Study Note—Euclid's *Elements*, Book I, Proposition 22

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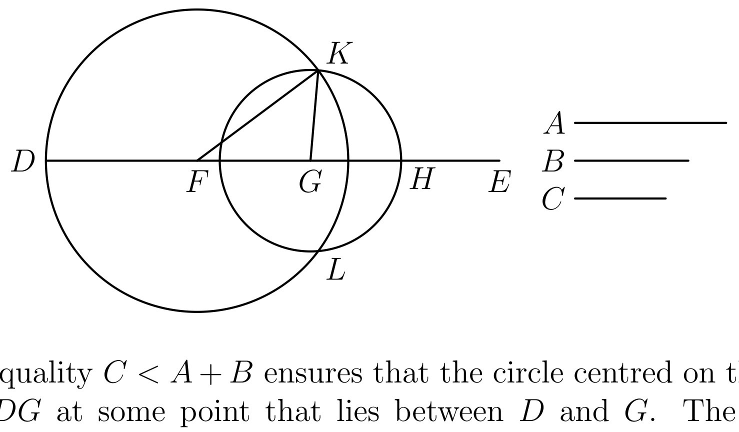
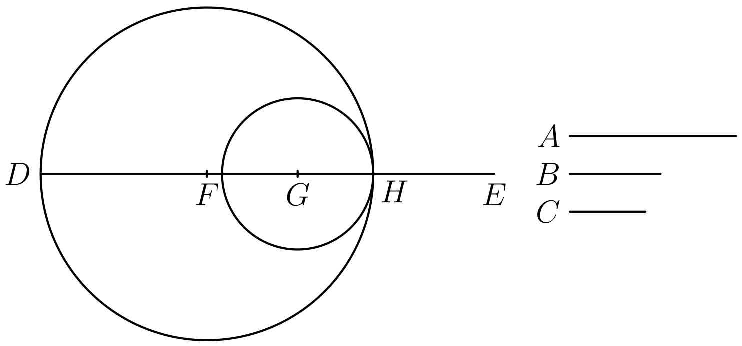
This proposition provides a construction that yields a triangle with sides equal in length to straight line segments A, B and C, where $A < B + C$, $B < A + C$ and $C < A + B$. Proposition 20 establishes that these conditions are necessary conditions for the existence of a triangle with its three sides equal in length to A, B and C respectively. The construction presented to achieve the construction of Proposition 22 shows that these three necessary conditions are also sufficient conditions for the existence of a triangle with its three sides equal in length to A, B and C respectively.

Euclid's construction is as depicted in the following figure, in which the straight line segments DF, FG and GH contained in the ray from D passing through the point E are equal in length to the given straight line segments A, B and C respectively.

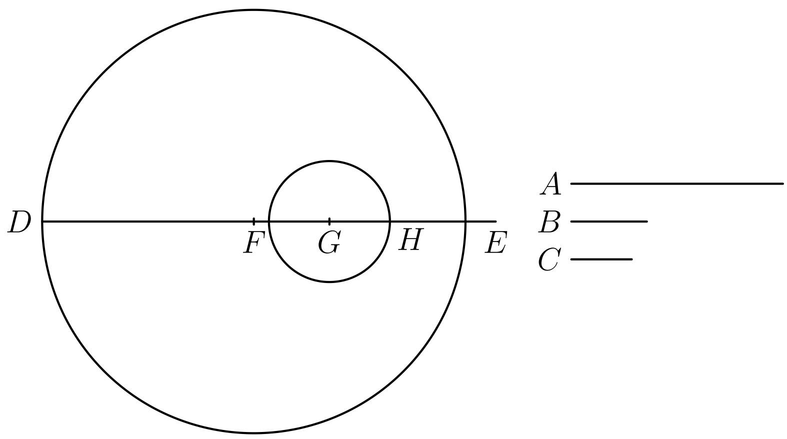


The inequality $C < A + B$ ensures that the circle centred on the point G intersects DG at some point that lies between D and G. The inequality $B < A + C$ ensures that the point on the straight line segment DG at which that segment meets the circle with radius equal to C centred on the point G lies within the circle with radius equal to A centered on the point F. The inequality $A < B + C$ ensures that the circle with radius C centred on the point G intersects the ray GE at a point that lies outside the circle with radius equal to A centered on the point F. Consequently the circle with radius equal to C centred on the point G passes both through points inside the circle with radius A centered on F and also through points outside that circle. Therefore the circles centered on the points F and G intersect one another and, taking K and L to be points of intersection of those circles, the triangles FGK and FGL have sides FK and FL equal to A, side FG equal to B, and sides GK and GL equal to C.

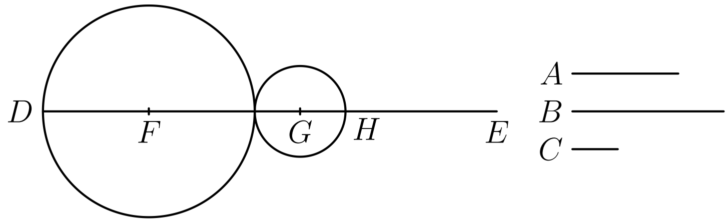
Were it the case that $A = B + C$, and if the same construction were attempted, the circle centred on the point G would touch the circle centred on the point F internally at the point H, as depicted in the following diagram, and consequently no triangle with sides equal to A, B and C would result from the construction.



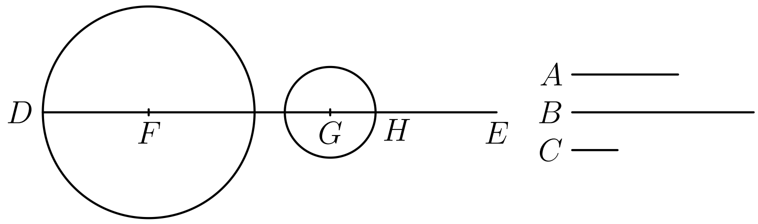
Were it the case that $A > B + C$, and if the same construction were attempted, the circle centred on the point G would lie inside the circle centred on the point F, as depicted in the following diagram, and consequently no triangle with sides equal to A, B and C would result from the construction.



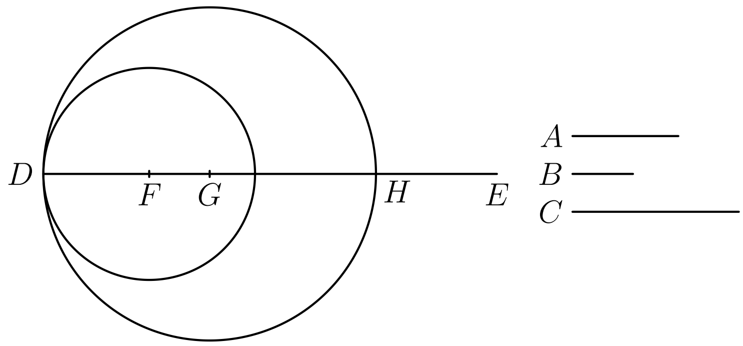
Were it the case that $B = A + C$, and if the same construction were attempted, the circles centred on the point F would touch the circle centred on the point G externally at a point between F and G, as depicted in the following diagram, and consequently no triangle with sides equal to A, B and C would result from the construction.



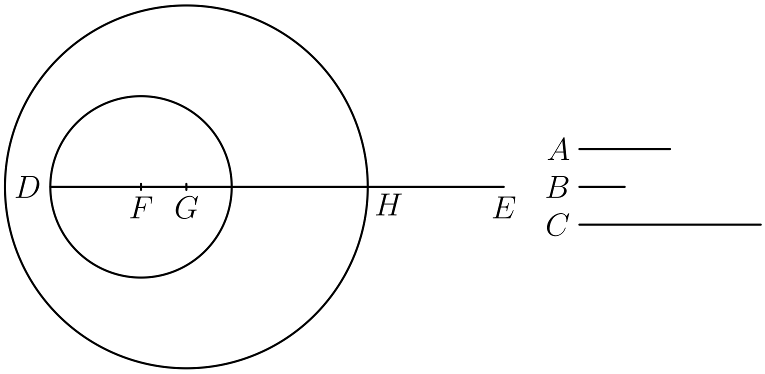
Were it the case that $B > A + C$, and if the same construction were attempted, the circles centred on the points F and G would be separated, and their circumferences would not meet, as depicted in the following diagram, and consequently no triangle with sides equal to A, B and C would result from the construction.



Were it the case that $C = A + B$, and if the same construction were attempted, the circle centred on the point F would touch the circle centred on the point G internally at the point D, as depicted in the following diagram, and consequently no triangle with sides equal to A, B and C would result from the construction.



Were it the case that $C > A + B$, and if the same construction were attempted, the circle centred on the point F would lie inside the circle centred on the point G, as depicted in the following diagram, and consequently no triangle with sides equal to A, B and C would result from the construction.



Thus an effective construction for constructing a triangle with sides of lengths equal to A, B and C results if and only if all inequalities $A < B + C$, $B < A + C$ and $C < A + B$ are valid.