Study Note—Euclid's *Elements*, Book I, Proposition 24

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In order to establish Proposition 24 of Book I of Euclid's *Elements of Geometry*, we must show that if ABC and DEF are triangles in the plane, and if $AB = DE$, $AC = DF$ and $\angle BAC > \angle EDF$, then $BC > EF$.

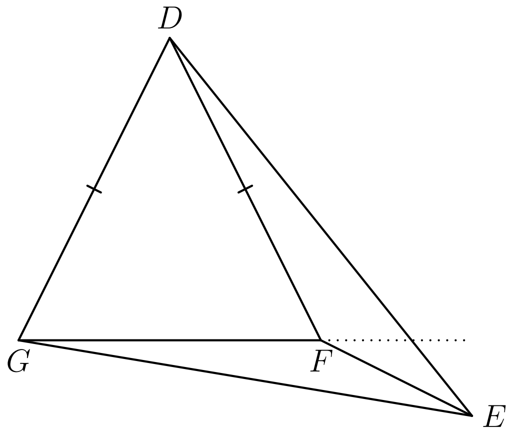
Now Proposition 23 enables the construction of a point G of the plane, on the same side of the line DE as the point F, with the properties that $DG = AC$ and $\angle GDE = \angle CAB$. Then the triangle ABC and DEG are congruent. Also $\angle GDE > \angle FDE$, and accordingly the point F lies in the interior of the angle GDE. The result stated in Proposition 24 will then follow immediately once it has been shown that $EG > EF$.

In this situation, three configurations arise:

- the configuration in which the points D and E lie on opposite sides of the infinite line that passes through the points G and F;
- the configuration in which the point E lies on the infinite line that passes through the points G and F;
- the configuration in which the points D and E lie on the same side of the infinite line that passes through the points G and F.

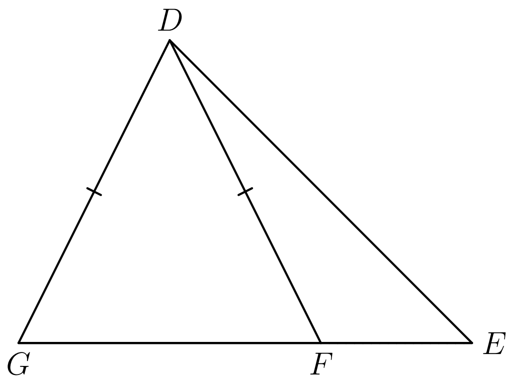
The proof given by Euclid only covers the third of these configurations. However proofs covering the other two configurations were supplied by Proclus, and are reproduced by later commentators.

We consider the first of the configurations, which is that in which the points D and E lie on opposite sides of the infinite line that passes through the points G and F. In this configuration, the point F lies in the interior of the angle DGE. But the point F also lies in the interior of the angle GDE. Consequently, in this configuration, the point F lies in the interior of the triangle DEG.

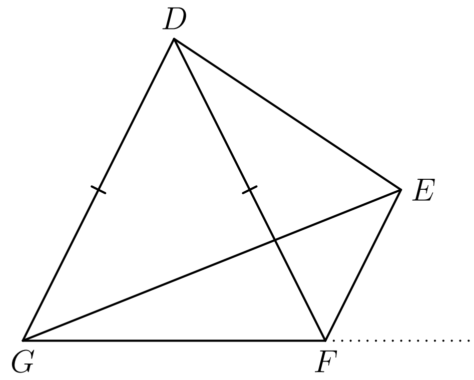


Applying Proposition 21 of Book I of Euclid's *Elements of Geometry*, we conclude that, in this configuration, $EG + GD > EF + FD$. But $GD = FD$. Consequently $EG > EF$, and therefore $BC > EF$, as required in this configuration.

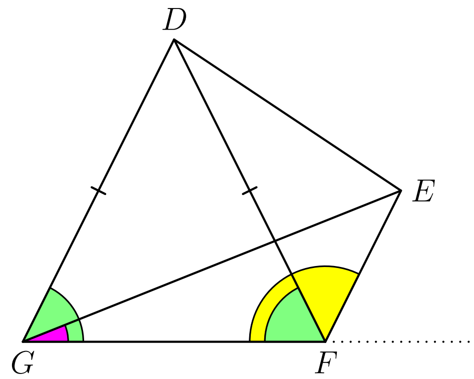
The next configurations to be considered is that in which the points E lies on the infinite line that passes through the points G and F. In this case the point F lies between the points G and E, and it follows immediately that $EG > EF$, and consequently $BC > EF$.



The complete proof of Proposition 24 is completed on considering the configuration in which the points D and E lie on the same side of the infinite line that passes through the points G and F. This is only configuration explicitly discussed by Euclid.



Now the triangle DGF is an isosceles triangle, with equal sides DG and DF. It follows that the angles DGF and DFG are equal to one another. Examination of the configuration then shows that



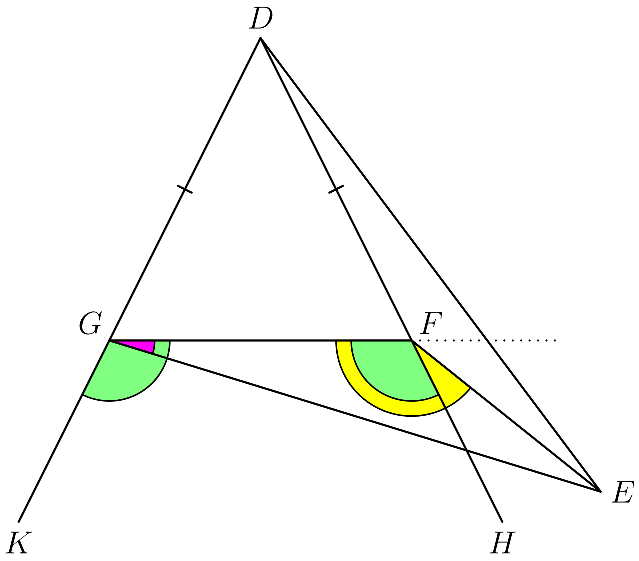
$$\angle EFG > \angle DFG = \angle DGF > \angle EGF$$

Now, in any triangle, the greater angle subtends the greater side (see Proposition 19 in Book I of Euclid's *Elements of Geometry*). Therefore $EG > EF$, and consequently $BC > EF$, as required in this configuration.

All relevant configurations have now been considered, and therefore a complete proof of the proposition has been achieved.

Nevertheless the proof strategies employed in the first and third configurations bear little resemblance to one another. One can however construct alternative proofs, valid for the first and third configurations respectively, which adapt the proof strategies used above for the third and first configurations respectively.

Let us then reconsider the first configuration, which is that in which the points D and E lie on opposite sides of the infinite line that passes through the points G and F, and let the equal sides DF and DG of the triangle DFG be produced in a straight line to points H and K respectively, as depicted in the diagram below.

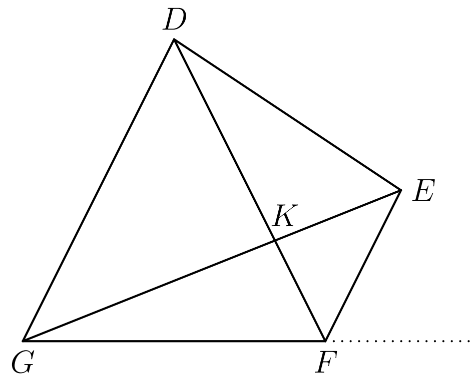


Now Proposition 5 of Book I of Euclid's *Elements of Geometry* ensures that the angles HFG and KGF below the base of the isosceles triangle are equal to one another. Examination of the configuration then shows that

$$\angle EFG > \angle HFG = \angle KGF > \angle EGF.$$

It follows, on applying Proposition 19 of Book I of Euclid's *Elements of Geometry*, that $EG > EF$, as required in this configuration.

Next let us then reconsider the third configuration, which is that in which the points D and E lie on the same side of the infinite line that passes through the points G and F, as depicted in the diagram below.



Now, applying Proposition 22 of Book I of Euclid's *Elements of Geometry* (establishing the *Triangle Inequality*), we see that

$$DG + EF < DK + KG + EK + KF = DF + EG.$$

But $DG = DF$. It follows that $EF < EG$, as required in this configuration.