Study Note—Euclid's *Elements*, Book I, Proposition 40

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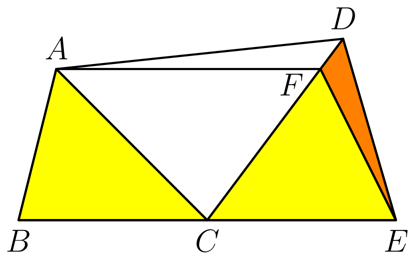
For this proposition, we are given two triangles on equal bases which are equal in area to one another, and where the bases are segments of a given straight line, and where the two triangles lie on the same side of the given straight line. We must prove that the two vertices that do not lie on the given straight line must lie on some other straight line parallel to the given straight line.

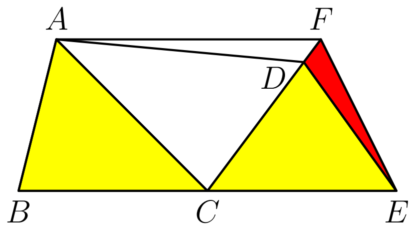
A complete proof covering all possible configurations involving two triangles with equal bases located within a given straight line, and with both triangles lying on the same side of that straight line, would rule out six configurations, only one of which was explicitly discussed by Euclid.

In the discussion that follows we first rule out the configuration, considered by Euclid, in which the bases BC and CE of the triangles ABC and DCE are located on the given straight line, with the point C located between A and E so as to ensure that $AC = CE$, and with the vertices B and D located on opposite sides of the straight line through the vertex A that is parallel to the given straight line BE on which the bases of the triangles lie.



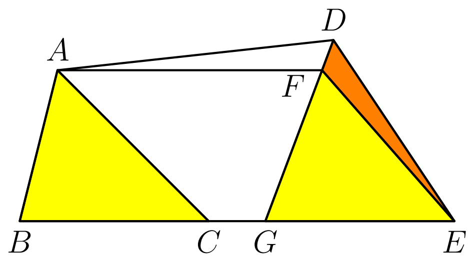
In this configuration, we take F to be the point where the side CD of the triangle DCE intersects the straight line passing through the point A that is parallel to the straight line BE, and we join the point F by straight lines to the vertices A and E. It then follows from Proposition I.36 that the triangles ABC and FCE are equal in area, and consequently the triangle DCE is greater in area than the triangle ABC.

Next we rule out the configuration, in which the bases BC and CE of the triangles ABC and DCE are located on the given straight line, with the point C located between A and E so as to ensure that $AC = CE$, and with the vertices B and D located on the same side of the straight line through the vertex A that is parallel to the given straight line BE on which the bases of the triangles lie.



In this configuration, we take F to be the point where the side CD of the triangle DCE, when produced in a straight line beyond the point D, intersects the straight line passing through the point A that is parallel to the straight line BE, and we join the point F by straight lines to the vertices A and E. It then follows from Proposition I.36 that the triangles ABC and FCE are equal in area, and consequently the triangle DCE is smaller in area than the triangle ABC.

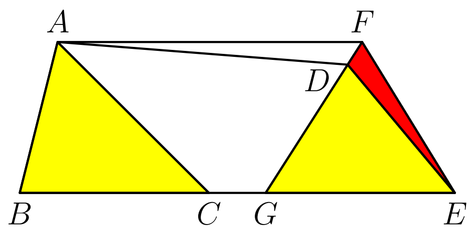
Next we rule out the configuration involving two triangles ABC and DGE with equal bases BC and GE, both contained in the same given straight line, and separate from one another, where the two triangles lie on the same side of the given straight line with points B and D on opposite sides of the the straight line passing through the point A that is parallel to the given straight line.



In this configuration, we take F to be the point where the side GD of the triangle DGE intersects the straight line passing through the point A

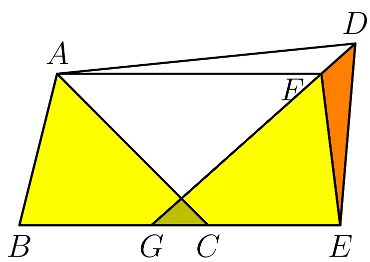
that is parallel to the straight line BE, and we join the point F by straight lines to the vertices A and E. It then follows from Proposition I.36 that the triangles ABC and FGE are equal in area, and consequently the triangle DGE is greater in area than the triangle ABC.

Next we rule out the configuration involving two triangles ABC and DGE with equal bases BC and GE, both contained in the same given straight line, and separate from one another, where the two triangles lie on the same side of the given straight line with points B and D on the same side of the the straight line passing through the point A that is parallel to the given straight line.



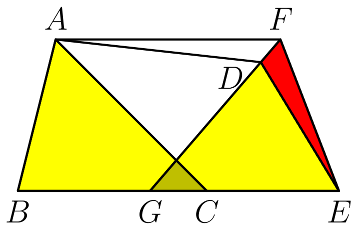
In this configuration, we take F to be the point where the side GD of the triangle DGE, when produced in a straight line beyond the point D, intersects the straight line passing through the point A that is parallel to the straight line BE, and we join the point F by straight lines to the vertices A and E. It then follows from Proposition I.36 that the triangles ABC and FGE are equal in area, and consequently the triangle DGE is smaller in area than the triangle ABC.

Next we rule out the configuration involving two triangles ABC and DGE with equal bases BC and GE, both contained in the same given straight line, and interlocking with one another so that the point G lies between B and C, and where the two triangles lie on the same side of the given straight line with points B and D on opposite sides of the the straight line passing through the point A that is parallel to the given straight line.



In this configuration the triangles ABC and FGE are again equal in area, and consequently the triangle DGE is greater in area than the triangle ABC.

Finally we rule out the configuration involving two triangles ABC and DGE with equal bases BC and GE, both contained in the same given straight line, and interlocking with one another so that the point G lies between B and C, and where the two triangles lie on the same side of the given straight line with points B and D on the same side of the the straight line passing through the point A that is parallel to the given straight line.



In this configuration the triangles ABC and FGE are again equal in area, and consequently the triangle DGE is smaller in area than the triangle ABC.

It follows from the preceding discussion that, in all cases where the two triangles have equal bases, those bases lying within a given straight line, and where both triangles lie on the same side of that straight line, but where the vertices of those triangles not located in the given straight line do not both lie on a single straight line parallel to the given straight line, one of the two triangles is greater in area than the other.

It follows from this that if we are given two triangles with equal bases, those bases lying within a given straight line, and where both triangles lie on the same side of that straight line, and if those two triangles are equal in area, then the vertices of those triangles not located in the given straight line must both lie on a single straight line parallel to the given straight line.