Study Note—Euclid's *Elements*, Book I, Proposition 7

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Let ABC and ABD be triangles for which the vertices C and D are distinct and both lie on the same side of the infinite straight line passing through the vertices A and B. We must show that, in this situation it is not possible both for AC and AD to be of equal length and also for BC and BD to be of equal length.

The proof of this result depends, implicitly or explicitly, on a number of underlying topological principles which include the following.

- The points along an infinite straight line can be ordered from one end to the other in the usual fashion so as to ensure that, given distinct points P, Q and R on the line, the point Q lies in the line segment PR if and only if either Q follows P and R follows Q in the ordering or else Q follows R and P follows Q in the ordering. The ordering of the points along the line is required to be *transitive*, so that if P, Q and R are distinct points on the line with Q following P and R following Q in the ordering, then the point R follows P in the ordering. The ordering of points along the line is also required to satisfy the *Trichotomy Law*, so that, given points P and Q on the line, exactly one of the following alternatives holds: P and Q coincide; Q follows P; P follows Q. There are moreover only two possible orderings of the points on the straight line satisfying these conditions, each being the reverse of the other.

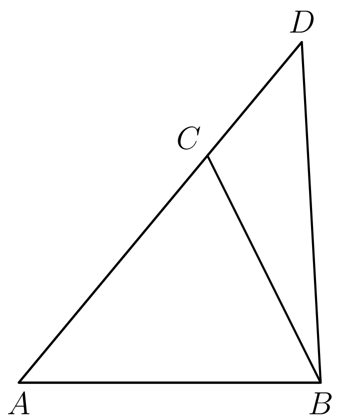
- Any infinite straight line in a plane divides that plane into exactly two half-planes, the *sides* of the line, that intersect along the straight line, and moreover two distinct points of the plane not on the line will lie on the same side of this infinite straight line if and only if the straight line segment joining them does not intersect the infinite straight line;

- Given an angle ABC in a plane, a point D lies in the interior of the angle ABC if and only if the point D lies on the same side of BC as the point A and on the same side of BA as the point C.

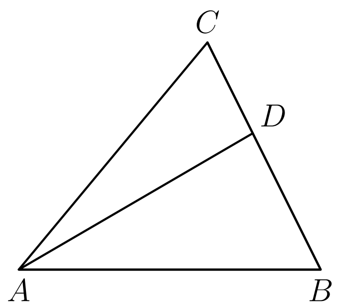
- Given a triangle ABC in a plane, a point D lies in the interior of the triangle angle ABC if and only if the point D lies on the same side of BC as the point A, on the same side of CA as the point B, and on the same side of AB as the point C.

A complete proof of this proposition requires one to study a number of configurations of which Euclid considers only one.

Now in configurations in which D lies on the ray starting at A and passing through C, but is distinct from C, the straight line segments AC and AD will be of unequal length.

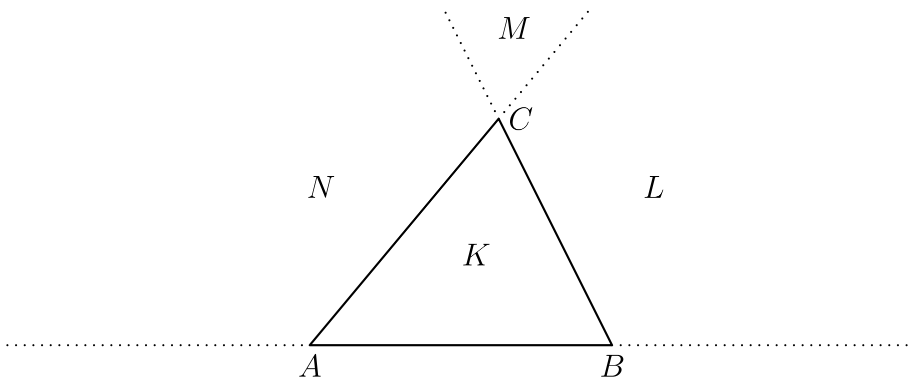


Similarly in configurations in which D lies on the ray starting at B and passing through C, but is distinct from C, the straight line segments BC and BD will be of unequal length.



It remains therefore to establish the stated result in configurations in which the point D lies on the same side of the line AB as the point C but does not lie on either of the rays starting at the points A and B that pass through the point C.

Now those points of the plane that do not lie on any of the infinite straight lines AB, AC and BC but lie on the same side of the infinite straight line AB as the point C belong to exactly one of the four regions into which the rays from points A and B passing though the point C divide that side of the line AB.



Thus, in configurations in which the point D lies on the same side of AB as the point C but does not lie on either of the rays from the points A and B that pass through the point C, the point D must lie in one of the regions K, L, M and N of the plane that are characterized as follows:

- **Region K** is the interior of the triangle ABC, consisting of those points of the plane of the triangle ABC that simultaneously lie on the same side of AB as the point C, on the same side of AC as the point B, and on the same side of BC as the point A;

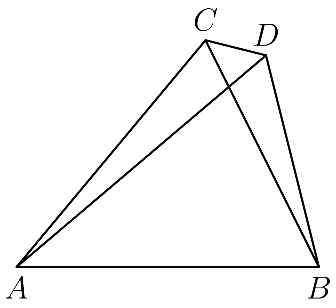
- **Region L** consists of those points of the plane of the triangle ABC that simultaneously lie on the same side of AB as the point C, on the same side of AC as the point B, and on the opposite side of BC to the point A;

- **Region M** consists of those points of the plane of the triangle ABC that simultaneously lie on the same side of AB as the point C, on the opposite side of AC to the point B, and on the opposite side of BC to the point A;

- **Region N** consists of those points of the plane of the triangle ABC that simultaneously lie on the same side of AB as the point C, on the opposite side of AC to the point B, and on the same side of BC as the point A.

Of these four configurations, Euclid only considers that in which the vertex D of the triangle ABD falls within region L.

The Proof of Proposition 7 for Configurations in which the Point D lies in Region L



In this configuration, the points A and D lie on opposite sides of the infinite straight line that passes through the points B and C, and therefore the straight line segment AD must intersect this infinite straight line. However the straight line segment AD cannot intersect that portion of the infinite straight line through the points B and C that lies on the opposite side of the line AB to the point C. Also the straight line segment AD cannot intersect that portion of the infinite straight line through the points B and C that lies on the opposite side of the line of the line AC as the point B. It follows that the straight line segment AD must intersect the straight line segment BC at some point that lies in the interior of both segments. It follows that the point B must lie in the interior of the angle ACD, and therefore

$$\angle BCD < \angle ACD.$$

Also the point A must lie in the interior of the angle BDC, and therefore

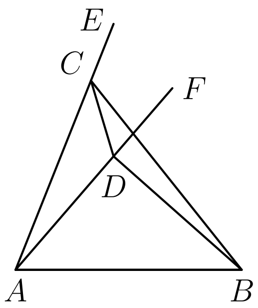
$$\angle ADC < \angle BDC.$$

Now, in any case where AC and AD are equal in length, the triangle ACD is isosceles, and therefore the angles the angles ACD and ADC opposite the equal sides must be equal (*Elements*, I.5). It then follows, in this configuration, that $\angle BCD < \angle BDC$, and consequently the line segments BC and BD cannot be equal in length. This argument proves that, in configurations in which the vertex D lies in Region L it cannot be the case that AC and AD are equal in length and also BC and BD are equal in length.

The Proof of Proposition 7 for Configurations in which the Point D lies in Region K

This configuration is discussed by Proclus and many other commentators.

In order to establish the required result in this case, let the side AC of the triangle ABC be produced in a straight line beyond the point C to a point E, and also let the side AD of the triangle ABD be produced in a straight line beyond the point D to some point F. (In the argument that follows, it is immaterial whether the point F lies inside or outside the triangle ABC.)



In this configuration, the point D lies in the interior of the angle BAC. Indeed the interior of that angle consists all points of the plane that lie both on the same side of the line AB as the point C and also on the same side of the line AC as the point B. The points B and C therefore lie on opposite sides of the line AF, and consequently

$$\angle FDC < \angle BDC.$$

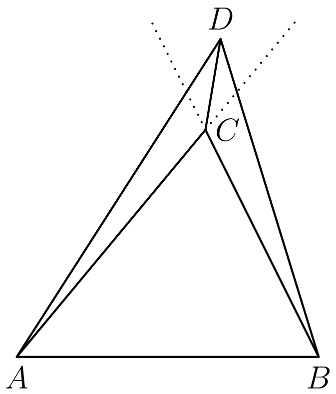
The point D, in this configuration, also lies in the interior of the angle ACB. Consequently the points A and B lie on opposite sides of the line CD, and therefore the points E and B lie on the same side of the line CD. Also the points D and B lie on the same side of the line AC, and therefore lie on the same side of the line CE. Consequently the point B lies in the interior of the angle DCE, and therefore

$$\angle BCD < \angle ECD.$$

Now, in any case where AC and AD are equal in length, the triangle ACD is isosceles, and therefore the angles the angles ECD and FDC under the base of that isosceles triangle must be equal (*Elements*, I.5). It then follows, in this configuration, that $\angle BCD < \angle BDC$, and consequently the line segments BC and BD cannot be equal in length. This argument proves that, in configurations in which the vertex D lies in Region K it cannot be the case that AC and AD are equal in length and also BC and BD are equal in length.

The Proof of Proposition 7 for Configurations in which the Point D lies in Region M

In this configuration the point D lies on the same side of AB as the point C, on the opposite side of AC to the point B and on the opposite side of BC to the point A. We show that, in this configuration, the point C lies in the interior of the triangle ABD.

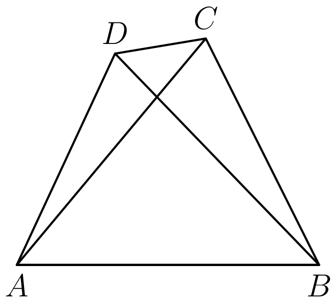


Now, in this configuration, the straight line segment BC and the ray starting at the point A that passes through the point D lie on opposite sides of the straight line AC, because the points B and D lie on opposite sides of that straight line. Consequently the straight line segment BC cannot intersect that part of the infinite straight line through A and D that lies on the same side of AB as the point C. Also the straight line segment BC cannot intersect that part of the infinite line through A and D that lies on the opposite side of AB to the point C. Consequently the straight line segment BC does not intersect the infinite straight line passing through the points A and D, and therefore the points B and C lie on the same side of the line AD. Similarly the condition that the points A and D lie on opposite sides of the straight line BC ensures that the points A and C lie on the same side of the line BD. Consequently the point C lies on the same side of AB as the point D, on the same side of AD as the point B, and on the same side of BD as the point A, and therefore the point C lies in the interior of the triangle ABD.

Now we have already shown that it is not possible for the sides AC and BC of the triangle ABC to be respectively equal to the sides AD and BD of the triangle ABD in the configuration in which the point D lies in the interior of the triangle ABC. Interchanging the roles of C and D, we conclude that it is not possible for the sides AC and BC of the triangle ABC to be respectively equal to the sides AD and BD of the triangle ABD in the configuration in which the the point C lies in the interior of the triangle ABD. Consequently AC and BC cannot be respectively equal to AD and BD in configurations in which the point D lies in the region M.

The Proof of Proposition 7 for Configurations in which the Point D lies in Region N

The proof in this configuration is completely analogous with that in the configuration in which the point D lies in the region L: the roles of the vertices A and B are interchanged.



In this case

$$\angle ACD < \angle BCD.$$

Also the point B must lie in the interior of the angle ADC, and therefore

$$\angle BDC < \angle ADC.$$

Now, in any case where BC and BD are equal in length, the triangle BCD is isosceles, and therefore the angles the angles BCD and BDC opposite the equal sides must be equal (*Elements*, I.5). It then follows, in this configuration, that $\angle ACD < \angle ADC$, and consequently the line segments AC and AD cannot be equal in length. This argument proves that, in configurations in which the vertex D lies in Region N it cannot be the case that BC and BD are equal in length and also AC and AD are equal in length.

This completes this review of the proof of Proposition 7 of Book I of Euclid's *Elements of Geometry*.