Study Note—Euclid's *Elements*, Book I, Proposition 26

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Proposition 26 of Book I of Euclid's *Elements of Geometry* establishes the *ASA Congruence Rule* and the *SAA Congruence Rule*. Previously, Euclid had established the *SAS Congruence Rule* and the *SSS Congruence Rule* in Propositions 4 and 8 respectively of Book I of the *Elements of Geometry*.

Euclid uses the same diagram for the proofs of the *ASA Congruence Rule* and the *SAA Congruence Rule*.

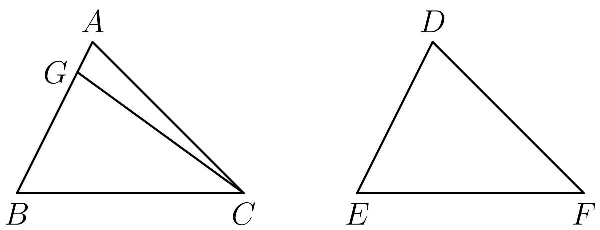
Following Euclid, we first establish the *ASA Congruence Rule*. Let ABC and DEF be triangles for which

$$\angle ABC = \angle DEF, \quad \angle BCA = \angle EFD \quad \text{and} \quad BC = EF.$$

We are required to show that

$$AB = DE, \quad AC = DF \quad \text{and} \quad \angle BAC = \angle EDF.$$

Suppose that AB and DE were unequal in length, and that AB were the longer. Then there would exist a point G between the points A and B for which $GB = DE$. Join G and C.



Then

$$GB = DE, \quad BC = EF \quad \text{and} \quad \angle GBC = \angle DEF.$$

It would then follow, applying the SAS Congruence Rule (*Elements*, I.4) that the triangles GBC and DEF would be congruent, and consequently

$$\angle BCG = \angle EFD = \angle BCA.$$

But the position of the point G between A and B ensures that $\angle BCG < \angle BCA$. Consequently a contradiction would arise were AB and DE unequal in length. We have now established that

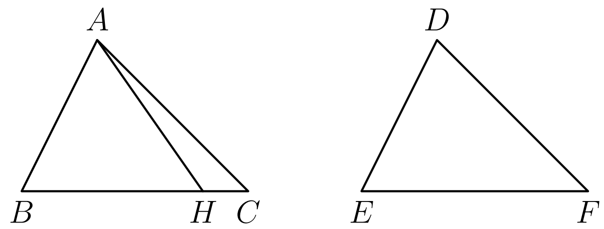
$$AB = DE, \quad BC = EF \quad \text{and} \quad \angle ABC = \angle DEF.$$

Applying the SAS Congruence Rule (*Elements*, I.4), we conclude that the triangles ABC and DEF are congruent, and consequently $AC = DF$ and $\angle BAC = \angle EDF$, as required.

The *SAA Congruence Rule* can be established using an analogous strategy, but it is necessary to apply Proposition 16 of Book I of Euclid's *Elements of Geometry* in order to complete the proof. Thus let ABC and DEF be triangles for which

$$\angle ABC = \angle DEF, \quad \angle BCA = \angle EFD \quad \text{and} \quad AB = DE.$$

Suppose then that BC and EF were unequal in length, and that BC were the longer. Then there would exist a point H between B and C for which $BH = EF$. Join A and H.



Then

$$AB = DE, \quad BH = EF \quad \text{and} \quad \angle ABH = \angle DEF.$$

It would then follow, applying the SAS Congruence Rule (*Elements*, I.4) that the triangles ABH and DEF would be congruent, and consequently

$$\angle BHA = \angle EFD = \angle BCA.$$

But the position of the point G between A and B ensures that the angle BHA is an exterior angle of the triangle AHC, and that the angle BCA is an interior and opposite angle of that triangle. Applying Proposition 16 of Book I of Euclid's *Elements of Geometry*, we conclude that $\angle BHA > \angle BCA$. Consequently a contradiction would arise were BC and EF unequal in length. We have now established that

$$AB = DE, \quad BC = EF \quad \text{and} \quad \angle ABC = \angle DEF.$$

Applying the SAS Congruence Rule (*Elements*, I.4), we conclude that the triangles ABC and DEF are congruent, and consequently $AC = DF$ and $\angle BAC = \angle EDF$, as required.