Study Note—Euclid's *Elements*, Book I, Proposition 20

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Proposition 20 of Book I of Euclid's *Elements of Geometry* establishes the *Triangle Inequality.*

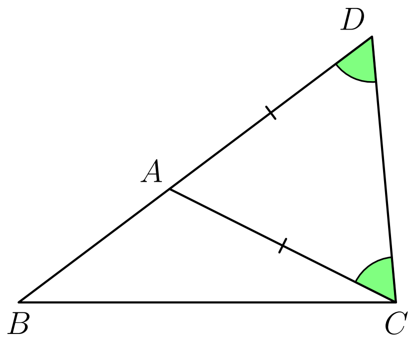
This proposition asserts that the sum of any two sides of a triangle is greater than the remaining side.

Proclus (as translated by Thomas Taylor, 1792) begins his commentary on this proposition as follows.

> The Epicureans oppose the present theorem, asserting that it is manifest even to an ass; and that it requires no demonstration: and besides this, that it is alike the employment of the ignorant, to consider things manifest as worthy of proof, and to assent to such as are of themselves immanifest and unknown; for he who confounds these, seems to be ignorant of the difference between demonstrable and indemonstrable. But that the present theorem is known even to an ass, they evince from hence, that grass being placed in one extremity of the sides, the ass seeking his food, wanders over one side, and not over two. Against these we reply, that the present theorem is indeed manifest to sense, but not to reason producing science: for this is the case in a variety of concerns.

We now present the proof of the proposition, following Euclid.

Let ABC be a triangle. We aim to show that $BC < BA + AC$. To prove this inequality, we produce the side BA of the triangle ABC in a straight line beyond A to the point D for which $DA = AC$.



Then the triangle ACD is an isosceles triangle. Consequently it follows (on applying Proposition 5 of Book I of Euclid's *Elements of Geometry*) that

$$\angle ACD = \angle ADC.$$

But the point A lies in the interior of the angle BCD. Consequently

$$\angle BCD > \angle ACD = \angle ADC = \angle BDC.$$

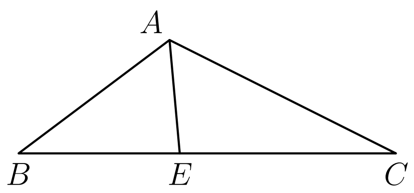
Now, given two unequal angles of a triangle, the larger angle subtends the longer side (by Proposition 19 of Book I of Euclid's *Elements of Geometry*). Consequently $BD > BC$. But $AD = AC$, and therefore

$$BD = BA + AD = BA + AC.$$

We conclude therefore that $BC < BA + AC$, as required.

We now describe an ancient alternative proof of the proposition (which, according to Proclus, was described either by Heron of Alexandria, or by Porphyry, or by both).

Let ABC be a triangle, and let the angle BAC be bisected by the straight line segment AE whose endpoint E lies on the side BC of the given triangle.



Now Proposition 16 of Book I of Euclid's *Elements of Geometry* ensures that an exterior angle of a triangle is greater than each of the interior and opposite angles of that triangle. Accordingly

$$\angle BEA > \angle EAC \quad \text{and} \quad CEA > \angle EAB.$$

But $\angle EAC = \angle EAB$, because the straight line AE bisects the angle BAC. Consequently

$$\angle BEA > \angle EAB \quad \text{and} \quad CEA > \angle EAC.$$

Now, in any triangle with unequal angles, the larger angle subtends the longer side (by Proposition 19 of Book I of Euclid's *Elements of Goemetry*). Consequently $BA > BE$ and $CA > EC$. Consequently

$$BA + AC > BE + EC = BC,$$

as required.