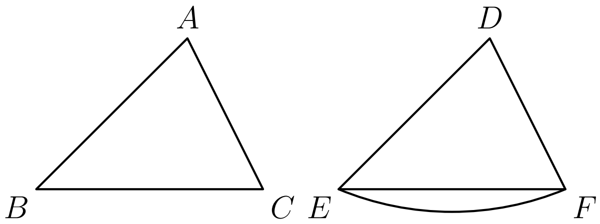
Study Note—Euclid’s *Elements*, Book I, Proposition 4

David R. Wilkins

Proposition 4 of Book I of Euclid’s *Elements of Geometry* establishes the *SAS Congruence Rule*, and moreover notes that, where the congruence of two triangles has been established through the application of this rule, the congruent triangles are equal to one another in area.



Thus suppose that, in the two triangles ABC and DEF,

$$AB = DE, \quad AC = DF \quad \text{and} \quad \angle BAC = \angle EDF,$$

We suppose that that triangle ABC is so placed (or ‘applied to the triangle DEF’) as to ensure that the point A coincides with the point D and the rays from D passing through the points B and E coincide in direction. Then, because $AB = DE$, the points B and E must coincide with one another. We also suppose (though Euclid does not express this in words) that, in placing the triangle ABC so as to make the side AB coincide with the side DE, we also ensure that the point C lies on the same side of DE as the point F. Then, because $\angle BAC = \angle EDF$ and $AC = DF$, the point C must coincide with the point F. Now the line segment joining the points E and F is uniquely determined by its endpoints. Thus, because the points B and E coincide with one another and the points C and F coincide with one another, the side BC of the triangle ABC coincides with the side EF of the triangle DEF. Accordingly if we place the triangle ABC so as to ensure that A is placed on D and B is placed on E, and if we ensure that the points C and F then lie on the same side of DE, then the triangle ABC will be placed so as to coincide with the triangle DEF, and the conclusions of the proposition follow immediately. Specifically

$$BC = EF, \quad \angle CBA = \angle FED \quad \text{and} \quad \angle ACB = \angle DFE,$$

and, in addition, the triangles ABC and DEF are equal in area.

Now the proof of this proposition is founded on use of the Fourth Common Notion set out in the first book of the *Elements*. The statement of that Common Notion, in the original Greek, employs a verb ἐφαρμοζω (**epharmozō**, meaning 'I fit, adapt, or put on') that is derived by applying the prefix ἐπι- (**epi-**, meaning 'on' or 'onto') to the verb ἁρμόζω (**harmosō**, meaning 'I fit together, join, or marry', and giving rise to the English word 'harmony' corresponding to the Greek word ἁρμονία, **harmonia**). Accordingly the Fourth Common Notion, in the Greek text, contemplates the possibility of placing or fitting one geometrical entity onto another of the same kind, thereby *applying* the first entity to the second, so as ensure that the first fits exactly on the second, and asserts that, if it would be possible to do this, then the two geometrical figures must be equal to one another in quantity or magnitude.

We now consider in more detail the wording of the Fourth Common Notion. In Greek:

> Καὶ τὰ ἐφαρμόζοντα ἐπ᾽ ἄλληλα ἴσα ἀλλήλοις ἐστίν.
> **Kai ta epharmozonta ep' allēla isa allēlois estin.**
> Also the [things that are] on-fitting onto one another equal to one another are.

Accordingly this common notion makes a stronger assertion than would result from merely asserting that things that are the same as one another are equal in quantity to one another.

Considering the properties of the flat Euclidean plane from the perspective of modern mathematics, we can introduce the concept of a *Euclidean motion* of the flat Euclidean plane. Such a *Euclidean motion* is by definition a transformation mapping the flat Euclidean plane onto itself that preserves lengths and rectilineal angles. Thus, if points A, B and C of the flat Euclidean plane are mapped by a Euclidean motion onto points D, E and F respectively, where A, B and C are not collinear, then the straight line segments AB and DE are equal to one another in length (and similarly the straight line segments AC and DF are equal to one another in length, and the straight line segments BC and EF are equal to one another in length). Also the angles BAC and EDF are equal to one another in magnitude (and similarly the angles CBA and FED are equal to one another in magnitude, and the angles ACB and DFE are equal to one another in magnitude).

It can be shown that there are four types of Euclidean motion that map the flat Euclidean plane onto itself: rotations; reflections; translations; glide reflections. (*Glide reflections* are the transformations of the flat Euclidean plane that arise when a reflection is preceded or followed by a translation in a direction parallel to the line of reflection.)

Now, suppose that we are given three points A, B and C in the flat Euclidean plane that are not collinear. Suppose also that we are given three further points D, E and F in that plane that are not collinear, where the line segment joining the points A and B is equal in length to that joining D and E. Then there exists a Euclidean motion of the flat Euclidean plane that maps the points A and B onto the points D and E respectively, and also maps the point C onto a point of the Euclidean plane that lies on the same side of the line DE as the point F. Indeed a translation of the flat Euclidean plane maps the point A onto the point D. If this translation did not map the point B onto the point E it could be composed with a rotation about the point D so as to obtain a Euclidean motion mapping the points A and B, onto the points D and E respectively. If this Euclidean motion happened to map the point C to a point on the same side of DE as the point F then it is the required Euclidean motion. If not, then it could be composed with a reflection in the straight line passing through the points D and E to obtain the required Euclidean motion mapping the points A and B to the points D and E respectively, whilst also mapping the point C to a point of the plane lying on the same side of the line DE as the point F.

If, furthermore, the straight line segments AC and DF are equal to one another in length, if the angles BAC and EDF are equal to one another in magnitude, and if a Euclidean motion is determined so as to ensure that it maps the points A and B onto the points D and E respectively and also maps the point C to a point that lies on the same side of DE as the point F, then that Euclidean motion must map the straight line passing through the points A and C onto the straight line passing through the points D and F, and therefore must map the point C onto the point F.

We conclude therefore that if A, B and C are points of the flat Euclidean plane, and if D, E and F are also points of the same flat Euclidean plane, if the straight line segments AB and DE are equal to one another in length, the straight line segments AC and DF are equal to one another in length, and if the angles BAC and DEF are equal to one another in magnitude, so that, symbolically,

$$AB = DE, \quad AC = DF \quad \text{and} \quad \angle BAC = \angle EDF,$$

then there exists a Euclidean motion that maps the points A, B and C onto the points D, E and F respectively. The Euclidean motion will then map the sides of the triangle ABC onto the sides of the triangle DEF, preserving lengths and angles, and consequently the line segments BC and EF will be equal in length, the angles CBA and FED will be equal to one another in magnitude, and the angles ACB and DFE will be equal to one another in magnitude, and thus, symbolically,

$$BC = EF, \quad \angle CBA = \angle FED \quad \text{and} \quad \angle ACB = \angle DFE.$$

These are the conclusions that follow when the *SAS Congruence Rule* is applied to the triangles ABC and DEF with respect to the angles of those triangles at the vertices A and D respectively.

Suppose furthermore that it has been established that Euclidean motions preserve the areas of triangles, so that under the action of any Euclidean motion, any triangle in the Euclidean plane is equal in area to the triangle onto which it is mapped by the Euclidean motion. Then if the hypotheses of the *SAS Congruence Rule* are satisfied in respect of two given triangles in the flat Euclidean plane, then those two congruent triangles must be equal in area.

The discussion of *Euclidean motions* is founded on concepts well-established in modern mathematics. It should be noted, however, that nothing in the text of Euclid's *Elements of Geometry* would suggest that Euclid was in possession of the notion of a *Euclidean motion* that considered such a motion to be a transformation mapping the entire flat Euclidean plane onto itself.

If we require that triangles in plane geometry satisfy the the *SAS Congruence Rule* established in Proposition 4 of the first book of Euclid's *Elements of Geometry*, then it is necessary that the plane itself be both *homogeneous* and *isotropic*. In other words, the geometrical properties of the plane in relation to any point within it should be exactly the same as apply in relation to any other point of the plane. And, given any particular point of the plane, the geometrical properties considered in relation to any direction determined by a straight ray directed away that point must correspond to those considered in relation to any other straight ray directed away from the point. For it is necesssary to ensure that, given two distinct points of the plane, a Euclidean motion preserving lengths and angles will map one point onto the other. And it is also necessary to ensure that, given two distinct rays directed away from a given point of the plane, a Euclidean motion will implement a rotation about the given point taking one ray onto the other.