Study Note—Euclid's *Elements*, Book I, Proposition 10

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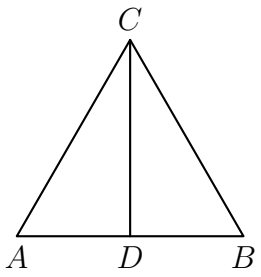
Proposition 10 of Book I of Euclid's *Elements of Geometry* establishes the feasibility of bisecting a straight line segment.

Euclid assumes the existence of a geometric construction for bisecting a given angle. Such a geometric construction is set out in the discussion of Proposition I.9. Euclid also assumes the existence of a geometric construction for constructing an equilateral triangle on a straight line segment. Such a geometric construction is set out in the discussion of Proposition I.1.

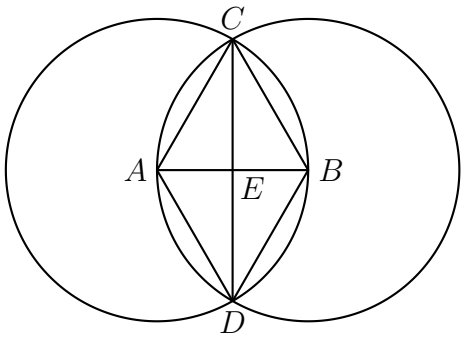


Let the given straight line segment be the line AB. It is required to bisect this line segment. In order to achieve this, one constructs an equilateral triangle ABC which has the line segment AB as one of its sides. One then bisects that angle of this equilateral triangle at the vertex C, thereby constructing a line segment CD, joining the vertex C to a point D on the given line segment AB, so as to ensure that the angles ACD and BCD are equal. Applying the SAS Congruence Rule (*Elements*, Proposition I.4), we can now conclude that the straight line segments AD and BD are equal in length. The line segment AB has consequently been bisected at the point D.

Proclus, in his commentary on the first book of Euclid's *Elements of Geometry*, describes a well-known self-contained construction for bisecting a line segment. Proclus's account is taken from an introduction to the elements

of geometry authored by Apollonius of Perga. (This work of Apollonius has not survived to the present day.)

Let AB be a straight line segment which one is required to bisect. As in the construction that Euclid discusses in his account of Proposition 1 of Book I of the *Elements*, one draws two circles: a circle centred on the point A and passing through the point B, and a circle centred on the point B and passing through the point A. These two circles intersect at points C and D that lie on opposite sides of the straight line that passes through the points A and B. Accordingly the line CD segment joining the points C and D will intersect the line segment AB at a point E. This point E bisects the line segment AB.



In order to justify this construction, one notes that the sides AC and CD and AD of the triangle ACD are respectively equal to the sides BC, CD and BD of the triangle BCD. It follows from the SSS Congruence Rule (*Elements*, Proposition I.8) that the angles ACD and BCD are equal to one another. But these two angles are the angles ACE and BCE respectively. Applying the SAS Congruence Rule (*Elements*, Proposition I.4), we conclude that the line segments AE and BE are equal in length. This establishes that the line segment AB is bisected at the point E, as required.