Notes on Real Analysis Convergence and Continuity in the Theory of Metric Spaces

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Metric Spaces

Metric spaces are sets provided with distance functions. There are criteria, expressible through the utilization of distance functions, that determine which infinite sequences in a metric space are convergent, and which functions between metric spaces are continuous. However any metric space has a collection of open sets, determined by the distance function, that gives the metric space the structure of a topological space. The concepts of convergence and continuity that arise within the theory of topological spaces are consistent with the criteria that characterize convergence and continuity in metric space contexts using distance functions.

Definition A *metric space* (X, d) consists of a set X together with a *distance function* $d: X \times X \to [0, +\infty)$ on X satisfying the following axioms:

(i) $d(x, y) \geq 0$ for all $x, y \in X$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$,

(iv) $d(x, y) = 0$ if and only if $x = y$.

The quantity $d(x, y)$ should be thought of as measuring the *distance* between the points x and y. The inequality $d(x, z) \leq d(x, y) + d(y, z)$ is referred to as the *Triangle Inequality*. The elements of a metric space are usually referred to as *points* of that metric space.

An n-dimensional Euclidean space $\mathbb{R}^n$ is a metric space with respect to the *Euclidean distance function* d, defined so that

$$d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Any subset X of $\mathbb{R}^n$ may be regarded as a metric space whose distance function is the restriction to X of the Euclidean distance function on $\mathbb{R}^n$.

Open Sets in Metric Spaces

Definition Let X be a metric space with distance function d. Given a point p of X and a positive real number η, the *open ball* $B_X(p, \eta)$ in X of *radius* η centred on the point p consists of all points of the set X whose distance from the point p is less than η.

We see therefore that

$$B_X(p, \eta) = \{x \in X : d(x, p) < \eta\}$$

for all points p of X and positive real numbers η.

Definition Let X be a metric space with distance function d. A subset V of X is said to be *open in* X if, given any point of V, there exists an open ball in X of positive radius, centred on that point, which is wholly contained within the set V.

By convention the empty set $\emptyset$ is also considered to be open in the given set X (on the grounds that there does not exist any point of the empty set that is not the centre of some open ball contained in the empty set).

Thus given any metric space X with distance function d, and given any subset V of X, the set V is said to be open in X if and only if, given any point p of V, there exists some strictly positive real number δ such that $B_X(p, \delta) \subset V$, where

$$B_X(p, \delta) = \{x \in X : d(x, p) < \delta\}.$$

Lemma A *Let X be a metric space, and let p be a point of X. Then, for any positive real number η, the open ball $B_X(p, \eta)$ in X of radius η centred on p is open in X.*

Proof Let d denote the distance function on the metric space X, and let q be an element of $B_X(p, \eta)$. We must show that there exists some positive real number δ such that $B_X(q, \delta) \subset B_X(p, \eta)$. Let $\delta = \eta - d(q, p)$. Then $\delta > 0$, since $d(q, p) < \eta$. Moreover if $x \in B_X(q, \delta)$ then

$$d(x, p) \leq d(x, q) + d(q, p) < \delta + d(q, p) = \eta,$$

by the Triangle Inequality, and hence $x \in B_X(p, \eta)$. Thus $B_X(q, \delta) \subset B_X(p, \eta)$. This shows that $B_X(p, \eta)$ is an open set, as required. ∎

Lemma B *Let X be a metric space with distance function d, and let p be a point of X. Then, for any non-negative real number η, the set $\{x \in X : d(x, p) > \eta\}$ is an open set in X.*

Proof Let q be a point of X satisfying $d(q, p) > \eta$, and let x be any point of X satisfying $d(x, q) < \delta$, where $\delta = d(q, p) - \eta$. Then

$$d(q, p) \leq d(q, x) + d(x, p),$$

by the Triangle Inequality. It follows that

$$d(x, p) \geq d(q, p) - d(x, q) > d(q, p) - \delta = \eta.$$

Thus $B_X(q, \delta)$ is contained in the given set. The result follows. ∎

Convergence of Sequences and Open Sets

Definition Let X be a metric space with distance function d, let p be point of X, and let $x_1, x_2, x_3, \ldots$ be an infinite sequence of points of X. The infinite sequence $x_1, x_2, x_3, \ldots$ is said to *converge* to the point p if, given any strictly positive real number ε, there exists some positive integer N with the property that $d(x_j, p) < \varepsilon$ for all integers j satisfying $j \geq N$.

Lemma C *An infinite sequence $x_1, x_2, x_3, \ldots$ of points in a metric space X converges to a point p of X if and only if, given any open set V which contains p, there exists some positive integer N such that $x_j \in V$ for all positive integers j satisfying $j \geq N$.*

Proof Let d denote the distance function on the metric space X. Suppose that the infinite sequence $x_1, x_2, x_3, \ldots$ of points in the metric space X has the property that, given any open set V which contains p, there exists some positive integer N such that $x_j \in V$ whenever $j \geq N$. Let some positive real number ε be given. The open ball $B_X(p, \varepsilon)$ of radius ε centred on the point p

is an open set by Lemma A. Therefore there exists some positive integer N such that $x_j \in B_X(p, \varepsilon)$ whenever $j \geq N$. Thus $d(x_j, p) < \varepsilon$ whenever $j \geq N$. This shows that the infinite sequence converges to the point p.

Conversely, suppose that the infinite sequence $x_1, x_2, x_3, \ldots$ of points of the metric space X converges to the point p. Let V be an open set to which that point p belongs. Then there exists some positive real number ε such that the open ball $B_X(p, \varepsilon)$ of radius ε centred on p is a subset of V. All points x of the metric space X that satisfy $d(x, p) < \varepsilon$ then belong to the open set V. But there exists some positive integer N with the property that $d(x_j, p) < \varepsilon$ whenever $j \geq N$, since the sequence converges to p. Therefore $x_j \in V$ whenever $j \geq N$, as required. ∎

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The Topology of Metric Spaces

Proposition D *Let X be a metric space with distance function d. The collection of open sets in X has the following properties:—*

(i) the empty set $\emptyset$ and the whole set X are both open in X;

(ii) the union of any collection of open sets in X is itself open in X;

(iii) the intersection of any finite *collection of open sets in X is itself open in X.*

Proof The empty set $\emptyset$ is an open set by convention. Moreover the definition of an open set is satisfied trivially by the whole set X. This proves (i).

Let $\mathcal{C}$ be any collection of open sets in X, and let W denote the union of all the open sets belonging to $\mathcal{C}$. We must show that W is itself open in X. Let $p \in W$. Then $p \in V$ for some set V belonging to the collection $\mathcal{C}$. It follows that there exists some positive real number δ such that $B_X(p, \delta) \subset V$. But $V \subset W$, and thus $B_X(p, \delta) \subset W$. This shows that W is open in X. This proves (ii).

Finally let $V_1, V_2, V_3, \ldots, V_k$ be a *finite* collection of subsets of X that are open in X, and let V denote the intersection $V_1 \cap V_2 \cap \cdots \cap V_k$ of these sets. Let $p \in V$. Now $p \in V_j$ for $j = 1, 2, \ldots, k$, and therefore there exist strictly positive real numbers $\delta_1, \delta_2, \ldots, \delta_k$ such that $B_X(p, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. (This is where we need the fact that we are dealing with a finite collection of sets.) Now $B_X(p, \delta) \subset B_X(p, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$, and thus $B_X(p, \delta) \subset V$. Thus the intersection V of the sets $V_1, V_2, \ldots, V_k$ is itself open in X. This proves (iii). ∎

Closed Sets in Metric Spaces

Definition Let X be a metric space with distance function d. A subset F of X is said to be *closed* in X if and only if its complement $X \setminus F$ in X is open in X.

(Recall that $X \setminus F = \{x \in X : x \notin F\}$.)

Let $\mathcal{A}$ be some collection of subsets of a set X. Then

$$X \setminus \bigcup_{S \in \mathcal{A}} S = \bigcap_{S \in \mathcal{A}} (X \setminus S), \qquad X \setminus \bigcap_{S \in \mathcal{A}} S = \bigcup_{S \in \mathcal{A}} (X \setminus S)$$

(i.e., the complement of the union of some collection of subsets of X is the intersection of the complements of those sets, and the complement of the intersection of some collection of subsets of X is the union of the complements of those sets).

Indeed let $\mathcal{A}$ be some collection of subsets of a set X, and let x be a point of X. Then

$$\begin{aligned} x \in X \setminus \bigcup_{S \in \mathcal{A}} S &\iff x \notin \bigcup_{S \in \mathcal{A}} S \\ &\iff \text{for all } S \in \mathcal{A},\ x \notin S \\ &\iff \text{for all } S \in \mathcal{A},\ x \in X \setminus S \\ &\iff x \in \bigcap_{S \in \mathcal{A}} (X \setminus S), \end{aligned}$$

and therefore

$$X \setminus \bigcup_{S \in \mathcal{A}} S = \bigcap_{S \in \mathcal{A}} (X \setminus S).$$

Again let x be a point of X. Then

$$\begin{aligned} x \in X \setminus \bigcap_{S \in \mathcal{A}} S &\iff x \notin \bigcap_{S \in \mathcal{A}} S \\ &\iff \text{there exists } S \in \mathcal{A} \text{ for which } x \notin S \\ &\iff \text{there exists } S \in \mathcal{A} \text{ for which } x \in X \setminus S \\ &\iff x \in \bigcup_{S \in \mathcal{A}} (X \setminus S), \end{aligned}$$

and therefore

$$X \setminus \bigcap_{S \in \mathcal{A}} S = \bigcup_{S \in \mathcal{A}} (X \setminus S).$$

The following result therefore follows directly from Proposition D.

Proposition E *Let X be a metric space with distance function d. The collection of closed sets in X has the following properties:—*

(i) the empty set $\emptyset$ and the whole set X are both closed in X;

(ii) the intersection of any collection of closed sets in X is itself closed in X;

(iii) the union of any finite *collection of closed sets in X is itself closed in X.*

Proof The empty set $\emptyset$ is the complement in X of the whole set X. The set X is open in itself. It follows that the empty set $\emptyset$ is closed in X.

The whole set X is the complement in X of the empty set. The empty set is open in X. It follows that the whole set X is closed in itself.

Next let $\mathcal{C}$ be a collection of subsets of X that are closed in X, and let G be the intersection of all the sets that are members of the collection $\mathcal{C}$. Now the complement in X of the set G, being the complement of the intersection of all the members of the collection $\mathcal{C}$ is the union of the complements of the members of this collection $\mathcal{C}$. Now the complement of each member of the collection $\mathcal{C}$ is open in X. Consequently the union of the complements of the members of the collection must also be open in X. Thus the complement of the set G is open in X, and therefore the set G itself is closed in X.

Now suppose that the collection $\mathcal{C}$ is a finite collection of subsets of X that are closed in X, and let H be the union of all the sets that are members of the finite collection $\mathcal{C}$. Now the complement in X of the set H, being the complement of the union of all the members of the finite collection $\mathcal{C}$ is the intersection of the complements of the members of this finite collection $\mathcal{C}$. Now the complement of each member of the finite collection $\mathcal{C}$ is open in X. Consequently the intersection of the complements of the members of the finite collection must also be open in X. Thus the complement of the set H is open in X, and therefore the set H itself is closed in X. This completes the proof. ∎

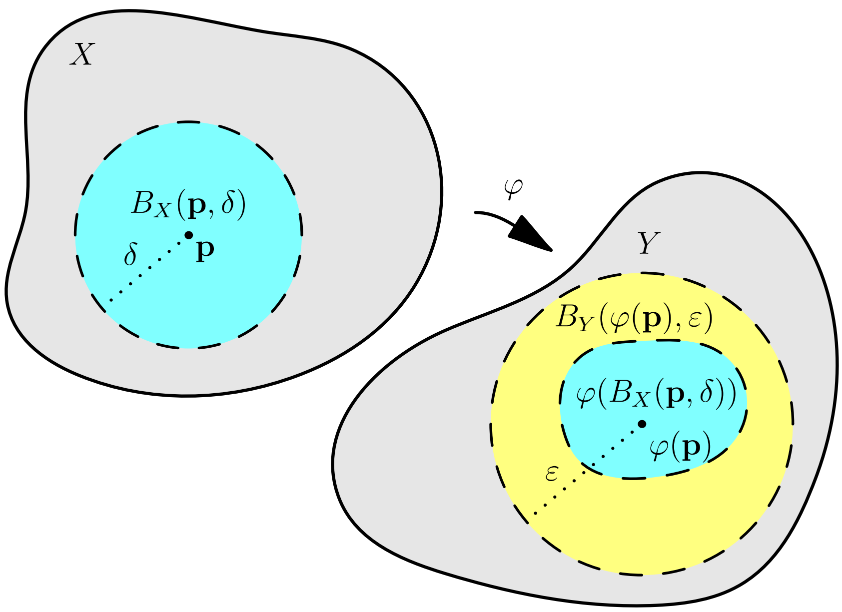
Lemma F *Let X be a metric space with distance function d, and let F be a subset of X which is closed in X. Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of points of F which converges to some point p of X. Then $p \in F$.*

Proof The complement $X \setminus F$ of F in X is open, since F is closed. Suppose that p were a point belonging to $X \setminus F$. It would then follow from Lemma C that $x_j \in X \setminus F$ for all values of j greater than some positive integer N, contradicting the fact that $x_j \in F$ for all j. This contradiction shows that p must belong to F, as required. ∎

The Concept and Basic Properties of Continuity

Definition Let X and Y be metric spaces with distance functions d_X and d_Y respectively. A function $\varphi\colon X \to Y$ from X to Y is said to be *continuous* at a point p of X if and only if, given any strictly positive real number ε, there exists some strictly positive real number δ such that $d_Y(\varphi(x), \varphi(p)) < \varepsilon$ whenever $x \in X$ satisfies $d_X(x,p) < \delta$.

The function $\varphi\colon X \to Y$ is said to be continuous on X if and only if it is continuous at every point p of X.

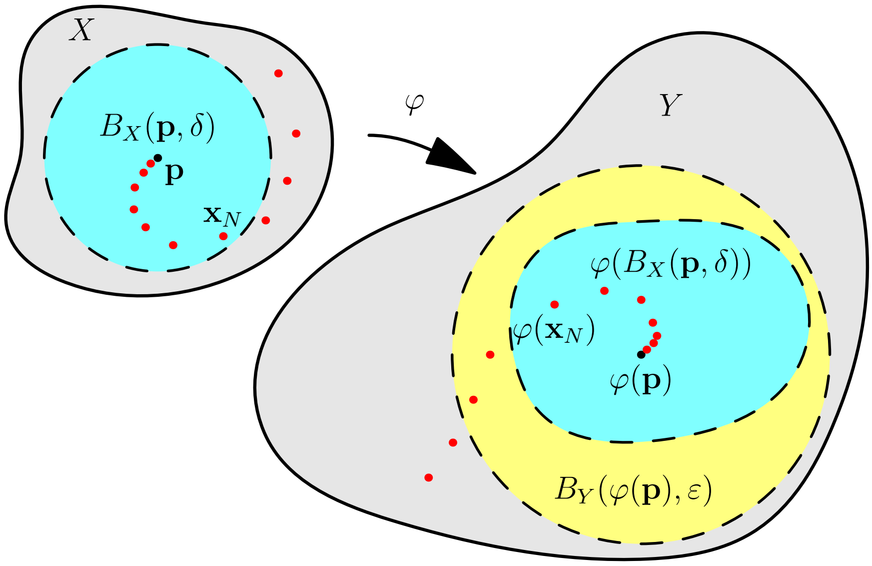


Proposition G *Let X, Y and Z be metric spaces, let $\varphi\colon X \to Y$ be a function from X to Y and let $\psi\colon Y \to Z$ be a function from Y to Z. Suppose that φ is continuous at some point p of X and that ψ is continuous at $\varphi(p)$. Then the composition function $\psi \circ \varphi\colon X \to Z$ is continuous at p.*

Proof Let d_X, d_Y and d_X denote the distance functions on the metric spaces X, Y and Z respectively, let $q = \varphi(p)$, and let some positive real number ε be given. Then there exists some positive real number η such that $d_Z(\psi(y), \psi(q)) < \varepsilon$ for all $y \in Y$ satisfying $d_Y(y,q) < \eta$. But then there exists some positive real number δ such that $d_Y(\varphi(x), q) < \eta$ for all $x \in X$ satisfying $d_X(x,p) < \delta$. It follows that $d_Z(\psi(\varphi(x)), \psi(\varphi(p))) < \varepsilon$ for all $x \in X$ satisfying $d_X(x,p) < \delta$, and thus $\psi \circ \varphi$ is continuous at p, as required. ∎

Proposition H *Let X and Y be metric spaces, and let $\varphi: X \to Y$ be a continuous function from X to Y. Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of points of X which converges to some point p of X. Then the sequence $\varphi(x_1), \varphi(x_2), \varphi(x_3), \ldots$ converges to $\varphi(p)$.*

Proof Let d_X and d_Y denote the distance functions on the metric spaces X and Y respectively, and let some positive real number ε be given. The function φ is continuous at p, and therefore there exists some positive real number δ such that $d_Y(\varphi(x), \varphi(p)) < \varepsilon$ for all $x \in X$ satisfying $d_X(x, p) < \delta$. Also the infinite sequence $x_1, x_2, x_3, \ldots$ converges to the point p, and therefore



there exists some positive integer N such that $d_X(x_j, p) < \delta$ whenever $j \geq N$. It follows that if $j \geq N$ then $d_Y(\varphi(x_j), \varphi(p)) < \varepsilon$. Thus the sequence $\varphi(x_1), \varphi(x_2), \varphi(x_3), \ldots$ converges to $\varphi(p)$, as required. ∎

Let X be a metric space with distance function d, and let $\varphi: X \to \mathbb{R}^n$ be a function mapping the metric space X into $\mathbb{R}^n$ for some positive integer n. Then

$$\varphi(x) = (f_1(x), f_2(x), \ldots, f_n(x))$$

for all $x \in X$, where $f_1, f_2, \ldots, f_n$ are functions from X to $\mathbb{R}$, referred to as the *components* of the function φ.

Proposition I *Let X be a metric space, let $\varphi: X \to \mathbb{R}^n$ be a function mapping the metric space X into $\mathbb{R}^n$, and let $p \in X$. Then the function $\varphi: X \to \mathbb{R}^n$ is continuous at the point p if and only if its components are all continuous at p.*

Proof Note that the ith component f_i of φ is given by $f_i = \pi_i \circ f$, where $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$ is the continuous function which maps $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ onto its ith component y_i. Now any composition of continuous functions is continuous, by Proposition G. Thus if φ is continuous at p, then so are the components of φ.

Conversely suppose that the components of φ are continuous at $p \in X$. Let d denote the distance function on the metric space X, and let some positive real number ε be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $|f_i(x) - f_i(p)| < \varepsilon/\sqrt{n}$ for $x \in X$ satisfying $d(x,p) < \delta_i$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. If $x \in X$ satisfies $d(x,p) < \delta$ then

$$|\varphi(x) - \varphi(p)|^2 = \sum_{i=1}^{n} |f_i(x) - f_i(p)|^2 < \varepsilon^2,$$

and hence $|\varphi(x) - \varphi(p)| < \varepsilon$. Thus the function φ is continuous at p, as required. ∎

Proposition J *Let X be a metric space with distance function d, and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be continuous functions from X to $\mathbb{R}$. Then the functions $f + g$, $f - g$ and $f \cdot g$ are continuous. If in addition $g(x) \neq 0$ for all $x \in X$ then the quotient function f/g is continuous.*

Proof Note that $f + g = s \circ \psi$ and $f \cdot g = m \circ \psi$, where the functions $\psi\colon X \to \mathbb{R}^2$, $s\colon \mathbb{R}^2 \to \mathbb{R}$ and $m\colon \mathbb{R}^2 \to \mathbb{R}$ are defined so that $\psi(x) = (f(x), g(x))$, $s(u,v) = u + v$ and $m(u,v) = uv$ for all $x \in X$ and $u, v \in \mathbb{R}$. One can readily show that the sum function s and the product function m are continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$. It follows from Proposition I and Proposition G that $f + g$ and $f \cdot g$ are continuous, being compositions of continuous functions. Now $f - g = f + (-g)$, and both f and $-g$ are continuous. Therefore $f - g$ is continuous.

Now suppose that $g(x) \neq 0$ for all $x \in X$. Note that $1/g = r \circ g$, where $r\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is the reciprocal function, defined so that $r(t) = 1/t$ for all non-zero real numbers t. Now the reciprocal function r is continuous. Thus the function $1/g$ is a composition of continuous functions and is thus continuous. But then, using the fact that a product of continuous real-valued functions is continuous, we deduce that f/g is continuous. ∎

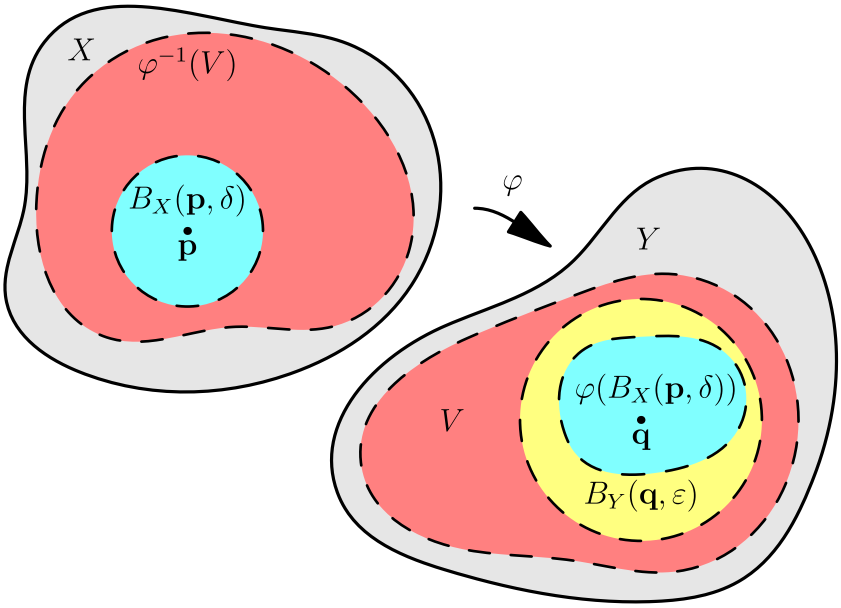
Continuous Functions and Open Sets

Let X and Y be metric spaces with distance functions d_X and d_Y respectively, and let $\varphi\colon X \to Y$ be a function from X to Y. We recall that the function φ is continuous at a point p of X if and only if, given any positive real number ε, there exists some positive real number δ such that $d_Y(\varphi(x), \varphi(p)) < \varepsilon$ for all points x of X satisfying $d_X(x, p) < \delta$. Thus the function $\varphi: X \to Y$ is continuous at p if and only if, given any positive real number ε, there exists some positive real number δ such that the function φ maps the open ball $B_X(p, \delta)$ in X of radius δ centred on the point p into the open ball $B_Y(q, \varepsilon)$ in Y of radius ε centered on the point q, where $q = \varphi(p)$.

Given any function $\varphi: X \to Y$, we denote by $\varphi^{-1}(V)$ the *preimage* of a subset V of Y under the map φ, defined so that $\varphi^{-1}(V) = \{x \in X : \varphi(x) \in V\}$.

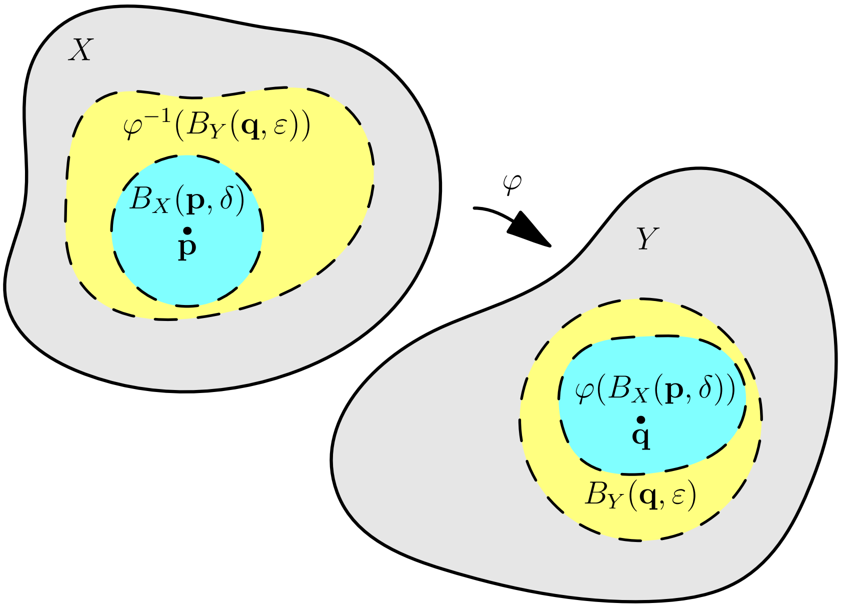
Proposition K *Let X and Y be metric spaces, and let $\varphi: X \to Y$ be a function from X to Y. The function φ is continuous if and only if $\varphi^{-1}(V)$ is open in X for every open subset V of Y.*

Proof Suppose that $\varphi: X \to Y$ is continuous. Let V be an open set in Y. We must show that $\varphi^{-1}(V)$ is open in X. Let p be a point of $\varphi^{-1}(V)$, and let $q = \varphi(p)$. Then $q \in V$. But V is open, hence there exists some positive real number ε with the property that $B_Y(q, \varepsilon) \subset V$. But φ is continuous at p. Therefore there exists some positive real number δ such that φ maps $B_X(p, \delta)$ into $B_Y(q, \varepsilon)$. Thus $\varphi(x) \in V$ for all $x \in B_X(p, \delta)$, showing that $B_X(p, \delta) \subset \varphi^{-1}(V)$. This shows that $\varphi^{-1}(V)$ is open in X for every open set V in Y.



Conversely suppose that $\varphi: X \to Y$ is a function with the property that $\varphi^{-1}(V)$ is open in X for every open set V in Y. Let $p \in X$, and let $q = \varphi(p)$.

We must show that φ is continuous at p. Let some positive real number ε be



given. Then $B_Y(q, \varepsilon)$ is an open set in Y, by Lemma A, hence $\varphi^{-1}(B_Y(q, \varepsilon))$ is an open set in X which contains p. It follows that there exists some positive real number δ such that $B_X(p, \delta) \subset \varphi^{-1}(B_Y(q, \varepsilon))$. Thus, given any positive real number ε, there exists some positive real number δ such that φ maps $B_X(p, \delta)$ into $B_Y(q, \varepsilon)$. We conclude that φ is continuous at the point p, as required. ∎

Let X be a metric space, let $f\colon X \to \mathbb{R}$ be continuous, and let c be some real number. Then the sets

$$\{x \in X : f(x) > c\}$$

and

$$\{x \in X : f(x) < c\}$$

are open in X, and, given real numbers a and b satisfying $a < b$, the set

$$\{x \in X : a < f(x) < b\}$$

is open in X.

Again let X be a metric space, let $f\colon X \to \mathbb{R}$ be continuous, and let c be some real number. Now a subset of X is closed in X if and only if its complement is open in X. Consequently the sets

$$\{x \in X : f(x) \leq c\}$$

and

$$\{x \in X : f(x) \geq c\},$$

being the complements in X of sets that are open in X, must themselves be closed in X. It follows that that set

$$\{x \in X : f(x) = c\},$$

being the intersection of two subsets X that are closed in X, must itself be closed in X.