Study Note—Euclid's *Elements*, Book I, Proposition 21

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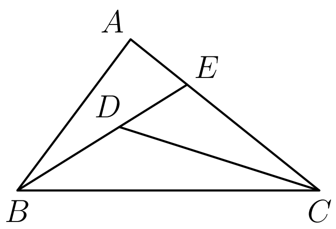
Let ABC and DBC be triangles with a common base BC, where the vertex D is located inside the triangle ABC. This proposition establishes that the sum of the sides BD and DC of the triangle BDC is less than the sum of the sides BA and AC of the triangls ABC, and the angle BDC is greater than the angle BAC.



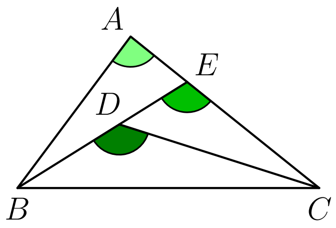
Proposition 20 of Book 1 of the *Elements* asserts that the sum of any two sides of a triangle is always less than the remaining side. (This result is frequently referred to as the *Triangle Inequality.* It therefore follows from this proposition that

$$DC < DE + EC \quad \text{and} \quad DE < DA + AE.$$

Consequently

$$\begin{aligned} BD + DC &< BD + DE + EC \quad \text{(Prop. I.20)} \\ &= BE + EC \\ &< BA + AE + EC \quad \text{(Prop. I.20)} \\ &= BA + AC. \end{aligned}$$

Proposition 16 of Book I of the *Elements* may be applied twice in order to establish the inequality involving the angles of the triangles ABC and DBC at vertices A and D respectively.



Indeed BDC is an external angle of the triangle CDE, and BEC is, with respect to this external angle, one of the internal and opposite (or remote) angles. Consequently it follows from Proposition 16 that $\angle BDC > \angle BEC$. In turn, the angle BEC is an external angle of the triangle BAE, and the angle BAC is, with respect to this external angle, one of the internal and opposite (or remote) angles. Consequently it follows from Proposition 16 that $\angle BEC > \angle BAC$. We conclude therefore that

$$\angle BDC > \angle BEC > \angle BAC,$$

as required.