Study Note—Euclid's *Elements*, Book I, Proposition 16

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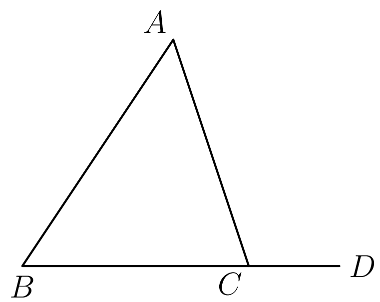
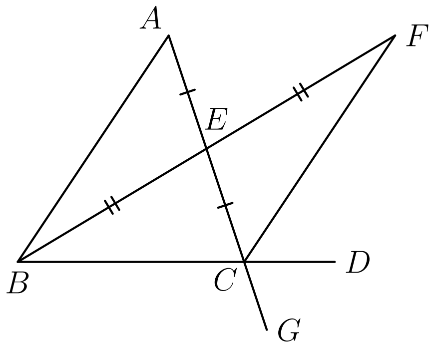
Let ABC be a triangle, let the side BC of that triangle be produced in a straight line beyond C to some point D. To establish this proposition, we must prove that the exterior angle ACD is greater than each of the interior and opposite angles CBA and BAC at B and A respectively.

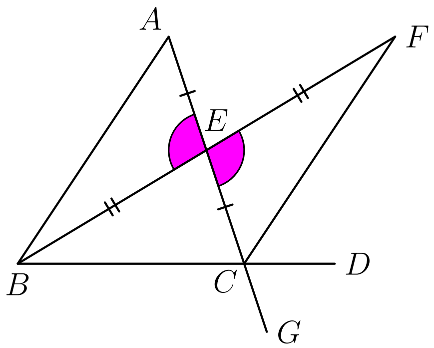


In order to prove this result, bisect AC at the point E, join B to E by a straight line segment, and produce this line segment in a straight line beyond E to a point F so that $BE = EF$, join F and C by a straight line segment FC, and produce the side AC of the given triangle in a straight line beyond C to some point G.

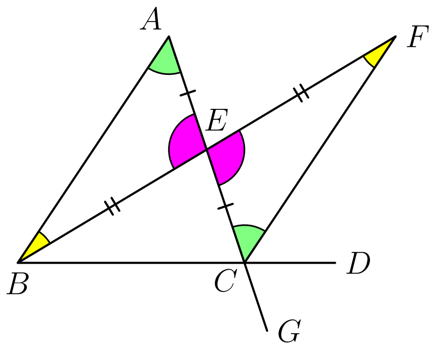


Now the straight line segments AC and BF are bisected at E, and moreover the angles AEB and FEC are vertically opposite angles. Thus

$$AE = CE, \quad EB = EF \quad \text{and} \quad \angle AEB = \angle FEC.$$



Applying the SAS Congruence Rule (Proposition 4 of Book I of Euclid's *Elements of Geometry*), we conclude that the triangles ABE and CFE are congruent, and therefore $\angle BAE = \angle ECF$.



We now present a fairly detailed argument to demonstrate that, as seems apparent from the diagram, the point F must lie in the interior of the angle ACD. First we note that the points A and E lie on the same side of the line BC, and consequently all points that lie on the straight ray starting at the point B and passing through the point E lie on the same side of BC as the point A. Consequently the points F and A lie on the same side of the straight line CD. Also the points B and F lie on opposite sides of AC, and the points B and D lie on opposite sides of AC, and therefore the points D and F lie on the same side of AC. Now the interior of the angle ACD consists of those points of the plane of the triangle ABC that lie on the same side of CD as the point A and on the same side of AC as the point D. Consequently the point F lies in the interior of the angle ACD, and therefore $\angle ECF < \angle ACD$. But we have established that $\angle ECF = \angle BAE$. It follows that the external angle ACD at C is greater than the internal and opposite angle BAE at A.

An analogous argument shows that the external angle BCG of the triangle ABC at C is greater than the internal and opposite angle CBA at B. But the angles BCG and ACD are equal, because they are vertically opposite angles. Therefore the external angle ACD is greater than the internal and opposite angle CBA at B. This completes the proof.