Study Note—Euclid's *Elements*, Book I, Proposition 39

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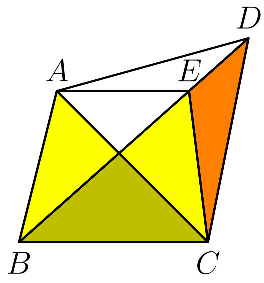
For this proposition, we are given two triangles on a common base which are equal in area to one another, where the two triangles lie on the same side of the common base. We must prove that the two vertices that not endpoints of the common base must lie on some straight line parallel to the common base of the two triangles.

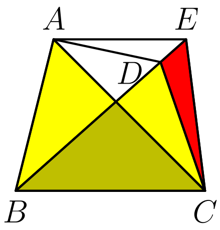
A complete proof covering all possible configurations involving two triangles with a common base located within a given straight line, and with both triangles lying on the same side of that straight line, would rule out two configurations, only one of which was explicitly discussed by Euclid.

In the discussion that follows we first rule out the configuration, considered by Euclid, in which vertices B and D are located on opposite sides of the straight line through the vertex A that is parallel to the common base BC of the two triangles.



In this configuration, we take E to be the point where the side BD of the triangle DBC intersects the straight line passing through the point A that is parallel to the common base BC of the two triangles, and we join the point E by straight lines to the vertices A and C. It then follows from Proposition I.36 that the triangles ABC and EBC are equal in area, and consequently the triangle DBC is greater in area than the triangle ABC.

Next we rule out the configuration in which vertices B and D are located on the same side of the straight line through the vertex A that is parallel to the common base BC of the two triangles.



In this configuration, we take E to be the point where the side BD of the triangle DBC, when produced in a straight line beyond the point D, intersects the straight line passing through the point A that is parallel to the common base BC of the two triangles, and we join the point E by straight lines to the vertices A and C. It then follows from Proposition I.36 that the triangles ABC and EBC are equal in area, and consequently the triangle DBC is smaller in area than the triangle ABC.

It follows from the preceding discussion that, in all cases where the two triangles have a common base, and where both triangles lie on the same side of the common base, but where the vertices of those triangles not located at the endpoints of the common base do not both lie on a single straight line parallel to the common base, one of the two triangles is greater in area than the other.

It follows from this that if we are given two triangles with a common base, where both triangles lie on the same side of the common base, and if those two triangles are equal in area, then the vertices of those triangles not located in at the endpoints of the common base must both lie on a single straight line parallel to the common base.