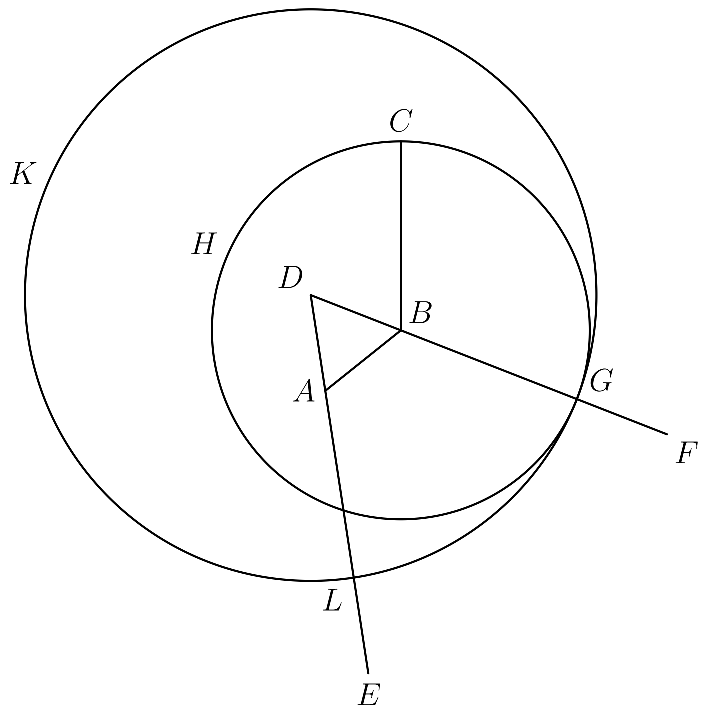
Study Note—Euclid's *Elements*, Book I, Proposition 2

David R. Wilkins

Proposition 2 of Book I of Euclid's *Elements of Geometry* establishes the feasibility of constructing a straight line segment in the plane, given one endpoint for the segment, where the straight line segment so constructed is equal in length to some other straight line segment elsewhere in the plane.

Thus let A be a point in the plane, and let BC be a straight line segment in that plane. We are required to construct a straight line segment with one endpoint at the point A which is equal in length to the line segment BC.



The construction is carried out as follows. First of all an equilateral triangle ABD is constructed with the line segment AB as one of its sides. (The construction that achieves this has already been described in the discussion of Proposition 1.) One draws the circle CGH centred on the point B that passes through the point C. (The Third Postulate ensures that this circle can be drawn.) The sides DB of the equilateral triangle ABD is produced in a straight line beyond the endpoint B so as to intersect the circle CGH at the point G. (The Second Postulate should ensure the possibility of so producing the side of the triangle.) The circle GKL is then drawn with centre D so as to pass through the point G. The side DA of the equilateral triangle ABD is then produced in a straight line beyond the endpoint A so as to intersect the circle GKL at the point L. One can then prove that the line segment AL is equal in length to the line segment BC.

Indeed, representing necessary proof symbolically, we see that

$$BC = BG, \quad DG = DB + BG, \quad DL = DA + AL,$$

$$DB = DA \quad \text{and} \quad DG = DL.$$

The Third Common Notion (asserting that if equals be subtracted from equals then the remainders are equal) then ensures that $BG = AL$. Consequently the line segments BC and AL are both equal in length to BG. It then follows, applying the First Common Notion, that the line segments BC and AL are equal to one another, as required.

Now the formal justification for the construction just discussed requires the Second Postulate, which, in Heath's translation, asserts that it is possible "to produce a finite straight line continuously in a straight line". (The term "finite straight line", in this context, is synonymous with "straight line segment"; the adjective "finite" indicates that the straight line is terminated by endpoints at both ends.) In fact, in order to achieve that which is necessary for geometrical constructions described in Euclid, it is an essential requirement that one can produce a straight line in this fashion so as to ensure that the resulting extended line equals or exceeds in length any other given straight line. Thus Dionysius Lardner, in his editions of the first six books of the Elements designed for use in schools and universities, states the requirements of the postulate as follows:

> Let it be granted that a finite right line may be produced to any length in a right line.

However this interpretation of the postulate is not clearly explicit in the Greek text, set out below with a transliteration and attempted literal translation:

Ἠιτήσθω [...] καὶ πεπερασμένην εὐθεῖαν κατὰ τὸ συνεχὲς ἐπ᾽ εὐθείας ἐκβαλεῖν
Ēitēsthō [...] **kai peperasmenēn eutheian kata tō suneches ep' eutheias ekbalein**
Let it have been requested [...] and also a finite straight line unbroken in a straight line to throw out.

Suppose in particular that a line segment lies in the interior of a circle with one endpoint located at the centre of the circle. Let the Second Postulate be interpreted to require that this line segment can be produced far enough to exceed in length any other given line segment. Then, in particular, the line segment can be produced so as to exceed in length the radii of the circle, and consequently the line segment will have one endpoint at the centre of the circle and the other endpoint outside the circle. A basic topological assumption, apparently tacitly assumed by Euclid, would then ensure that the line segment thus produced intersects the circle. Thus the geometrical construction descibed by Euclid in order to establish Proposition 2 is indeed feasible.