Study Note—Euclid's *Elements*, Book I, Proposition 12

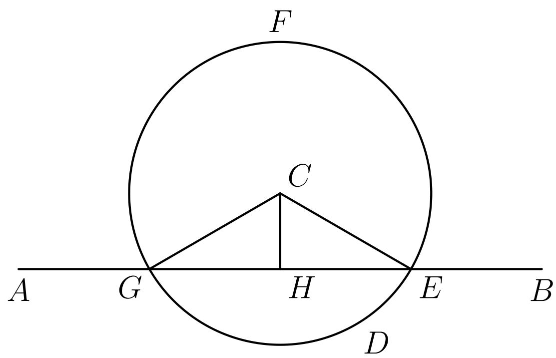
David R. Wilkins

Proposition 12 of Book I of Euclid's *Elements of Geometry* establishes the feasibility of dropping a perpendicular to a given infinite straight line from some given point that does not lie on the line.

Proclus attributes the construction described by Euclid to Oenopides, who was a geometer and astronomer active in the 5th century B.C.E..

In this context, we regard an 'infinite straight line' (εὐθεῖα ἄπειρος, **eutheia apeiros**) as being a straight line (εὐθεῖα γραμμὴ, **eutheia grammē**) that lacks extremities (πέρατα, **perata**). In other words, the straight line is 'infinite' in the sense of lacking endpoints that serve as bounds or extremities.

Let such an infinite straight line AB be given, together with a point C that does not lie on the line AB. A point D is taken on the opposite side of the line AB to the point C. (It is apparently tacitly assumed by Euclid that a line has two sides.) A circle EFG centred on the point C and passing through the point D is first constructed. This circle intersects the straight line AB in two distinct points E and G that are equidistant from the point C. The line segment GE is bisected at the point H, and the line segment CH is drawn. This line segment CH is the required perpendicular dropped from the point C to the infinite straight line AB.



In order to justify the validity of this construction, we note, following Euclid, that the sides CE, EH and CH of the triangle CEH are respectively equal to the sides CG, GH and CH of the triangle CGH. Applying the SSS Congruence Rule (*Elements*, I.8), we conclude that the triangles CEH and CGH are congruent, and consequently the angles CHE and CHG are equal to one another. It now follows from the definition of right angles that the angle CHE is a right angle, and thus the line CH is a perpendicular dropped from the point C to the infinite straight line AB.

The Commentary on the First Book of Euclid's *Elements of Geometry* written by Proclus (in the fifth century of the Common Era) includes a lengthy discussion of possible cases that might occur were it admitted possible that a circle might intersect an infinite straight line in more than two points. Parts of this discussion were paraphrased by Heath in his commentary on this proposition. Nevertheless the validity of the construction desribed by Euclid merely requires the existence of two distinct points on the straight line AB that are equidistant from the point C. The following discussion indicates how the existence of two distinct points on the line AB equidistant from the point C might be established on the basis of the postulates explicitly stated by Euclid and the other basic principles tacitly assumed by Euclid.

The Third Postulate specified in the first book of Euclid's *Elements* is the following.

> Ἠιτήσθω [...] καὶ παντὶ κέντρῳ καὶ διαστήματι κύκλον γράφεσθαι
> **Ēitēsthō** [...] **kai panti kentrō kai diastēmati kuklon graphesthai**
> Let it have been requested [...] and also for any centre and [radial] distance a circle to be drawn.

Euclid also tacitly assumes a basic topological principle, or postulate, which might be stated formally as follows:

> *Given an infinite straight line in a plane, and given a straight line segment or circular arc in that plane, where the endpoints of the straight line segment or circular arc lie on opposite sides of the given straight line, there exists a point at which the straight line segment or circular arc under consideration intersects the given infinite straight line.*

Now in describing the circle EFG starting and finishing at the point F, one first traces a first circular arc joining the point F to the point D, and then returns to the point F by tracing a second circular arc, where, with the exception of the endpoints, the points on the second circular arc are distinct from those on the first arc. The first and second circular arcs both join points F and D on opposite sides of the infinite straight line AB. Consequently we establish the existence of a point E on the straight line AB at which the straight line intersects one of the two circular arcs and also the existence of a second point G on the straight line AB, distinct from the point E, at which the straight line intersects the other circular arc. We thus obtain distinct points E and G on the straight line AB at which that straight line intersects the circle EFG. This provides an argument to justify the existence of an isosceles triangle CEG whose base EG is a segment of the given straight line AB, and the construction of the perpendicular AH then proceeds as described by Euclid.