Accounts of Topics in Real Analysis: Ordered Fields and the Real Number System

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Sets

A *set* is a collection of objects. These objects are referred to as the *elements* of the set. One can specify a set by enclosing a list of suitable objects within braces. Thus, for example, $\{1, 2, 3, 7\}$ denotes the set whose elements are the numbers 1, 2, 3 and 7. If x is an element of some set X then we denote this fact by writing $x \in X$. Conversely, if x is not an element of the set X then we write $x \notin X$. We denote by $\emptyset$ the *empty set*, which is defined to be the set with no elements.

We denote by $\mathbb{N}$ the set $\{1, 2, 3, 4, 5 \ldots\}$ of all *positive integers* (also known as *natural numbers*), and we denote by $\mathbb{Z}$ the set

$$\{\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$$

of all *integers* (or 'whole numbers'). We denote by $\mathbb{Q}$ the set of *rational numbers* (i.e., numbers of the form p/q where p and q are integers and $q \neq 0$), and we denote be $\mathbb{R}$ and $\mathbb{C}$ the sets of real numbers and complex numbers respectively.

If X and Y are sets then the *union* $X \cup Y$ of X and Y is defined to be the set of all elements that belong either to X or to Y (or to both), the *intersection* $X \cap Y$ of X and Y is defined to be the set of all elements that belong to both X and Y, and the *difference* $X \setminus Y$ of X and Y is defined to be the set of all elements that belong to X but do not belong to Y. Thus, for example, if

$$X = \{2, 4, 6, 8\}, \qquad Y = \{3, 4, 5, 6, 7\}$$

then

$$X \cup Y = \{2, 3, 4, 5, 6, 7, 8\}, \qquad X \cap Y = \{4, 6\},$$

$$X \setminus Y = \{2, 8\}, \qquad Y \setminus X = \{3, 5, 7\}.$$

If X and Y are sets, and if every element of X is also an element of Y then we say that X is a *subset* of Y, and we write $X \subset Y$. We use the notation $\{y \in Y : P(y)\}$ to denote the subset of a given set Y consisting of all elements y of Y with some given property $P(y)$. Thus for example $\{n \in \mathbb{Z} : n > 0\}$ denotes the set of all integers n satisfying $n > 0$ (i.e., the set $\mathbb{N}$ of all positive integers).

Rational and Irrational Numbers

Rational numbers are numbers that can be expressed as fractions of the form p/q, where p and q are integers (i.e., 'whole numbers') and $q \neq 0$. The set of rational numbers is denoted by $\mathbb{Q}$. Operations of addition, subtraction, multiplication and division are defined on $\mathbb{Q}$ in the usual manner. In addition the set of rational numbers is ordered.

There are however certain familiar numbers which cannot be represented in the form p/q, where p and q are integers. These include $\sqrt{2}$, $\sqrt{3}$, π and e. Such numbers are referred to as *irrational numbers*. The irrationality of $\sqrt{2}$ is an immediate consequence of the following famous result, which was discovered by the Ancient Greeks.

> *There do not exist non-zero integers p and q with the property that $p^2 = 2q^2$.*

Proof. Let us suppose that there exist non-zero integers p and q with the property that $p^2 = 2q^2$. We show that this leads to a contradiction. Without loss of generality we may assume that p and q are not both even (since if both p and q were even then we could replace p and q by $p/2^k$ and $q/2^k$ respectively, where k is the largest positive integer with the property that 2^k divides both p and q). Now $p^2 = 2q^2$, hence p^2 is even. It follows from this that p is even (since the square of an odd integer is odd). Therefore $p = 2r$ for some integer r. But then $2q^2 = 4r^2$, so that $q^2 = 2r^2$. Therefore q^2 is even, and hence q is even. We have thus shown that both p and q are even. But this contradicts our assumption that p and q are not both even. This contradiction shows that there cannot exist integers p and q with the property that $p^2 = 2q^2$, and thus proves that $\sqrt{2}$ is an irrational number. ∎

This result shows that the rational numbers are not sufficient for the purpose of representing lengths arising in familiar Euclidean geometry. Indeed consider the right-angled isosceles triangle whose short sides are q units long. Then the hypotenuse is $\sqrt{2}q$ units long, by Pythagoras' Theorem. Proposition **??** shows that it is not possible to find a unit of length for which the two short sides of this right-angled isosceles triangle are q units long and the hypotenuse is p units long, where both p and q are integers. We must therefore enlarge the system of rational numbers to obtain a number system which contains irrational numbers such as $\sqrt{2}$, $\sqrt{3}$, π and e, and which is capable of representing the lengths of line segments and similar quantities arising in geometry and physics. The rational and irrational numbers belonging to this number system are known as *real numbers*.

Ordered Fields

An *ordered field* $\mathbb{F}$ consists of a set $\mathbb{F}$ on which are defined binary operations $+$ of addition and $\times$ of multiplication, together with an ordering relation $<$, where these binary operations and ordering relation satisfy the following axioms:—

1. if u and v are elements of $\mathbb{F}$ then their sum $u+v$ is also a element of $\mathbb{F}$;
2. (*the Commutative Law for addition*) $u + v = v + u$ for all elements u and v of $\mathbb{F}$;
3. (*the Associative Law for addition*) $(u + v) + w = u + (v + w)$ for all elements u, v and w of $\mathbb{F}$;
4. there exists an element of $\mathbb{F}$, denoted by 0, with the property that $u + 0 = x = 0 + u$ for all elements u of $\mathbb{F}$;
5. for each element u of $\mathbb{F}$ there exists some element $-u$ of $\mathbb{F}$ with the property that $u + (-u) = 0 = (-u) + u$;
6. if u and v are elements of $\mathbb{F}$ then their product $u \times v$ is also a element of $\mathbb{F}$;
7. (*the Commutative Law for multiplication*) $u \times v = v \times u$ for all elements u and v of $\mathbb{F}$;
8. (*the Associative Law for multiplication*) $(u \times v) \times w = u \times (v \times w)$ for all elements u, v and w of $\mathbb{F}$,
9. there exists an element of $\mathbb{F}$, denoted by 1, with the property that $u \times 1 = u = 1 \times u$ for all elements u of $\mathbb{F}$, and moreover $1 \neq 0$,
10. for each element u of $\mathbb{F}$ satisfying $u \neq 0$ there exists some element u^{-1} of $\mathbb{F}$ with the property that $u \times u^{-1} = 1 = u^{-1} \times u$,

11. (*the Distributive Law*) $u \times (v + w) = (u \times v) + (u \times w)$ for all elements u, v and w of $\mathbb{F}$,

12. (*the Trichotomy Law*) if u and v are elements of $\mathbb{F}$ then one and only one of the three statements $u < v$, $u = v$ and $u < v$ is true,

13. (*transitivity of the ordering*) if u, v and w are elements of $\mathbb{F}$ and if $u < v$ and $v < w$ then $u < w$,

14. if u, v and w are elements of $\mathbb{F}$ and if $u < v$ then $u + w < v + w$,

15. if u and v are elements of $\mathbb{F}$ which satisfy $0 < u$ and $0 < v$ then $0 < u \times v$,

The operations of subtraction and division are defined on an ordered field $\mathbb{F}$ in terms of the operations of addition and multiplication on that field in the obvious fashion: $u - v = u + (-v)$ for all elements u and v of $\mathbb{F}$, and moreover $u/v = uv^{-1}$ provided that $v \neq 0$.

Example The rational numbers, with the standard ordering, and the standard operations of addition, subtraction, multiplication, and division constitute an ordered field.

Example Let $\mathbb{Q}(\sqrt{2})$ denote the set of all numbers that can be represented in the form $b+c\sqrt{2}$, where b and c are rational numbers. The sum and difference of any two numbers belonging to $\mathbb{Q}(\sqrt{2})$ themselves belong to $\mathbb{Q}(\sqrt{2})$. Also the product of any two numbers $\mathbb{Q}(\sqrt{2})$ itself belongs to $\mathbb{Q}(\sqrt{2})$ because, for any rational numbers b, c, e and f,

$$(b + c\sqrt{2})(e + f\sqrt{2}) = (be + 2cf) + (bf + ce)\sqrt{2},$$

and both $be + 2cf$ and $bf + ce$ are rational numbers. The reciprocal of any non-zero element of $\mathbb{Q}(\sqrt{2})$ itself belongs to $\mathbb{Q}(\sqrt{2})$, because

$$\frac{1}{b + c\sqrt{2}} = \frac{b - c\sqrt{2}}{b^2 - 2c^2}.$$

for all rational numbers b and c. It is then a straightforward exercise to verify that $\mathbb{Q}(\sqrt{2})$ is an ordered field.

The *absolute value* $|x|$ of an element number x of an ordered field $\mathbb{F}$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

Note that $|x| \geq 0$ for all x and that $|x| = 0$ if and only if $x = 0$. Also $|x + y| \leq |x| + |y|$ and $|xy| = |x||y|$ for all elements x and y of the ordered field $\mathbb{F}$.

Let D be a subset of an ordered field $\mathbb{F}$. An element u of $\mathbb{F}$ is said to be an *upper bound* of the set D if $x \leq u$ for all $x \in D$. The set D is said to be *bounded above* if such an upper bound exists.

Definition Let $\mathbb{F}$ be an ordered field, and let D be some subset of $\mathbb{F}$ which is bounded above. An element s of $\mathbb{F}$ is said to be the *least upper bound* (or *supremum*) of D (denoted by $\sup D$) if s is an upper bound of D and $s \leq u$ for all upper bounds u of D.

Example The rational number 2 is the least upper bound, in the ordered field of rational numbers, of the sets $\{x \in \mathbb{Q} : x \leq 2\}$ and $\{x \in \mathbb{Q} : x < 2\}$. Note that the first of these sets contains its least upper bound, whereas the second set does not.

The axioms (1)–(15) listed above that characterize ordered fields are not in themselves sufficient to fully characterize the real number system. (Indeed any property of real numbers that could be derived solely from these axioms would be equally valid in any ordered field whatsoever, and in particular would be valid were the system of real numbers replaced by the system of rational numbers.) We require in addition the following axiom:—

> *the Least Upper Bound Axiom*: given any non-empty set D of real numbers that is bounded above, there exists a real number $\sup D$ that is the least upper bound for the set D.

A *lower bound* of a set D of real numbers is a real number l with the property that $l \leq x$ for all $x \in D$. A set D of real numbers is said to be *bounded below* if such a lower bound exists. If D is bounded below, then there exists a greatest lower bound (or *infimum*) $\inf D$ of the set D. Indeed $\inf D = -\sup\{x \in \mathbb{R} : -x \in D\}$.

Remark We have simply listed above a complete set of axioms for the real number system. We have not however proved the existence of a system of real numbers satisfying these axioms. There are in fact several constructions of the real number system: one of the most popular of these is the representation of real numbers as *Dedekind sections* of the set of rational numbers. For an account of the this construction, and for a proof that these axioms are sufficient to characterize the real number system, see chapters 27–29 of *Calculus*, by M. Spivak. The construction of the real number system using Dedekind cuts is also described in detail in the Appendix to Chapter 1 of *Principles of Real Analysis* by W. Rudin.

Remarks on the Existence of Least Upper Bounds

We present an argument here that is intended to show that if the system of real numbers has all the properties that one would expect it to possess, then it must satisfy the Least Upper Bound Axiom.

Let $\mathbb{F}$ be an ordered field that contains the field $\mathbb{Q}$ of rational numbers. The set $\mathbb{Z}$ is a subset of $\mathbb{Q}$. Thus $\mathbb{Z} \subset \mathbb{Q}$ and $\mathbb{Q} \subset \mathbb{F}$, and therefore $\mathbb{Z} \subset \mathbb{F}$.

Definition Let $\mathbb{F}$ be an ordered field that contains the field of rational numbers. The field $\mathbb{F}$ is said to satisfy the *Axiom of Archimedes* if, given any element x of $\mathbb{F}$, there exists some integer n satisfying $n \geq x$.

The Axiom of Archimedes excludes the possibility of "infinitely large" elements of the ordered field $\mathbb{F}$. Given that all real numbers should be representable in decimal arithmetic, any real number must be less than some positive integer. Thus we expect the system of real numbers to satisfy the Axiom of Archimedes.

Lemma 0.1 *Let $\mathbb{F}$ be an ordered field that satisfies the Axiom of Archimedes. Then, given any element x of $\mathbb{F}$ satisfying $x > 0$, there exists some positive integer n such that $x > \dfrac{1}{n} > 0$.*

Proof The Axiom of Archimedes ensures the existence of a positive integer n satisfying $n > \dfrac{1}{x}$. Then

$$n - \frac{1}{x} > 0 \quad \text{and} \quad \frac{x}{n} = x \times \frac{1}{n} > 0,$$

and therefore

$$x - \frac{1}{n} = \left(n - \frac{1}{x}\right) \times \frac{x}{n} > 0,$$

and thus $x > \dfrac{1}{n}$, as required. ∎

Now let $\mathbb{F}$ be an ordered field containing as a subfield the field $\mathbb{Q}$ of rational numbers. We suppose also that $\mathbb{F}$ satisfies the Axiom of Archimedes. Let D be a subset of $\mathbb{F}$ which is bounded above. The Axiom of Archimedes then ensures that there exists some integer that is an upper bound for the set D. It follows from this that there exists some integer m that is the largest integer that is *not* an upper bound for the set D. Then m is not an upper bound for D, but $m + 1$ is. Let

$$E = \{x \in \mathbb{F} : x \geq 0 \quad \text{and} \quad m + x \in D\}.$$

Then E is non-empty and $x \leq 1$ for all $x \in E$. Suppose that there exists a least upper bound $\sup E$ in $\mathbb{F}$ for the set E. Then $m + \sup E$ is a least upper bound for the set D, and thus $\sup D$ exists, and $\sup D = m + \sup E$. Thus, in order to show that every non-empty subset of D that is bounded above has a least upper bound, it suffices to show this for subsets D of $\mathbb{F}$ with the property that $0 \leq x \leq 1$ for all $x \in D$.

Now let $\mathbb{F}$ be an ordered field containing the field $\mathbb{Q}$ of rational numbers that satisfies the Axiom of Archimedes, and let D be a subset of $\mathbb{F}$ with the property that $0 \leq x \leq 1$ for all $x \in D$. Then, for each positive integer m, let u_m denote the largest non-negative integer for which $u_m \times (10)^{-m}$ is *not* an upper bound for the set D. Then $0 \leq u_m < (10)^m$ and $(u_m + 1)(10)^{-m}$ is an upper bound for the set D. Thus if there were to exist a least upper bound s for the set D, then s would have to satisfy

$$\frac{u_m}{(10)^m} < s \leq \frac{u_m}{(10)^m} + \frac{1}{(10)^m}$$

for $m = 1, 2, 3, \ldots$. Now if $m > 1$ then definitions of u_m and u_{m-1} ensure that $(10u_{m-1}) \times (10)^{-m}$ is not an upper bound for the set D but $(10u_{m-1} + 10) \times (10)^{-m}$ is an upper bound for the set D. It follows that

$$10u_{m-1} \leq u_m < 10u_{m-1} + 10.$$

Let $d_1 = u_1$, and let $d_m = u_m - 10u_{m-1}$ for all integers m satisfying $m > 1$. Then d_m is an integer satisfying $0 \leq d_m < 10$ for $m = 1, 2, 3, \ldots$, and

$$\frac{u_m}{(10)^m} = \frac{d_m}{(10)^m} + \frac{u_{m-1}}{(10)^{m-1}}.$$

It follows that

$$\frac{u_m}{(10)^m} = \sum_{k=1}^{m} \frac{d_k}{(10)^k}.$$

Any least upper bound t for the set D would therefore have to satisfy the inequalities

$$\sum_{k=1}^{m} \frac{d_k}{(10)^k} < t \leq \sum_{k=1}^{m} \frac{d_k}{(10)^k} + \frac{1}{(10)^m}$$

for all positive integers m.

Now suppose that every well-formed decimal expansion determines a corresponding element of the ordered field $\mathbb{F}$. Assuming this, we conclude that there must exist some element s of the ordered field $\mathbb{F}$ whose decimal expansion takes the form

$$0.d_1\, d_2\, d_3\, d_4\, d_5, \ldots$$

The basic properties of decimal expansions then ensure that

$$\sum_{k=1}^{m} \frac{d_k}{(10)^k} \leq s \leq \sum_{k=1}^{m} \frac{d_k}{(10)^k} + \frac{1}{(10)^m}.$$

Let ε be an element of $\mathbb{F}$ satisfying $\varepsilon > 0$. Then, because the ordered field $\mathbb{F}$ is required to satisfy the Axiom of Archimedes, a positive integer m can be chosen large enough to ensure that $0 < (10)^{-m} < \varepsilon$. Then

$$s - \varepsilon < \sum_{k=1}^{m} \frac{d_k}{(10)^k} = \frac{u_m}{(10)^m},$$

and therefore $s - \varepsilon$ cannot be an upper bound for the set D. Also

$$s + \varepsilon > \sum_{k=1}^{m} \frac{d_k}{(10)^k} + \frac{1}{(10)^m} = \frac{u_m}{(10)^m} + \frac{1}{(10)^m},$$

and therefore $s + \varepsilon$ is an upper bound for the set D. We see therefore if s is an element of $\mathbb{F}$ satisfying $0 \leq s \leq 1$, and if s is determined by the decimal expansion whose successive decimal digits are $d_1, d_2, d_3, \ldots$, where these digits are determined by D as described above, then $s - \varepsilon$ cannot be an upper bound for the set D for any $\varepsilon > 0$, but $s + \varepsilon$ must be an upper bound for the set D for all $\varepsilon > 0$.

Now if there were to exist any element x of D satisfying $x > s$, then we could obtain a contradiction on choosing $\varepsilon \in \mathbb{F}$ such that $0 < \varepsilon < x - s$. It follows that $x \leq s$ for all $x \in D$, and thus s is an upper bound for the set D. But if $\varepsilon > 0$ then $s - \varepsilon$ is not an upper bound for the set D. Therefore s must be the least upper bound for the set D.

This analysis shows that if $\mathbb{F}$ is an ordered field, containing the field of rational numbers, that satisfies the Axiom of Archimedes, and if every decimal expansion determines a corresponding element of $\mathbb{F}$, then every non-empty subset of $\mathbb{F}$ that is bounded above must have a least upper bound. The ordered field $\mathbb{F}$ must therefore satisfy the Least Upper Bound Axiom.

This justifies the characterization of the field $\mathbb{R}$ of real numbers as an ordered field that satisfies the Least Upper Bound Axiom.

Intervals

Given real numbers a and b satisfying $a \leq b$, we define

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}.$$

If $a < b$ then we define

$$(a,b) = \{x \in \mathbb{R} : a < x < b\}, \qquad [a,b) = \{x \in \mathbb{R} : a \leq x < b\},$$
$$(a,b] = \{x \in \mathbb{R} : a < x \leq b\}.$$

For each real number c, we also define

$$[c,+\infty) = \{x \in \mathbb{R} : c \leq x\}, \qquad (c,+\infty) = \{x \in \mathbb{R} : c < x\},$$
$$(-\infty,c] = \{x \in \mathbb{R} : x \leq c\}, \qquad (-\infty,c) = \{x \in \mathbb{R} : x < c\}.$$

All these subsets of $\mathbb{R}$ are referred to as *intervals*. An *interval* I may be defined as a non-empty set of real numbers with the following property: if s, t and u are real numbers satisfying $s < t < u$ and if s and u both belong to the interval I then t also belongs to the interval I. Using the Least Upper Bound Axiom, one can prove that every interval in $\mathbb{R}$ is either one of the intervals defined above, or else is the whole of $\mathbb{R}$.

The Real Number System

From the time of the ancient Greeks to the present day, mathematicians have recognized the necessity of establishing rigorous foundations for the discipline. This led mathematicians such as Bolzano, Cauchy and Weierstrass to establish in the nineteenth century the definitions of continuity, limits and convergence that are required in order to establish a secure foundation upon which to build theories of real and complex analysis that underpin the application of standard techiques of the differential calculus in one or more variables.

But mathematicians in the nineteenth century realised that, in order to obtain satisfactory proofs of basic theorems underlying the applications of calculus, they needed a deeper understanding of the nature of the real number system. Accordingly Dedekind developed a theory in which real numbers were represented by *Dedekind sections*, in which each real number was characterized by means of a partition of the set of rational numbers into two subsets, where every rational number belonging to the first subset is less than every rational number belonging to the second. Dedekind published his construction of the real number system in 1872, in the work *Stetigkeit und irrationale Zahlen*. In the same year, Georg Cantor published a construction of the real number system in which real numbers are represented by sequences of rational numbers satisfying an appropriate convegence criterion.

It has since been shown that the system of real numbers is completely characterized by the statement that the real numbers constitute an ordered field which satisfies the Least Upper Bound Axiom.