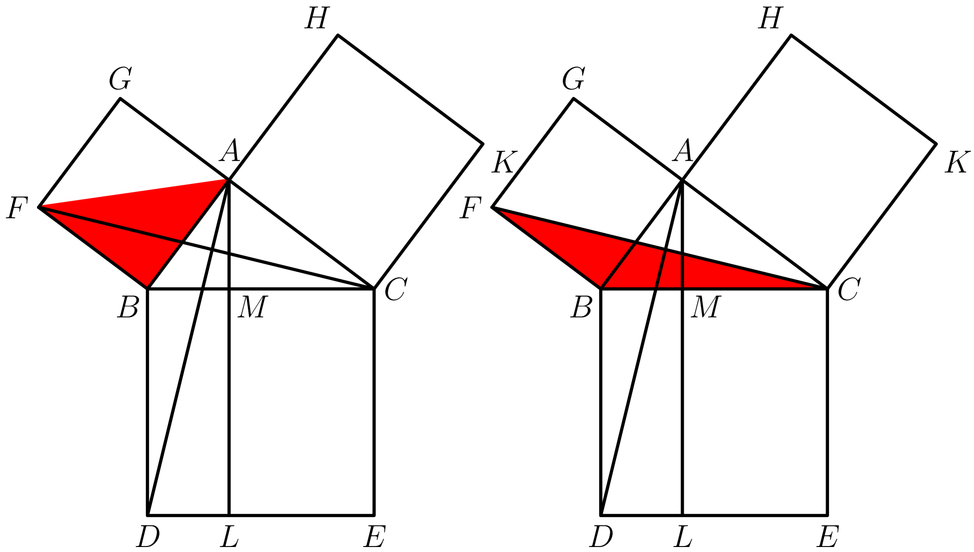
(The squares $GFBA$ and $HKCA$ are identified in Euclid's *Elements of Geometry* with the designations GB and HC respectively: in order to identify a square or rectangle on a geometric diagram, it suffices to list two diagonally opposite vertices of the square or rectangle.)

Let a straight line AL be drawn from the point A to the line DE, so as to ensure that AL is parallel to the sides BD and CE of the square on BC, and meets that side DE of that square at the point L. Also let CF and AD be joined. Let M denote the point at which the straight line segments BC and AL intersect. (The point M is not labelled or referred to in Euclid's proof, but has been labelled here of the purposes of the following discussion.)

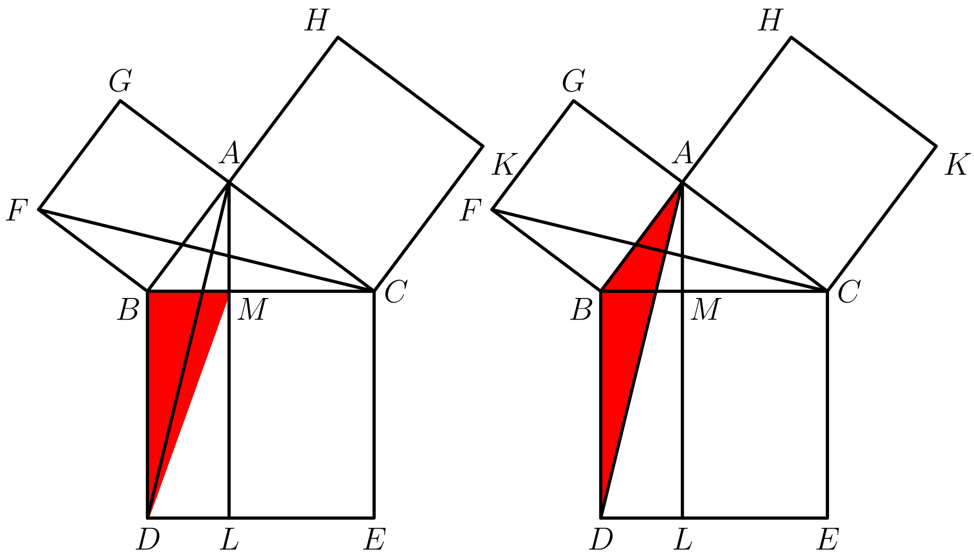


Now the angles FBA, CAB and BAG are right angles. Consequently it follows (on applying Propositions I.14 of Euclid's *Elements*) that the straight line segments CA and AG together constitute a straight line CG with end-points C and G. Similarly the line segments BA and AH together constitute a straight line BH. As Euclid says: "CA is in a straight line with AG. For the same reason BA is also in a straight line with AH." Moreover CG is parallel to BF (by Proposition I.27 of Euclid's *Elements*), and similarly BH is parallel to CK.

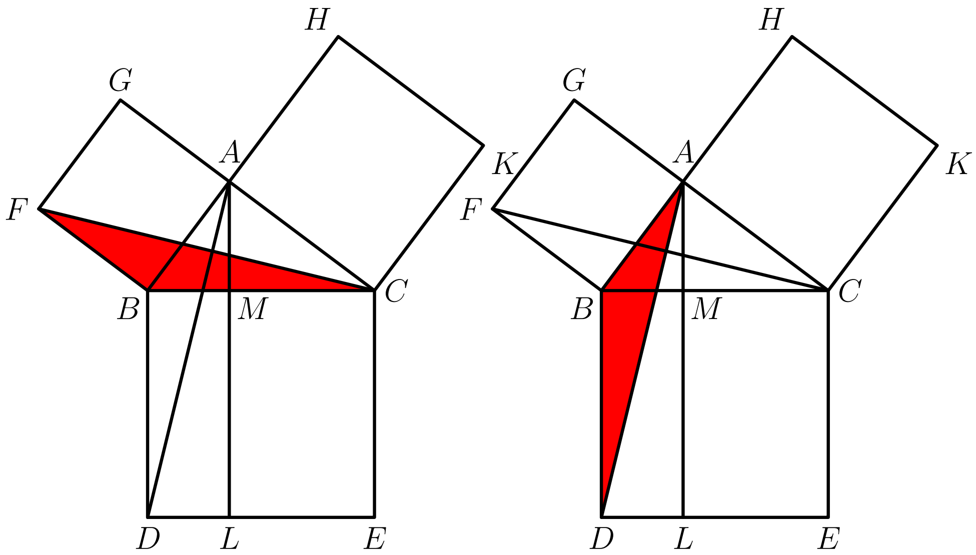
Next we note (departing from the strict ordering of the statements in Euclid's proof) that triangles FBA and FBC are on the same base BF, and between the same parallels FB and GC. Moreover the square $GFBA$ (or GB) is double the triangle FBA in area. Consequently the square $GFBA$ is double the triangle FBC in area.



Similarly the triangles MBD and ABD are on the same base BD, and between the same parallels BD and AL. Moreover the rectangle $MBDL$ (or BL) is double the triangle MBD in area. Consequently the rectangle $MBDL$ is double the triangle ABD in area.



We now compare the triangles FBC and ABD. The angles FBC and ABD are each formed by adding a right angle to to the angle ABC. It follows that the angles FBC and ABD are equal to one another. Also the sides FB and BC are respectively equal to the sides AB and BD. Applying the SAS Congruence Rule (Euclid's *Elements*, Proposition I.4), we conclude that the triangles FBC and ABD are congruent to one another, and are therefore equal to one another in area.



(Note that a clockwise rotation through a right angle about the point B would bring the triangle FBC onto the triangle ABD. Consequently those two triangles should be equal in area, and equality of area, in the theory developed in Book I of Euclid's *Elements of Geometry*, is formally justified, in this situation, and in analogous situations, as a consequence of the SAS Congruence Rule established in Proposition 4 of that book.)

But the square $GFBA$ has been shown to be double the triangle FBC in area, and also the rectangle $MBDL$ has been shown to be double the triangle ABD in area. We have just noted that the triangles FBC and ABD are equal to one another in area. It follows that the rectangle $MBDL$ (or BL) is equal in area to the square $GFBA$ (or GB).

A similar argument shows that the rectangle $MCEL$ (or CL) is equal in area to the square $HKCA$ (or HC). Now the square $BDEC$ is equal in area to the sum of the rectangles $MBDL$ and $MCEL$. The square $BDEC$ on the side BC of the right-angled triangle ABC opposite the right angle at the vertex A is consequently equal in area to the sum of the squares $GFBA$ and $HKCA$ on the other two sides of that right-angled triangle, as required.