Study Note—Euclid's *Elements*, Book I, Proposition 6

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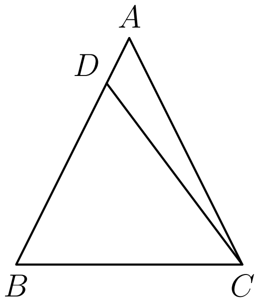
Proposition 6 of Book I of Euclid's *Elements of Geometry* ensures that if two angles of a triangle are equal to one another then the sides subtended by the equal angles are equal to one another.

Euclid proves the proposition by showing that a contradiction would arise were the stated result not true. Thus suppose that there were to exist a triangle ABC for which

$$\angle ABC = \angle ACB \quad \text{and} \quad AB \neq AC.$$

Then either $AB > AC$ or else $AC > AB$. We may suppose, without loss of generality that $AB > AC$. Applying Proposition I.3 of the *Elements*, we are assured of the existence of a point D lying on the side AB for which $DB = AC$. Then DB and BC are respectively equal to AC and CB, and $\angle DBC = \angle ABC = \angle ACB$.



Applying the the result that two triangles satisfying the hypotheses of the SAS Congruence Rule must necessarily be equal in area to one another (*Elements*, Proposition I.4), Euclid would conclude that the triangles DBC and ABC would be equal in area. But the triangle DBC would also be a part of ABC, and therefore it would be smaller in area. Thus the assumption that

$AB > AC$ would lead to a contradiction. The assumption that $AC > AB$ would also lead to a contradition. Consequently it must be the case that $AB = AC$ as required.

As an alternative to the consideration of areas, one could also argue as follows. Were it the case that $\angle ABC = \angle ACB$ and $AB > AC$ we could again determine the point D on AB for which $DB = AC$. It would then follow, applying the SAS Congruence Rule, that $\angle DCB = \angle ABC$, and thus $\angle DCB = \angle ACB$. But the angle DCB would also be a part of the angle ACB, and consequently, applying the Sixth Common Notion, $\angle DCB < \angle ACB$. Thus a contradiction would arise from assuming that $\angle ABC = \angle ACB$ and also $AB > AC$. The required result therefore follows.