Notes on Real Analysis
The Contraction Mappping Theorem

David R. Wilkins

Complete Metric Spaces

The following definitions and remarks define the concept of a *complete metric space*, and set out the reasons why any closed subset of a Euclidean space is a complete metric space.

Definition A *metric space* (X, d) consists of a set X together with a *distance function* $d: X \times X \to [0, +\infty)$ on X satisfying the following axioms:

(i) $d(x, y) \geq 0$ for all $x, y \in X$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$,

(iv) $d(x, y) = 0$ if and only if $x = y$.

Definition Let X be a metric space with distance function d, let p be point of X, and let $x_1, x_2, x_3, \ldots$ be an infinite sequence of points of X. The infinite sequence $x_1, x_2, x_3, \ldots$ is said to *converge* to the point p if, given any strictly positive real number ε, there exists some positive integer N with the property that $d(x_j, p) < \varepsilon$ for all integers j satisfying $j \geq N$.

Definition Let X be a metric space with distance function d, let p be point of X, and let $x_1, x_2, x_3, \ldots$ be an infinite sequence of points of X. The infinite sequence $x_1, x_2, x_3, \ldots$ is said to be a *Cauchy sequence* of points of X if, given any strictly positive real number ε, there exists some positive integer N with the property that $d(x_j, x_k) < \varepsilon$ for all integers j and k satisfying $j \geq N$ and $k \geq N$.

Definition A metric space is said to be *complete* if every Cauchy sequence of points belonging to that metric space is convergent.

Cauchy's Criterion for Convergence (also known as the *General Principle of Convergence*) ensures that all finite-dimensional Euclidean spaces are complete metric spaces. Moreover if an infinite sequence of points belonging to some given closed set in a Eucldidean space converges to some point of the Euclidean space, then the limit of the infinite sequence belongs to the given closed set. Consequently any Cauchy sequence of points belonging to a given closed set in a Euclidean space must converge to some point of that closed set. We conclude from this that every closed subset of a finite-dimensional Euclidean space is a complete metric space.

The Contraction Mapping Theorem

Definition Let X be a metric space. A function $\varphi\colon X \to X$ mapping that set X into itself is said to be a *contraction mapping* on X if there exists some non-negative real number λ satisfying $\lambda < 1$ that is such as to ensure that

$$d(\varphi(u), \varphi(v)) \leq \lambda d(u, v)$$

for all points u and v of X.

Theorem A *Let X be a complete metric space, and let $\varphi\colon X \to X$ be a contraction mapping on the set X. Then there exists a unique point p of X for which $\varphi(p) = p$.*

Proof The function $\varphi\colon X \to X$ is a contraction mapping. Therefore a non-negative real number λ satisfying $\lambda < 1$ can be associated with the function φ so as to ensure that

$$d(\varphi(u), \varphi(v)) \leq \lambda d(u, v)$$

for all points u and v of X.

Choose $x_0 \in X$, and let $x_1, x_2, x_3, \ldots$ be the infinite sequence of points of X defined such that $x_j = \varphi(x_{j-1})$ for all positive integers j. Then

$$d(x_j, x_{j+1}) \leq \lambda d(x_j, x_{j-1})$$

for all positive integers j. It follows that

$$d(x_j, x_{j+1}) \leq \lambda^j d(x_0, x_1)$$

for all positive integers j, and therefore

$$\begin{aligned} d(x_j, x_k) &\leq \left(\sum_{m=j}^{k-1} \lambda^m \right) d(x_0, x_1) \leq \frac{\lambda^j - \lambda^k}{1-\lambda} d(x_0, x_1) \\ &\leq \frac{\lambda^j}{1-\lambda} d(x_0, x_1) \end{aligned}$$

for all positive integers j and k satisfying $j < k$.

Now the inequality $\lambda < 1$ ensures that, given any positive real number ε, there exists a positive integer N large enough to ensure that $\lambda^j d(x_0, x_1) < (1-\lambda)\varepsilon$ for all integers j satisfying $j \geq N$. Then $d(x_j, x_k) < \varepsilon$ for all positive integers j and k satisfying $k > j \geq N$. The infinite sequence $x_1, x_2, x_3, \ldots$ is thus a Cauchy sequence of points of X. The completeness of the metric space X then ensures that the infinite sequence $x_1, x_2, x_3, \ldots$ converges to some point p of X. Moreover the continuity of the contraction mapping φ ensures that

$$p = \lim_{j \to +\infty} x_{j+1} = \lim_{j \to +\infty} \varphi(x_j) = \varphi\left(\lim_{j \to +\infty} x_j \right) = \varphi(p).$$

We have thus proved the existence of a point p of X for which $\varphi(p) = p$.

Now let q be any point of the complete metric space X with the property that $\varphi(q) = q$. Then

$$d(p, q) = d(\varphi(q), \varphi(p)) \leq \lambda d(p, q).$$

But $\lambda < 1$. It follows that the Euclidean distance $d(p, q)$ from p to q cannot be strictly positive, and therefore $p = q$. We conclude therefore that p is the unique point of X for which $\varphi(p) = p$, as required. ∎