Study Note—Euclid's *Elements*, Book I, Proposition 1

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Proposition 1 of Book I of Euclid's *Elements of Geometry* establishes the feasibility of constructing, using straightedge and compass, an equilaterial triangle in the plane, given a line segment to serve as one of the sides of the constructed triangle.

It should be noted that, in ancient Greek geometry, such "constructions" concern purely mathematical concepts and abstractions, not the sorts of reasonably round and thin circles and reasonably straight and thin straight lines that one might draw in the physical world with, for example, pens and pencils. Indeed the definitions prefixed to Book I of Euclid's *Elements of Geometry* make it clear that straight lines and circles are purely one-dimensional objects, having longitudinal extension, and characterized in magnitude by their lengths, but without either breadth or depth. The propositions in Euclid's *Elements of Geometry* that establish the feasibility of constructions "using straightedge and compass" are concerned with the possibility of forming, in some conceptual universe of mathematical concepts, triangles and other geometric figures that satisfy exactly the properties that the construction is required to establish.

Plato explained the nature of such geometrical reasoning, about a century before Euclid, in the following terms:—

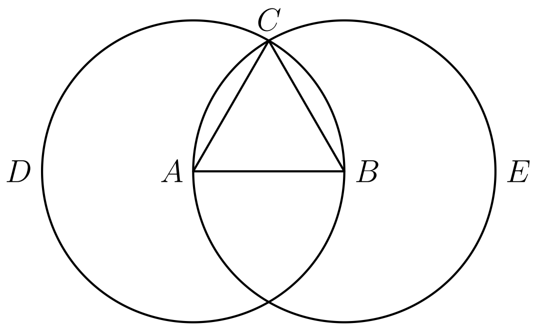
> You also know how they [i.e., students of geometry] make use of visible figures and discourse about them, though what they really have in mind is the originals of which these figures are images: they are not reasoning, for instance, about this particular square and diagonal which they have drawn, but about *the* Square and *the* Diagonal; and so in all cases. The diagrams they draw and the models they make are actual things, which may have their shadows or images in water; but now they serve in their turn

> as images, while the student is seeking to behold those realities which only thought can apprehend.
>
> Plato, *The Republic*, Book VI, 510D–E.
> Translated by Francis Cornford, 1941.

In the first book of Euclid's *Elements of Geometry*, geometric constructions involving both straight lines and circles are specified and justified in Propositions 1, 2, 3, 12 and 22. Circles are not explicitly employed in the proof of any other proposition in Book I, though proofs of other propositions may rely on the feasibility of geometrical constructions justified by means of the propositions just cited.

We consider now the particular geometric construction whose feasibility is established through this proposition.

We are given a straight line segment AB in some plane, with endpoints A and B, and the assigned task is to construct an equilateral triangle ABC which has the line segment AB as one of its sides. In order to achieve this, one draws two circles, the first centred on the point A and passing through the point B, and the second centred on the point B and passing through the point A. It is tacitly assumed in Euclid's proof that the two circles will intersect at some point C. This could be explained on the basis that the semicircular arc with the point B as its centre and the point A as one of its endpoints will join a point inside the circle BCD centred on the point A to a point E that lies outside the circle BCD. In consequence the semicircular arc ACE just specified will intersect the circle BCD at some point C. The points A and C are then joined by a straight line segment, and the points B and C are similarly joined. There results a triangle ABC with vertices A, B and C.



The definition of circles now ensures that the line segments AB and AC are equal in length. Similarly the line segments AB and BC are equal in length. Now the First Common Notion set out in the first book of the *Elements* guarantees that when two geometrical entities of the same species are equal in magnitude to a third geometrical entity of that same species, then the two geometrical entities are equal in magnitude to one another. In Greek:

> Τὰ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα.
> **Ta tō autō isa kai allēlois estin isa.**
> The [things that are] equal to the same [thing] are also equal to one another.

Accordingly, because the sides AC and BC of the constructed are both equal in length to the side AB, they must also be equal in length to one another. Consequently the triangle ABC must be equilateral, as required.

It should be noted that the proof of this proposition given in Euclid's *Elements* tacitly assumes a basic topological postulate concerning the existence of points where straight lines and circular arcs intersect circles which might be codified as follows:

> *Given a circle in a plane, and given a straight line segment or circular arc in that plane, where one endpoint of that straight line segment or circular arc lies inside the given circle, and the other endpoint lies outside the given circle, there exists a point at which the straight line segment or circular arc under consideration intersects the given circle.*