Study Note—Euclid's *Elements*, Book I, Proposition 35

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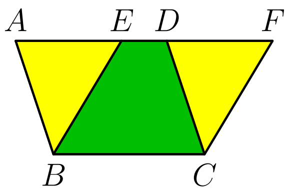
To establish this proposition, we must prove that two parallelograms on the same base and between the same parallels have the same area. The common base of the two parallelograms is a segment of some given infinite straight line, and the sides of the parallelograms opposite the common base are segments of some other given straight line parallel to that which contains the common base of the parallelograms.

Three configurations arise, and need to be considered in order to establish a complete proof of the proposition. The first configuration to be considered is that in which the sides of the two parallelograms that are opposite the common base overlap with one another. The second configuration to be considered is that in which the sides of the two parallelograms that are opposite the common base meet at a common endpoint. The third configuration to be considered, and the only configuration covered by the proof explicitly given by Euclid, is that in which the sides of the two parallelograms that are opposite the common base are disjoint from one another.

We first consider the configuration in which the sides of the parallelograms opposite the common side overlap one another. In this configuration, $ABCD$ and $EBCF$ are parallelograms on the same base BC, and the endpoints of the sides opposite the base are collinear and occur in the order A, E, D and F on the straight line that passes through all four.



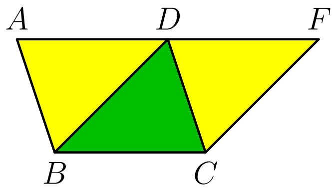
First we show, adapting Euclid's argument in the case considered by him, that the triangles EAB and FDC are congruent. Now opposite sides of parallelograms are equal in length (Euclid, *Elements*, Book I, Proposition 34.) Accordingly $AD = BC = DF$. It follows that

$$AE + ED = AD = EF = ED + DF.$$

Consequently $AE = DF$. Also $AB = DC$, because these straight line segments are opposite sides of the parallelogram $ABCD$. The angles EAB and FDC are equal because they are the corresponding angles formed where the straight line AF intersects the parallel lines AB and DC (Euclid, *Elements*, Book I, Proposition 29). Applying the SAS Congruence Rule (Euclid, *Elements*, Book I, Proposition 4), we find that the triangles EAB and FDC are congruent, and are consequently equal in area.

Adding the trapezium $EBCD$ to the equal triangles EAB and FDC, we conclude (applying Common Notion 2) that the parallelograms $ABCD$ and $EBCF$ are equal in area, as required. This completes the proof of Proposition 35 in which the sides opposite the common side overlap.

We next consider the configuration in which the sides of the parallelograms opposite the common side meet at a common endpoint. In this configuration, $ABCD$ and $DBCF$ are parallelograms on the same base BC, and the endpoints of the sides opposite the base are collinear and occur in the order A, D and F on the straight line that passes through all three.

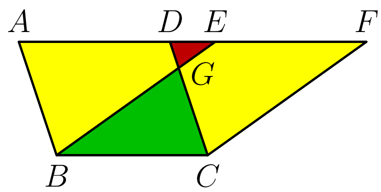


We show, adapting Euclid's argument in the case considered by him, that the triangles DAB and FDC are congruent. Now opposite sides of parallelograms are equal in length (Euclid, *Elements*, Book I, Proposition 34.) Accordingly $AD = BC = DF$. Also $AB = DC$, because these straight line segments are opposite sides of the parallelogram $ABCD$. The angles DAB and FDC are equal because they are the corresponding angles formed where the straight line AF intersects the parallel lines AB and DC (Euclid, *Elements*, Book I, Proposition 29). Applying the SAS Congruence Rule (Euclid, *Elements*, Book I, Proposition 4), we find that the triangles DAB and FDC are congruent, and are consequently equal in area.

Adding the triangle DBC to the equal triangles DAB and FDC, we conclude (applying Common Notion 2) that the parallelograms $ABCD$ and

$DBCF$ are equal in area, as required. This completes the proof of Proposition 35 in which the sides opposite the common side meet at a common endpoint.

Finally we turn our attention the configuration explicitly considered by Euclid. This is the configuration in which the sides of the parallelograms opposite the common side are disjoint. In this configuration, $ABCD$ and $EBCF$ are parallelograms on the same base BC, and the endpoints of the sides opposite the base are collinear and occur in the order A, D, E and F on the straight line that passes through all four.



First we show, following Euclid, that the triangles EAB and FDC are congruent. Now opposite sides of parallelograms are equal in length (Euclid, *Elements*, Book I, Proposition 34.) Accordingly $AD = BC = EF$. Now DE is a part of both AE and DF, and accordingly

$$AE = AD + DE \quad \text{and} \quad DF = DE + EF.$$

Consequently $AE = DF$. Also $AB = DC$, because these straight line segments are opposite sides of the parallelogram $ABCD$. The angles EAB and FDC are equal because they are the corresponding angles formed where the straight line AF intersects the parallel lines AB and DC (Euclid, *Elements*, Book I, Proposition 29). Applying the SAS Congruence Rule (Euclid, *Elements*, Book I, Proposition 4), we find that the triangles EAB and FDC are congruent, and are consequently equal in area.

Subtracting the triangle DGE from the equal triangles EAB and FDC, we conclude (applying Common Notion 3) that the trapezia $ABGD$ and $EGCF$ are equal in area. Then adding the triangle GBC to these equal trapezia, we conclude (applying Common Notion 2) that the parallelograms $ABCD$ and $EBCF$ are equal in area, as required. This completes the proof of Proposition 35 in the configuration considered by Euclid.