Study Note—Euclid's *Elements*, Book I, Proposition 32

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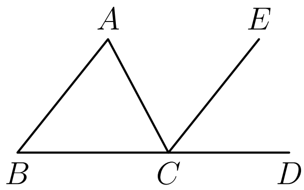
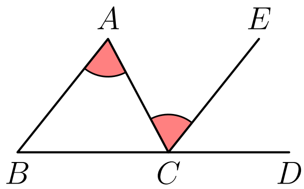
For this proposition we are required to show that the internal angles of a given triangle ABC add up to two right angles. Moreover each external angle of the triangle is equal to the sum of the two opposite (or remote) internal angles.

The construction underlying Euclid's proof requires the side BC of that triangle to be produced in a straight line beyond the point C to some point D, and also requires a line segment to be constructed (as permitted by Proposition I.31), parallel to the side AB of the triangle, with one endpoint located at the vertex C of the triangle, and with the other endpoint E located on the same side of BC as the vertex A of the triangle, as depicted in the following figure.



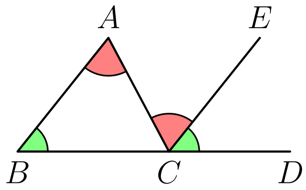
The requirement that the lines AB and EC be parallel ensures (by Proposition I.29) that the resulting alternate angles at the vertices A and C are equal. Accordingly

$$\angle BAC = \angle ACE.$$



The requirement that the lines AB and EC be parallel also ensures (again applying Proposition I.29), that the corresponding angles at the vertices B and C are equal, and accordingly

$$\angle ABC = \angle ECD.$$



It follows that

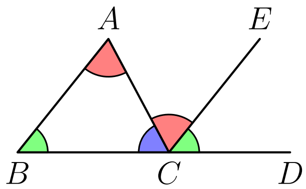
$$\angle BAC + \angle ABC = \angle ACE + \angle ECD = \angle ACD.$$

Thus the external angle ACD of the triangle is equal to the sum of the two opposite (or remote) angles BAC and ABC.

If we now add in the angle ACB, we find that

$$\angle BAC + \angle ABC + \angle ACB = \angle ACD + \angle ACB.$$

But angles ACD and ACB are supplementary angles that together add up to two right angles (Proposition I.13).



Accordingly the three internal angles of the triangle ABC at the vertices A, B and C add up to two right angles, as required.

In his commentary on the first book of Euclid's *Elements of Geometry*, Proclus (402–485) presented a proof described in the history of mathematics written by Eudemus of Rhodes, who was a younger colleague of Aristotle living in the fourth century before the Common Era. (The history of mathematics written by Eudemus has not survived, but it would have been written before Euclid's *Elements of Geometry*.) Eudemus attributed the proof he described to the Pythagoreans.

Thomas Taylor (in a translation of Proclus's Commentary published in 1792) translated Proclus's account of the proof described by Eudemus as follows:

> But Eudemus, the Peripatetic, ascribes the invention of this theorem to the Pythagoreans, I mean that every triangle has its internal angles equal to two right, and says that they demonstrate it in the following manner. Let there be a triangle ABC, and let there be drawn through the point A, a line DE, parallel to BC. Because, therefore, the right lines DE, BC, are parallel,
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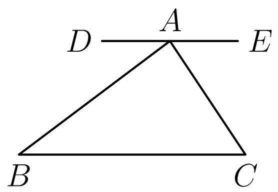
> 
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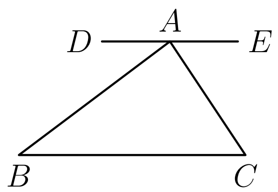
> the alternate angles are equal. Hence, the angle DAB, is equal to the angle ABC; and the angle EAC, to the angle ACB. Let the common angle BAC, be added. The angles, therefore, DAB, BAC, CAE, that is, the angles DAB, BAE, and that is two right, are equal to the three angles of the triangle. And such is the demonstration of the Pythagoreans.

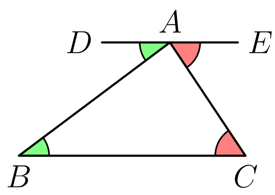
(Vertices have been represented above, as is traditional, by upper case letters, rather than by lower case italic letters as they appear in Thomas Taylor's published version of 1792.)

Thus, in the proof described by Eudemus, a straight line DE is drawn through the vertex A of the triangle ABC parallel to the side BC of the triangle opposite that vertex, as depicted in the following figure.



Representing this proof in a more symbolic fashion, we apply the result that the alternate angles determined by any transversal to a pair of parallel lines are equal (Proposition I.29) to deduce that

$$\angle DAB = \angle ABC \quad \text{and} \quad \angle EAC = \angle ACB.$$

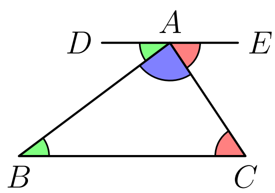


Consequently

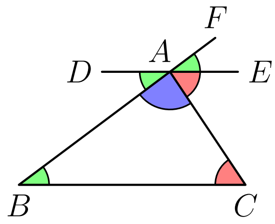
$$\angle ABC + \angle ACB = \angle DAB + \angle EAC.$$

Adding in the angle CAB, we find that

$$\begin{aligned} \angle ABC + \angle CAB + \angle BCA &= \angle DAB + \angle BAC + \angle CAE \\ &= \text{two right angles.} \end{aligned}$$



The proof presented by Euclid and the proof attributed to the Pythagoreans are in fact closely related. Suppose that, in the diagram associated with the proof attributed to the Pythagoreans, the side BA of the triangle is produced in a straight line past A to some point F. Then Proposition I.15 ensures that the vertically opposite angles DAB and EAF are equal to one another. Consequently the three angles of the triangle ABC are equal both



to the sum of the three angles below the line DE at the point A, as demonstrated in the proof attributed to the Pythagoreans, and also to the sum of the three angles at the point A that lie on the same side of the line BF as the points C and E, as demonstrated in the proof presented by Euclid (as may be verified by implementing an appropriate reorientation of the diagram associated with Euclid's proof and relabelling the vertices within it).