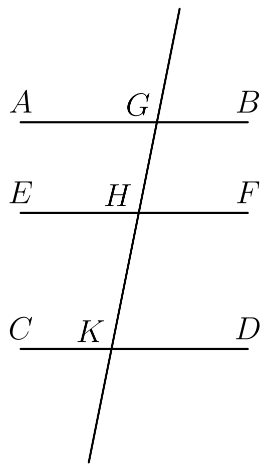
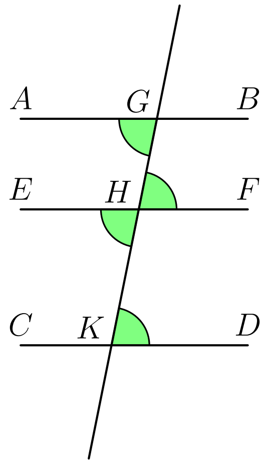
Study Note—Euclid's *Elements*, Book I, Proposition 30

David R. Wilkins

Proposition 30 of Book I of Euclid's *Elements of Geometry* asssserts that if two straight lines in a given plane are each parallel to a third straight line in that plane, then the two straight lines must be parallel to one another.



Euclid's proof of Proposition 30 may be summarized as follows. Referring to the angles on the associated figure, we see that



$$\begin{aligned} \angle AGH &= \angle GHF \quad \text{(Proposition 29, alternate angles)} \\ \angle GHF &= \angle HKD \quad \text{(Proposition 29, corresponding angles).} \end{aligned}$$

Accordingly

$$\angle AGK = \angle AGH = \angle GHF = \angle HKD = \angle GKD.$$

Applying Proposition 27, we conclude from the equality of the alternate angles AGK and GKD that the lines AB and CD are parallel to one another, as required.

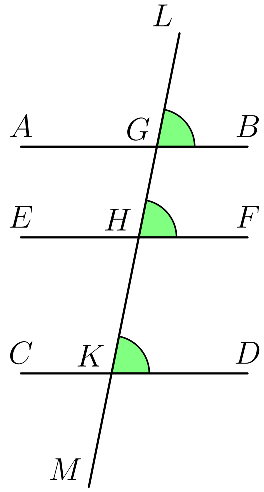
Replacing the step involving corresponding angles in the above argument with steps that involve alternate and vertically-opposite angles, we obtain the following more detailed argument:

$$\begin{aligned} \angle AGH &= \angle GHF \quad \text{(Proposition 29, alternate angles)} \\ \angle GHF &= \angle EHK \quad \text{(Proposition 15, vertically-opposite angles)} \\ \angle EHK &= \angle HKD \quad \text{(Proposition 29, alternate angles).} \end{aligned}$$

Accordingly

$$\angle AGK = \angle AGH = \angle GHF = \angle EHK = \angle HKD = \angle GKD.$$

A variant of Euclid's proof may be described as follows. The straight lines AB and CD in the following figure are each supposed parallel to the line EF. Accordingly, applying Proposition I.29, it follows that corresponding angles



at G and H are equal to one another, and also corresponding angles at H and K are equal to one another, and thus

$$\angle LGB = \angle GHF \quad \text{and} \quad \angle GHF = \angle HKD.$$

It follows that the corresponding angles LGB and GKD are equal to one another. Applying Proposition I.28 (which is the converse of the Corresponding Angles Theorem), we conclude that the lines AB and CD are parallel to one another.

In 1795, John Playfair, Professor of Mathematics at the University of Edinburgh, published a textbook on the Elements of Geometry, based on Euclid's *Elements*, but not aiming for a literal translation of Euclid's text. Playfair replaced Euclid's Fifth Postulate (required for the theory of parallels) by the following assertion, which has become known as *Playfair's Axiom*:

> **Playfair's Axiom.** *Two straight lines cannot be drawn through the same point, parallel to the same straight line, without coinciding with one another.*

Playfair's axiom is logically equivalent to the result stated by Euclid in Proposition 30 of Book I of the *Elements of Geometry.*

Indeed suppose that the result stated in Proposition 30 is true. Then straight lines parallel to the same straight line must be parallel to each other. Let some straight line be given, and let some point also be given, where this point does not lie on the given straight line. Two distinct straight lines passing through this point could not then both be parallel to the given straight line, because, if they were, they would then be parallel to one another, contradicting the condition that they both pass through the given point. Consequently, as asserted by Playfair's Axiom, two straight lines cannot be drawn through the same point, parallel to the same straight line, without coinciding with one another.

Now suppose that the statement of Playfair's Axiom is true. Let two straight lines in a given plane each be parallel to a third straight line in that plane. Suppose that those two straight lines were to intersect at some point of the plane. Then there would exist two straight lines through that point parallel to the same straight line but not coinciding with one another, contrary to the assertion of Playfair's Axiom. Therefore the two straight lines in the plane that are parallel to the third straight line must also be parallel to one another, as asserted in Proposition 30 of Book I of Euclid's *Elements of Geometry.*

Accordingly Playfair's axiom is indeed logically equivalent to the result stated by Euclid in Proposition 30 of Book I of the *Elements of Geometry.*

Next we note that Playfair's axiom together with Proposition 27 of Book I of Euclid's *Elements of Geometry* together imply Proposition 29 of that book. Indeed suppose that we are given two straight lines in a given plane that are intersected by a third straight line at points P and Q of that plane respectively. Now a straight line can be constructed through the point Q so as to ensure that alternate angles determined with respect to the given straight line through the point P and the constructed straight line through the point Q are equal to one another. Proposition 27 then ensures that the constructed straight line through the point Q is parallel to the given straight line through the point P. But the given straight line through the point Q is also parallel to the given straight line through the point P. Playfair's Axiom then ensures that the given straight line through the point Q must coincide with the constructed straight line through that point, and therefore alternate angles determined by the two given straight lines must be equal to one another. The result stated in Proposition 29 then follows.